EE 8510 Thursday, Jan. 20, 2005
Advanced Topics in Communications Prof. N. Jindal

Homework Set # 1
Due: Thursday, Jan. 27, 2005

1. *Entropy.* Show the following expression holds for any probability mass function (p_1, p_2, p_3):

$$H(p_1, p_2, p_3) = H(p_1) + (p_2 + p_3)H\left(\frac{p_2}{p_2 + p_3}, \frac{p_3}{p_2 + p_3}\right).$$

2. *Entropy of functions.* Let X be a random variable taking on a finite number of values. What is the inequality relationship between $H(X)$ and $H(Y)$ if:

   (a) $Y = 2^X$

   (b) Y = cos X

   (Cover & Thomas 2.2)

3. *Zero conditional entropy.* Show that if $H(Y|X) = 0$, then Y is a function of X, i.e., for all x with $p(x) > 0$, there is only one possible value of y with $p(x, y) > 0$. (Cover & Thomas 2.6)

4. *Conditional mutual information vs. unconditional mutual information.* Give examples of joint random variables X, Y, and Z such that

   (a) $I(X; Y|Z) < I(X; Y)$

   (b) $I(X; Y|Z) > I(X; Y)$

   (Cover & Thomas 2.10)

5. *Entropy of a sum.* Let X and Y be random variables that take on values $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_s$, respectively. Let $Z = X + Y$.

   (a) Show that $H(Z|X) = H(Y|X)$. Argue that if X, Y are independent, then $H(Y) \leq H(Z)$ and $H(X) \leq H(Z)$. Thus the addition of independent random variables adds uncertainty.

   (b) Give an example of random variables for which $H(X) > H(Z)$ and $H(Y) > H(Z)$.

(c) Under what conditions does $H(Z) = H(X) + H(Y)$?

(Cover & Thomas 2.18)

6. *Mixing increases entropy.* Show that the entropy of the probability distribution $(p_1, \ldots, p_i, \ldots, p_j, \ldots, p_m)$ is less than the entropy of the probability distribution $(p_1, \ldots, \frac{p_i+p_j}{2}, \ldots, \frac{p_i+p_j}{2}, \ldots, p_m)$. (Cover & Thomas 2.28)
   Hint: Use Problem 1 or the concavity of $H(p)$.

7. *Inequalities* Let X, Y, and Z be joint random variables. Prove the following inequalities and find conditions for equality. (Cover & Thomas 2.29)

   (a) $H(X, Y|Z) \geq H(X|Z)$

   (b) $I(X, Y; Z) \geq I(X; Z)$

   (c) $H(X, Y, Z) - H(X, Y) \leq H(X, Z) - H(X)$

   (d) $I(X; Z|Y) \geq I(Z; Y|X) - I(Z; Y) + I(X; Z)$

8. *Cards.* An ordinary deck of cards containing 26 red cards and 26 black cards is shuffled and dealt out one card at a time without replacement. Let X_i be the color of the ith card dealt. (Cover & Thomas 6.3)

   (a) Determine $H(X_1)$.

   (b) Determine $H(X_2)$.

   (c) Determine $H(X_1, X_2, \ldots, X_52)$.

   (d) Does $H(X_k|X_1, \ldots, X_{k-1})$ increase or decrease as a function of k?