Capacity and Dirty Paper Coding for Gaussian Broadcast Channels with Common Information

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Abstract — **We consider a set of parallel, two-user scalar Gaussian broadcast channels, where the transmitter wishes to send independent information to each of the receivers and common information to both receivers. The capacity region of this channel is implicitly characterized in [1]. Here, we provide an explicit characterization of the power and rate allocation schemes that achieve the boundary of the three-dimensional rate region. We also propose a dirty-paper coding achievable region for MIMO broadcast channels with common information.**

I. Scalar Channels

We consider a set of N parallel, scalar broadcast channels:

$$y_j(i) \quad = \quad x(i) + z_j(i) \quad i = 1, \ldots, N \tag{1}$$

for $j = 1, 2$, where $z_1(i) \sim N(0, N_1(i))$ and $z_2(i) \sim N(0, N_2(i))$. We consider the non-degraded scenario where for some i we have $N_1(i) \leq N_2(i)$ and for some other i we have $N_2(i) < N_1(i)$. We impose an average power constraint P on the input, i.e. $\sum_{i=1}^{N} E[x(i)^2] \leq P$.

In [1], the capacity region of the 2-user broadcast channel with $N = 2$ is found. This derivation can easily be extended to arbitrary N to give:

$$C_{BC}(\overline{P}) = Co\left(\bigcup_{\mathbf{P}(i)} C(\mathbf{P}(i))\right), \tag{2}$$

where the union is taken over all power allocations $\mathbf{P}(i)$ such that $\sum_{i=1}^{N} P_0(i) + P_1(i) + P_2(i) \leq \overline{P}$. The quantity $C(\mathbf{P}(i))$ is the set below (component-wise) the triplet $(R_1(\mathbf{P}(i)), R_2(\mathbf{P}(i)), R_0(\mathbf{P}(i)))$, with $R_1(\mathbf{P}(i))$ and $R_2(\mathbf{P}(i))$ defined as

$$R_j(\mathbf{P}(i)) \quad = \quad \sum_{i=1}^{N} \log\left(1 + \frac{P_j(i)}{N_j(i) + P_l(i)\mathbf{1}[N_j(i) \geq N_l(i)]}\right)$$

with $l \neq j$ and $R_0(\mathbf{P}(i)) = \min(R_{01}(\mathbf{P}(i)), R_{02}(\mathbf{P}(i)))$ with

$$R_{0j}(\mathbf{P}(i)) \quad = \quad \sum_{i=1}^{N} \log\left(1 + \frac{P_0(i)}{N_j(i) + P_1(i) + P_2(i)}\right)$$

for $j = 1, 2$. Successive decoding is used at the receivers to achieve the capacity region, and the common information (corresponding to power $P_0(i)$) is decoded *before* the independent information codewords (powers $P_1(i)$ and $P_2(i)$).

Since the capacity region is convex, it can be fully characterized by maximizing the weighted sum of rates:

$$\max_{(R_0, R_1, R_2) \in C_{BC}(\overline{P})} \quad \mu_1 R_1 + \mu_2 R_2 + \mu_0 R_0. \tag{3}$$

We wish to find the optimal power allocation policy that achieves the maximum for any weights (μ_0, μ_1, μ_2). Using standard convex optimization techniques, it can be shown that the optimal power allocation policy achieving the maximum in (3) is also the solution to

$$\max_{\mathbf{P}(i)} \quad \mu_1 R_1(\mathbf{P}(i)) + \mu_2 R_2(\mathbf{P}(i)) + \lambda_1 R_{01}(\mathbf{P}(i)) + \lambda_2 R_{02}(\mathbf{P}(i)) - \lambda\left(\sum_{i=1}^{N} P(i) - \overline{P}\right) \tag{4}$$

for the optimal Lagrangian multipliers $(\lambda, \lambda_1, \lambda_2)$. The optimal Lagrangians must be non-negative and must satisfy $\lambda_1 + \lambda_2 = \mu_0$. Furthermore, the optimal Lagrangians yield a power allocation with either $R_{01} = R_{02}$ or $\lambda_i = 0$ for one of the users. Unlike the broadcast channel without common information [2, 3], there is no general analytical solution to (4), but the maximization can easily be solved numerically.

II. MIMO Channels

In this section we consider multiple-input, multiple-output (MIMO) broadcast channels. Since MIMO broadcast channels are not in general degraded, the capacity region with common and independent information is unknown.

An achievable region for the MIMO broadcast channel can be established using dirty paper coding. By first encoding the common message followed by the independent messages, the following rate triplet is achievable:

$$\begin{aligned}
R_0 &= \min_{j=1,2} \log \frac{\left|\mathbf{I} + \mathbf{H_j}(\Sigma_0 + \Sigma_1 + \Sigma_2)\mathbf{H_j}^T\right|}{\left|\mathbf{I} + \mathbf{H_j}(\Sigma_1 + \Sigma_2)\mathbf{H_j}^T\right|} \\
R_1 &= \log \frac{\left|\mathbf{I} + \mathbf{H_1}(\Sigma_1 + \Sigma_2)\mathbf{H_1}^T\right|}{\left|\mathbf{I} + \mathbf{H_1}\Sigma_2\mathbf{H_1}^T\right|} \\
R_2 &= \log\left|\mathbf{I} + \mathbf{H_2}\Sigma_2\mathbf{H_2}^T\right|
\end{aligned}$$

for any set of positive semi-definite covariances satisfying $Tr(\Sigma_0 + \Sigma_1 + \Sigma_2) \leq P$. Additionally, the ordering of users 1 and 2 can be switched. This region can easily be extended to parallel broadcast channels. However, the rate equations given above are not concave functions of the covariances, and thus finding this region for even $N = 1$ is numerically difficult.

If only common information is to be transmitted, the capacity of a K-user MIMO broadcast channel is given by:

$$\mathcal{C}_0 = \max_{\mathbf{\Sigma} \geq 0, Tr(\mathbf{\Sigma}) \leq P} \quad \min_{i=1,\ldots,K} \log\left|\mathbf{I} + \mathbf{H}_i \mathbf{\Sigma} \mathbf{H}_i^T\right|. \tag{5}$$

Since this is a concave maximization, standard numerical techniques can be used to solve (5).

References

[1] A. El Gamal, "Capacity of the product and sum of two unmatched broadcast channels", *Probl. Information Transmission*, Jan-March 1980, pp. 3-23

[2] L. Li and A. Goldsmith, "Capacity and optimal resource allocation for fading broadcast channels–Part I: Ergodic capacity", *IEEE Trans. Inform. Theory*, March 2001, pp. 1083-1102.

[3] D. Tse, "Optimal Power Allocation over Parallel Gaussian Broadcast Channels", unpublished.