Diversity-Multiplexing tradeoffs in the Rayleigh Fading Relay Channel

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Abstract

We study the Rayleigh fading relay channel. We first show that relay cooperation offers a small capacity increase with respect to the direct channel and then focus on studying the diversity-multiplexing tradeoff. We find a bound on the optimal diversity-rate function and establish that this bound is achieved by an adaptive version of the classical Markov coding scheme.

Keywords: Relay channel, cooperative networking, diversity-multiplexing tradeoff, Markov coding.

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I. INTRODUCTION

Recently, there has been a renewed interest in the Gaussian Relay Channel in the context of Cooperative Networking in wireless fading environments; see e.g., [4], [5], [6], [7]. First analyzed in [2], the relay channel models a situation in which an ongoing transmission between a source S and a destination D receives cooperation from a "relay" terminal R. The broadcast nature of the wireless environment makes this setting particularly appealing, [3].

While relay cooperation can increase the capacity of the $S \to D$ link, diversity not capacity is the main advantage that relay cooperation offers for wireless networks. Indeed, we will later show that the capacity increase is not significant in general, but that the outage probability decays with the second power of the signal to noise ratio (SNR). This has to be contrasted to the inversely proportional decay of the direct link, showing that in non-ergodic scenarios the relay may offer a significant advantage.

To clearly asses the diversity advantage we adopt the diversity-multiplexing tradeoff tool introduced in [8] for point to point multiple input multiple output (MIMO) channels. Constraining the communications to orthogonal channels this tradeoff was studied in [4]. For the general case of half-duplex relays it was studied in [1]. In this report we study full-duplex relays in non-orthogonal channels and show that similar conclusions hold true.

II. THE RAYLEIGH FADING RELAY CHANNEL AND THE RATE DIVERSITY TRADEOFF

Depicted in Fig. 1, the Rayleigh fading Gaussian relay channel consists of a Source S transmitting to a Destination D with the help of a Relay R. Due to the broadcast nature of the wireless channel, the source's signal X_S is received by D and R, based on the information received, the relay constructs a signal X_R which in turn sends to the destination. Letting Y_D and Y_R be the signals received at D and R respectively the input-output equations for the relay channel can be written as

$$Y_D = \sqrt{h_{SD}}\, e^{j\phi_{SD}} X_S + \sqrt{h_{RD}}\, e^{j\phi_{RD}} X_R + Z_D, \tag{1}$$

$$Y_R = \sqrt{h_{SR}}\, e^{j\phi_{SR}} X_S + Z_R \tag{2}$$

where $h_{SD}\, e^{j\phi_{SD}}$, $h_{SR}\, e^{j\phi_{SR}}$ and $h_{RD} e^{j\phi_{RD}}$ denote the channels $S \to R$, $S \to D$ and $R \to D$ respectively; and; Z_D and Z_R represent Additive White Gaussian Noise (AWGN) with zero mean and variances N_D and N_R respectively.

The Rayleigh fading assumption implies that the channels $\sqrt{h}\, e^{j\phi}$ (with $he^{j\phi}$ denoting any of the $S \to R$, $S \to D$ and $R \to D$ channels) are complex normally distributed random variables. Equivalently, this implies that ϕ_{SD} is uniform in $[0, 2\pi]$ and that h is exponentially distributed (see also Appendix A).

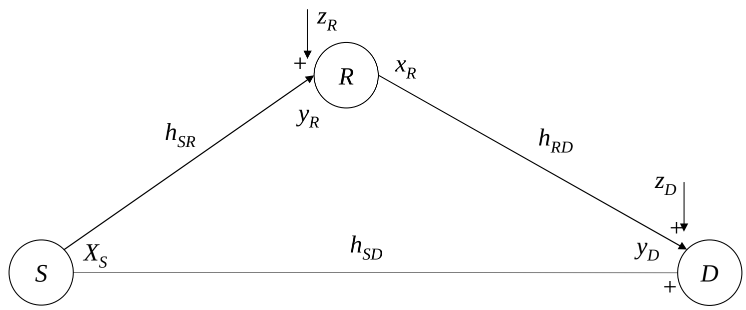

Fig. 1. The Gaussian Relay channel. A relay R cooperates with the ongoing transmission between the source S and the destination D.

The mean channel powers will be denoted by $\bar{h}_{SD} = \mathrm{E}(h_{SD})$, $\bar{h}_{SR} = \mathrm{E}(h_{SR})$ and $\bar{h}_{RD} = \mathrm{E}(h_{RD})$. We will further assume that $P_S : \mathrm{E}(X_S^2) = P_R : \mathrm{E}(X_R^2)$ and define SNR as

$$\gamma = \frac{P_S}{N_D} = \frac{P_S}{N_R} = \frac{P_R}{N_D}. \tag{3}$$

which can be done without loss of generality if we incorporate the difference in noise and signal powers in the channel coefficients.

It can be proven that relay cooperation does not yield a significant ergodic capacity increase in a Rayleigh fading environment (see Section III) but it can offer a significant advantage in terms of outage capacity. This motivates studying the rate-multiplexing tradeoff in the relay channel, following the formulation and solution for the MIMO channel introduced in [8]. Consider a family of codes $\{\mathcal{C}_\gamma\}$ indexed by their operating SNR γ such that the code $\mathcal{C}_\gamma$ has rate $R(\gamma)$ bits per channel use and error probability $P_e(\gamma)$. For this family we define the multiplexing gain r and the diversity gain d as follows

$$r := \lim_{\gamma\to\infty} \frac{R(\gamma)}{\log\gamma}, \qquad d := -\lim_{\gamma\to\infty} \frac{\log[P_{out}(\gamma)]}{\log\gamma}. \tag{4}$$

The purpose of this report is to study the diversity-rate function $d(r)$ for some specific protocols, study the best achievable diversity for a given rate $d^*(r)$, and show that this optimum curve is achieved by and adaptive decode and forward protocol.

Notation: In the remaining of the paper we will say that $f(x) \sim Cx^n$ if $\lim_{x\to 0} f(x)/x^n = C$ and that $f(x) \sim C/x^n$ if $\lim_{x\to\infty} f(x)x^n = C$

III. Ergodic Capacity of the Rayleigh Fading Relay channel

When considering fading with channel state information available only at the receivers we have to distinguish between the so called instantaneous, ergodic and outage capacity. If we consider given realizations of the fading coefficients, we say that $C = C(h_{SD}\ e^{j\phi_{SD}}, h_{SR}\ e^{j\phi_{SR}}, h_{RD}\ e^{j\phi_{RD}})$ is the instantaneous channel capacity. The ergodic capacity is defined as the expected value over the channel realizations of the instantaneous capacity:

$$\bar{C} = \mathrm{E}[C(h_{SD}\ e^{j\phi_{SD}}, h_{SR}\ e^{j\phi_{SR}}, h_{RD}\ e^{j\phi_{RD}})] \tag{5}$$

and is interpreted as the capacity achievable when packets are long with respect to the channel coherence time. When the channel varies slowly with respect to the coherence time a more compelling measure is the outage probability P_{out} defined as the probability that a given rate R_{out} can be transmitted using the given channel

$$P_{out}(R_{out}) = \Pr\{C(h_{SD}\ e^{j\phi_{SD}}, h_{SR}\ e^{j\phi_{SR}}, h_{RD}\ e^{j\phi_{RD}}) < R_{out}\} \tag{6}$$

Even though the capacity region of the relay channel is unknown we can obtain interesting conclusions by using the well-known max-flow min-cut bound [2]. This bound, when specialized to an AWGN channel yields the expression

$$C < \max_{\rho\in[0,1]} \min \left\{ \begin{array}{l} \log\left(1 + \left(h_{SD} + h_{SR} + 2\rho\sqrt{h_{SD}h_{RD}}\ |e^{j(\phi_{SD}+\phi_{RD})}|\right)\gamma\right), \\ \log(1 + (h_{SD} + h_{RD})(1-\rho^2)\gamma) \end{array} \right\} \tag{7}$$

for the instantaneous channel capacity. Note that the first term of the bound in (7) increases as ρ increases, while the second one decreases.

Let us consider the ergodic capacity as defined in (6), and apply it to the capacity bound (7). An important first observation is that the channel phases appear only in the first term and that taking expected value over them yields

$$\bar{C} < \mathrm{E} \max_{\rho\in[0,1]} \min \left[\log\left(1 + (h_{SD} + h_{SR} +)\gamma\right), \log(1 + (h_{SD} + h_{RD})(1-\rho^2)\gamma)\right], \tag{8}$$

which is just a formal statement that in a wireless fading channel we cannot expect coherent superposition of source's and relay's signals. Interestingly, the first term does not depend on ρ yielding a simple upper bound on the ergodic capacity of rayleigh fading relay channels

$$\bar{C} < \mathrm{E}\left[\min\left[\log(1 + (h_{SD} + h_{SR})\gamma), \log(1 + (h_{SD} + h_{RD})\gamma)\right]\right] \tag{9}$$

Comparing (19) with the direct transmission capacity $\bar{C}_{DT} := \mathrm{E}[\log(1 + h_{SD}\gamma)]$ we can see that the capacity increase in the relay channel stems form the extra power transmitted by the relay channel; or; from the better channels h_{SR} and h_{RD}.

The increase is in any case small and suggests that the advantage of relays, if any, are not from an ergodic capacity point of view. This leads us to the study of outage probability in the next section.

IV. OUTAGE CAPACITY

In order to study outage probability we will introduce achievable rates in the relay channel by considering specific protocols. Studying the outage behavior of this specific protocols will lend a lower bound in the outage capacity.

The classical achievable region for the relay channel, is the one achieved by Markov coding (MC) defined as follows.

Definition 1 *Markov coding protocol.* In the Markov coding protocol the source sends information at a rate R_1 such that the packets are perfectly decoded by the relay. The relay later sends information to let the destination resolve the uncertainty in the received message.

The details of how this may be implemented are to be found in [2]. Note that different form the usual definition we are not requiring Source and Relay to cooperate in resolving the uncertainty at the destination but letting this task to the relay alone.

The instantaneous transmission rate achievable by the MC protocol of Definition 1 is well known and given by

$$C > I_{MC} = \min\left[\log(1 + h_{SR}\gamma), \log(1 + (h_{SD} + h_{RD})\gamma)\right] \tag{10}$$

where we also stated that being achievable by a certain coding scheme I_{MC} is a lower bound on capacity.

Combining (10) with (6) we can obtain the following upper bound in the outage probability of the relay channel,

$$\begin{aligned} P_{out}(R_{out}) < P_{out}^{MC}(R_{out}) &:= \Pr\{I_{MC} < R_{out}\} \\ &= \Pr\{\min\left[\log(1 + h_{SR}\gamma), \log(1 + (h_{SD} + h_{RD})\gamma)\right] < R_{out}\} \end{aligned} \tag{11}$$

To study the outage probability upper bound in (11) we introduce the channel power outage variable defined as

$$h_{out} = \frac{2^{R_{out}} - 1}{\gamma} \tag{12}$$

and we stress that $\lim_{\gamma\to\infty} h_{out} = 0$. Using (12) we can reduce (11) to the expression,

$$P_{out}^{MC}(h_{out}) = \Pr\{\min(h_{SR}, h_{SD} + h_{RD}) < h_{out}\} \tag{13}$$

It is not difficult to obtain a closed form expression for $P_{out}^{MC}(h_{out})$ in (13). However since we are only interested in the high SNR behavior it suffices to study the probability of h_{SR} and $h_{SD} + h_{RD}$ to be very small. This can be done using results about exponential random variables summarized in Appendix A; indeed, using (31), and then (29) and (30) we obtain

$$P_{out}^{MC}(h_{out}) \sim \frac{h_{out}}{\bar{h}_{SR}} + \frac{h_{out}^2}{2\bar{h}_{SD}\bar{h}_{RD}} \sim \frac{h_{out}}{\bar{h}_{SR}} \tag{14}$$

since the linear term dominates the quadratic one as $h_{out} \to 0$.

While the discussion about the diversity/multiplexing tradeoff will have to wait until Section VI, (13) already hints that MC coding does not achieve diversity gains since as $\gamma \to \infty$ the outage probability behaves like the outage probability of direct transmission. This is, perhaps, to be expected since the rate of MC coding is limited by the $S \to R$ channel.

A possible solution for the lack of diversity of MC is an adaptive version of the protocol. Indeed, given that the relay measures the channel h_{SR} it can adapt its cooperation strategy to the quality of this link. Thus, if the channel is good it cooperates as in Definition 1 and if it is not it just remains silent. This is better stated in the following definition.

Definition 2 *Adaptive Markov coding protocol (AMC).* Consider h_{out} as defined in (12). If $h_{SR} < h_{out}$ the relay does not cooperate in the transmission; if $h_{SR} < h_{out}$ the relay cooperates as in MC (Definition 1

This is unmistakeably a hybrid between MC and direct transmission and the achievable rate is thus given by,

$$C > I_{AMC} = \begin{cases} \log(1 + h_{SD}\gamma), & h_{SR} < h_{out} \\ \log(1 + (h_{SD} + h_{RD})\gamma), & h_{SR} > h_{out} \end{cases} \tag{15}$$

As well as we did for MC coding we can study the outage capacity which in this case is given by

$$P_{out}(h_{out}) < P_{out}^{AMC}(h_{out}) = \Pr\{h_{SD} < h_{out}, h_{SR} < h_{out}\} + \Pr\{h_{SD} + h_{RD} < h_{out}, h_{SR} > h_{out}\}. \tag{16}$$

Using independence of the fading coefficients we reduce the former to

$$P_{out}^{AMC}(h_{out}) = \Pr\{h_{SD} < h_{out}\} \Pr\{h_{SR} < h_{out}\} + \Pr\{h_{SD} + h_{RD} < h_{out}\} \Pr\{h_{SR} > h_{out}\}. \tag{17}$$

Finally, considering the high SNR (low h_{out}) behavior we can use (30) to obtain

$$P_{out}^{AMC}(h_{out}) \sim \left[\frac{1}{\bar{h}_{SD}\bar{h}_{SR}} + \frac{1}{2\bar{h}_{SD}\bar{h}_{RD}}\right] h_{out}^2. \tag{18}$$

Different from (14), the outage probability of AMC as given in (18) behaves as γ^{-2} hinting to the diversity gains of AMC coding.

Finally, let us introduce a lower bound in the outage capacity by considering the capacity upper bound in (7). Consider the second expression only and set $\rho = 1$ to obtain that the instantaneous capacity must satisfy,

$$C < I_{TA} \log(1 + (h_{SD} + h_{RD})\gamma), \tag{19}$$

which, by the way, can be interpreted as the capacity of a single antenna transmitter two antenna receiver channel. This capacity upper bound leads to an outage probability lower bound that can be expressed as

$$P_{out}(h_{out}) > P_{out}^{TA}(h_{out}) = \Pr\{h_{SD} + h_{RD} < h_{out}\} \tag{20}$$

As before, we are interested in the high SNR behavior; for what we let $h_{out} \to 0$ to obtain

$$P_{out}^{TA}(h_{out}) \sim \left[\frac{h_{out}^2}{2\bar{h}_{SD}\bar{h}_{RD}}\right]. \tag{21}$$

As expected, the relay channel cannot offer a diversity order greater than 2.

Remark 1 The diversity order of the relay channel is not better than 2 as per (21). Since this diversity order is achieved by AMC as per (18) we conclude that AMC is optimal from a diversity order perspective

V. Rate multiplexing tradeoff

In this section we will use the outage probability lower bound $P_{out}^{TA}(h_{out})$ in (21) to obtain a bound in the optimum diversity-multiplexing tradeoff of the relay channel. We will then show that this bound is achieved by AMC coding but not by MC coding

So, consider the diversity gain definition in (4) and the outage capacity bound $P_{out}^{TA}(h_{out})$ in (20) to obtain,

$$d := -\lim_{\gamma\to\infty} \frac{\log[P_{out}(\gamma)]}{\log\gamma} \leq -\lim_{\gamma\to\infty} \frac{\log[P_{out}^{TA}(\gamma)]}{\log\gamma}. \tag{22}$$

But now use the large SNR behavior of $P_{out}^{TA}(h_{out})$ as described by (21) to obtain

$$d \leq -\lim_{\gamma\to\infty} \frac{\log(h_{out}^2)}{\log\gamma} = -2\lim_{\gamma\to\infty} \frac{R_{out} - \log(\gamma)}{\log\gamma}. \tag{23}$$

where for the las equality we used the definition of h_{out} in (12). Finally, note that according to (4) $r := \lim_{\gamma\to\infty}(R_{out}/\log\gamma)$ which upon substitution in (23) yields,

$$d \leq 2(1 - r). \tag{24}$$

A parallel line of reasoning can be used for the MC protocol; using the high SNR outage behavior summarized in (14) we obtain

$$d^{MC} := -\lim_{\gamma\to\infty} \frac{\log(P_{out}^{MC}(\gamma))}{\log\gamma} = -\lim_{\gamma\to\infty} \frac{R_{out} - \log(\gamma)}{\log\gamma}. \tag{25}$$

From where we use the multiplexing gain definition (4) to obtain

$$d^{MC} = (1 - r). \tag{26}$$

Finally we can repeat the steps for the AMC protocol to obtain the expression [c.f. (18), (4)]

$$d^{AMC} := -\lim_{\gamma\to\infty} \frac{\log(P_{out}^{AMC}(\gamma))}{\log\gamma} = 2(1 - r). \tag{27}$$

The rate-multiplexing curves (24), (26), and (27) are depicted in Fig. 2. It can be seen that the best possible diversity gain is achieved when the multiplexing gain approaches 0, in which case the diversity gain of MC approaches 1 and the diversity gain of AMC approaches 2. When the multiplexing gain approaches 1 on the other hand, the three curves yield 0 diversity gain. This is in all consistent with the intuition that we can either get a diversity gain or a multiplexing gain but not both at the same time. It is also consistent with the facts hinted in (14), (18), and (21) that the relay channel cannot achieve a diversity greater than 2 and that this diversity is achieved by AMC but not by MC.

Finally, note that upper bound for the diversity-rate function in (26) coincides with the diversity-rate function for AMC in (27). This fact allows to claim that the optimum rate function is $d^*(r) = 2(1 - r)$ as stated in the following proposition.

Proposition 1 *The optimum diversity multiplexing tradeoff of the relay channel is*

$$d^*(r) = 2(1 - r) \tag{28}$$

Proof: Due to (24) $d^*(r) \le 2(1-r)$, but due to (27) $d^{AMC} = 2(1-r)$. Since the bound is achieved by at least one coding strategy $d^*(r)$ is given by (28) ■

This proposition establishes the main claim of this report that the AMC protocol is optimal from a diversity-multiplexing tradeoff point of view. We finish this section with some reamrks.

Remark 2 Similar studies for half-duplex relay channels were done in [4], and [1], with similar conclusions. The half-duplex constraint is motivated by the fact that due to involuntary feedback form transmission to reception the relay cannot receive and transmit at the same time. A way to circumvent with restriction using two physical relays to simulate a single logical relay has been described in [5].

Remark 3 Quite surprisingly, analog repetition of the signal Y_R is also optimal for the diversity-multiplexing tradeoff, as shown in [4] for half-duplex relays. Alas, implementation of this protocol requires storage of the analog waveform, and for this reason we favor the AMC protocol presented here.

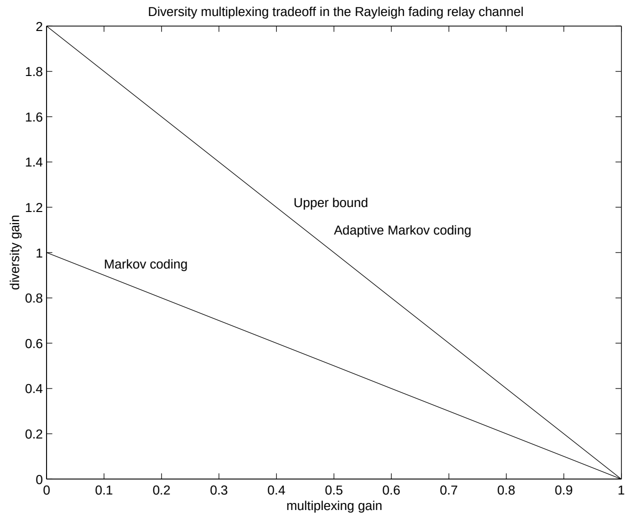

Fig. 2. Multiplexing-diversity tradeoff for the relay channel. The optimum diversity tradeoff $d^*(r) = 2(1-r)$ is achieved by the AMC protocol in Definition 2.

Remark 4 The original definition of rate in [8] considers packet error probability instead of outage capacity. The definitions are equivalent for the problem at hand and the one in (4) was chosen for simplicity of exposition.

VI. CONCLUSION

In this report we have studied the multiplexing-diversity tradeoff in the relay channel. We considered the classical Markov coding scheme and showed that this protocol does not achieve any diversity advantage with respect to direct transmission. We then showed that this is not an inherent limitation of the relay channel by establishing that an upper bound for the optimum diversity-rate curve behaves like a two antenna receiver. Finally, we showed that this optimum diversity-rate curve can be achieved by an adaptive version of the Markov coding scheme.

VII. APPENDIX

A. Limiting probabilities of exponential random variables

That the fading is Rayleigh implies that the channel variables $h_{(\cdot)}$ are exponentially distributed. Letting h denote any of this variables we have that its probability density function (pdf) is $p_h(x) = 1/\bar{h}\exp(x/\bar{h})$ with $\bar{h} = \mathrm{E}(h)$ the average channel power.

The outage probabilities in (20), (13), and (16) can be found from the exponential distribution. However, since we are interested in high SNR behavior we can obtain even easier expressions by looking at small values of h,

$$\Pr\{h < \epsilon\} = \frac{1}{\bar{h}} \exp\left(\frac{x}{\bar{h}}\right) \sim \frac{\epsilon}{\bar{h}} \tag{29}$$

If we consider the sum of two exponential variables $h = h_1 + h_2$ with h_1, h_2 exponentially distributed we have

$$\begin{aligned} \Pr\{h < \epsilon\} &= \int_0^{\epsilon} \Pr\{h_1 < \epsilon - h_2\} p_{h_2}(h_2) dh_2 \\ &\sim \frac{1}{\bar{h}_2} \int_0^{\epsilon} \Pr\{h_1 < \epsilon - h_2\} dh_2 \\ &\sim \frac{\epsilon^2}{2\bar{h}_1 \bar{h}_2} \end{aligned} \tag{30}$$

Finally, if we consider the minimum of two random variables $h = \min(h_1, h_2)$ with h_1, h_2 having arbitrary distributions such that $\lim_{\epsilon \to 0} \Pr\{h_1 < \epsilon\} = 0$, we obtain

$$\Pr\{h < \epsilon\} \sim \Pr\{h_1 < \epsilon\} + \Pr\{h_2 < \epsilon\} \tag{31}$$

Equations (29), (30), (31) are used to obtain the outage probability limiting behavior for large SNR in (21), (14) and (18).

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