EE 5581 Prof. Jindal
Information Theory & Coding December 21, 2005

Final Exam

You have 2 hours to complete this exam. You must show your work to receive credit.

1. **Basics** (15 pts)
   Let X_1 and X_2 be i.i.d. Bern(1/2) random variables, and let $Y = \max(X_1, X_2)$. Compute: (5 pts each)

   (a) $H(Y)$

   (b) $I(X_1; Y)$

   (c) $I(X_1, X_2; Y)$

2. **Source Coding** (20 pts)

   (a) Which of the following codes are optimal prefix-free codes for the given source distribution? Briefly justify each answer. (10 pts total)

   i.

x	p(x)	C(x)
1	0.25	110
2	0.5	0
3	0.1	10
4	0.1	111

   ii.

x	p(x)	C(x)
1	0.25	0
2	0.25	10
3	0.25	110
4	0.25	111

   iii.

x	p(x)	C(x)
1	0.3	00
2	0.3	01
3	0.2	10
4	0.2	11

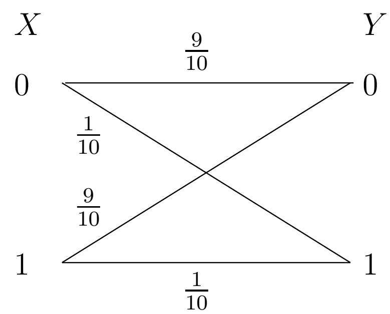
   (b) Consider the following source X:

$$X = \begin{cases} 1, & \text{with probability } 0.25 \\ 2, & \text{with probability } 0.25 \\ 3, & \text{with probability } p \\ 4, & \text{with probability } (0.5 - p) \end{cases}$$

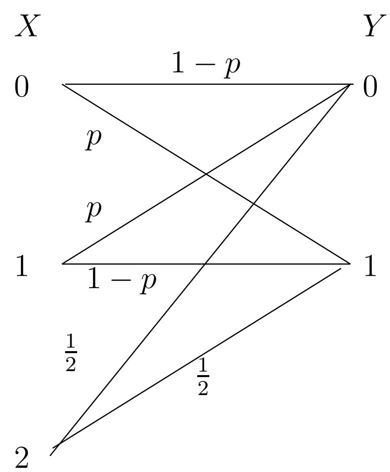
   for $0 < p < 0.5$. For what values of p does the optimal prefix-free code have expected length equal to 2? (10 pts)

3. **Channel Capacity** (25 pts)
Compute the capacity of the following channels:

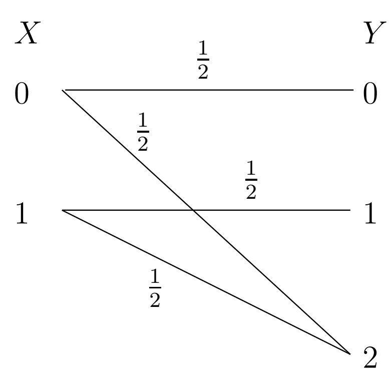
(a) $C =$? (5 pts)



(b) $C =$? (10 pts)



(c) $C =$? (10 pts)



4. **Differential Entropy** (15 pts)

Let X be a continuous random variable with support $S = [-1, 1]$ (i.e., $f(x) > 0$ for $-1 \leq x \leq 1$ and $f(x) = 0$ for $x < -1$ and $x > 1$). Assume $h(X)$ is finite. Define the random variable Y as:

$$Y = \begin{cases} +a & \text{with probability } 1/2 \\ -a & \text{with probability } 1/2 \end{cases}$$

for some constant $a \geq 0$. Assume X and Y are independent. Let $Z = X + Y$.

(a) Compute $h(Z)$ in terms of $h(X)$ for $a > 1$. (10 pts)
(Hint: The quantity $h(Z)$ is finite.)

(b) Does the same answer hold if $a < 1$? Why or why not? (5 pts)

5. **Rate Distortion** (25 pts)

Consider a ternary source and reconstruction alphabet ($\mathcal{X} = \{0, 1, 2\}, \hat{\mathcal{X}} = \{0, 1, 2\}$). Assume the source has a uniform distribution, i.e. $p(X = 0) = p(X = 1) = p(X = 2) = 1/3$, and let the distortion measure be given by the following matrix:

$$d(x, \hat{x}) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

or equivalently

$$d(x, \hat{x}) = \begin{cases} 1 & \text{if } (x = 0, \hat{x} = 2) \text{ or } (x = 1, \hat{x} = 0) \text{ or } (x = 2, \hat{x} = 1) \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute an expression for the expected distortion $E[d(x, \hat{x})]$. (5 pts)

(b) Compute the rate distortion function at $D = 1/3$, i.e., $R(D = 1/3)$. (10 pts)

(c) Compute the rate distortion function at $D = 0$, i.e., $R(D = 0)$. (10 pts)
(Hint: $R(0)$ is strictly smaller than $H(X)$ because there are two zero-distortion reconstructions for each source symbol.)