On the Existence of a General Multiple-Access/Broadcast Channel Duality

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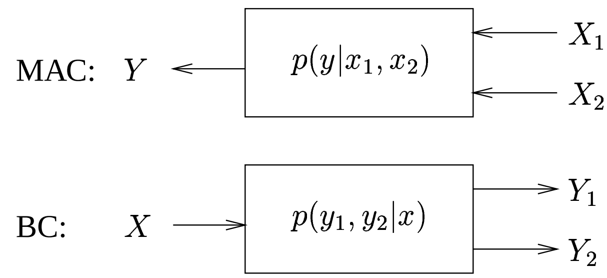
Abstract— **We investigate expanding the recently established Gaussian multiple-access/broadcast channel duality to a duality between discrete memoryless broadcast and multiple-access channels. More specifically, we attempt to find a relationship between the capacity regions of broadcast and multiple-access channels. We are able to establish a capacity region relationship for a class of deterministic broadcast and multiple-access channels, and we conjecture it exists for a slightly expanded class of deterministic channels. However, we are also able to find a counter-example for which no such capacity region relationship can exist. In the process of finding this counter-example, we prove the interesting result that random (i.e. non-deterministic) broadcast or multiple-access channels cannot have a larger capacity region than deterministic channels (i.e. channels in which the channel outputs are a deterministic function of the channel input) with the same input/output alphabets. We show this by upper bounding the capacity region of a discrete memoryless broadcast channel by the the capacity region of a finite-state broadcast channel, in which the channel is deterministic in every state.**



$$|\mathcal{Y}| = |\mathcal{X}| = M, \ |\mathcal{X}_1| = |\mathcal{X}_2| = |\mathcal{Y}_1| = |\mathcal{Y}_2| = N$$

Fig. 1. Discrete memoryless MAC and BC.

I. Introduction

In [1], a *duality* was established between the Gaussian multiple-access channel and the Gaussian broadcast channel. The dual channels considered had the same noise power and the same channel gains on the uplink and downlink. The capacity region of the Gaussian broadcast channel was found to equal a union of capacity regions of the dual Gaussian multiple-access channel. A natural question to ask given this result is the following: does a dual multiple-access channel exist for every broadcast channel? In the strongest form, duality would mean that the capacity region of any discrete memoryless (DM) broadcast channel could be written as a union of multiple-access channel capacity regions. It is not known if such a relationship exists, and the fact that the capacity region of the general broadcast channel is not known makes it extremely difficult to prove a duality for the most general case. We therefore wish to investigate some basic relationships between the BC and MAC.

We consider a two-user discrete memoryless broadcast channel consisting of an input alphabet $\mathcal{X}$ of cardinality M, output alphabets $\mathcal{Y}_1$ and $\mathcal{Y}_2$ each of cardinality N, and a probability transition function $p(y_1, y_2|x)$. Similarly, we consider a multiple-access channel consisting of input alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$ each of cardinality N, an output alphabet $\mathcal{Y}$ of cardinality M, and a probability transition function $p(y|x_1, x_2)$. A simple model for these channels is shown in Figure 1. In the broadcast channel we consider the situation where the transmitter sends independent information to each receiver, and in the multiple-access channel we consider the situation where each transmitter sends independent information to the receiver. Our ultimate goal is to discover if there is a fundamental connection between the broadcast and multiple-access channel beyond the obvious symmetries in the channel models. Since the capacity region of the general broadcast channel remains unknown, we explore this duality by considering *deterministic* channels, for which the BC capacity region is known. Deterministic channels are channels in which the output(s) are uniquely defined by the input(s). Alternatively, all entries in the probability transition matrix are either 0 or 1 for a deterministic channel.

For the case where $M = aN$, where a is an integer greater than or equal to 1, we find that a duality can be established between the capacity region of the broadcast and multiple-access channel. More explicitly, we find that the convex hull of the capacity regions of all deterministic broadcast channels (with the given input/output alphabets) is equal to the convex hull of the capacity regions of all deterministic multiple-access channels. We also conjecture that this capacity region equivalence holds for any $N < M < 2N$. However, we are able to show that this duality *does not exist* for $M = 8$ and $N = 3$ for which the capacity region of a deterministic BC is *strictly* larger than the capacity region of any deterministic MAC.

We also consider non-deterministic MAC's and BC's from a deterministic point of view. We decompose a non-deterministic channel into a finite-state channel in which the channel is deterministic in each state. This decomposition allows us to upper bound the the capacity region of any non-deterministic BC by a convex hull of deterministic BC capacity regions.

The remainder of this paper is structured as follows. In Section II we state the capacity region of the deterministic BC. In Section III, we establish a duality between the BC and MAC for the case where $M = aN$. In Section IV we extend this duality to a slightly broader class, and we provide a counter-example to this duality in Section V. Finally, in Section VI we upper bound the capacity region of a random BC by the capacity region of a series of related deterministic broadcast channels.

II. Deterministic Broadcast Channels

Deterministic broadcast channels are discrete memoryless multi-user channels in which entries in the probability transition matrix $p(y_1, y_2|x)$ are either 1 or 0 (i.e. every input maps deterministically to a *single* output pair). The transmitter in the broadcast channel is assumed to have alphabet $\mathcal{X} = \{1, \ldots, M\}$ and *each* receiver is assumed to have alphabet $\mathcal{Y}_i = \{1, \ldots, N\}$. The input to the channel is denoted by x. The deterministic nature of the channel allows us to write the channel output as a function (i.e. deterministic mapping) of the channel input:

$$(y_1, y_2) = f_{BC}(x). \tag{1}$$

Here, the channel transition matrix is entirely captured by the function $f(\cdot)$ which is a mapping from an input (in $\mathcal{X}$) to an output pair (in $\mathcal{Y}_1 \times \mathcal{Y}_2$). The Blackwell channel (see [2]) is the classical example of such a channel with $M = 3$ and $N = 2$ and the following channel function:

$$f(1) = (1,1), \;\; f(2) = (2,2), \;\; f(3) = (2,1). \tag{2}$$

The capacity region of this specific channel was found in [3] and the capacity region of the general deterministic channel was later derived in [4] and [5].

For a two-user deterministic channel, the capacity region is given by the convex closure of the union (over all input distributions $p(x)$) of the rate pairs (R_1, R_2) satisfying

$$R_1 \leq H(Y_1) \tag{3}$$
$$R_2 \leq H(Y_2) \tag{4}$$
$$R_1 + R_2 \leq H(Y_1, Y_2). \tag{5}$$

where the distributions of y_1 and y_2 are induced by the input distribution $p(x)$. Each input distribution $p(x)$ corresponds to a pentagon region, and the capacity region is the closure of the convex hull of all such pentagons. If different inputs map to the same output pair, then the inputs are identical. Thus, we only consider channels for which each input maps to a different output pair (i.e. $f(a) \neq f(b) \; \forall a \neq b$). Therefore it follows that $H(Y_1, Y_2) = H(X)$.

Since the deterministic BC is fully characterized by the mapping from input to output pair, we characterize the channel by an input/output table, in which the location of an input in the table indicates the output pair (y_1, y_2) which it maps to. Notice that no two inputs correspond to the same output pair by our earlier assumption, so exactly M of the N^2 output pairs in the table contain inputs. The Blackwell channel in (2) is described by the following channel table:

$x \mapsto (y_1, y_2)$:

y_1, y_2	1	2
1	1	
2	3	2

(6)

The deterministic multiple-access channel has a very similar structure to the deterministic broadcast channel. The deterministic nature of the channel allows us to write the output of the channel as a function of the two channel inputs:

$$y = f_{MAC}(x_1, x_2). \tag{7}$$

We also use an input/output table to characterize the MAC. In the MAC, the table entries correspond to the outputs of each input pair (x_1, x_2). Notice that in the MAC every input pair must have a corresponding output.

The capacity region of a MAC is equal to the convex closure of the union (over all product distributions $p(x_1)p(x_2)$) of rate pairs that satisfy $R_1 \leq I(X_1; Y|X_2)$, $R_2 \leq I(X_2; Y|X_1)$, and $R_1 + R_2 \leq I(X_1, X_2; Y)$. For the deterministic MAC, these inequalities simplify to $R_1 \leq H(Y|X_2)$, $R_2 \leq H(Y|X_1)$, and $R_1 + R_2 \leq H(Y)$.

III. Duality for $M = aN$

In this section we show that there exists a duality between the deterministic BC and MAC for $M = aN$, where a is an integer and $1 \leq a \leq N$. By first principles, it follows that in the BC, R_1 and R_2 are bounded by $\log(N)$ and $R_1 + R_2 \leq \log(M) = \log(aN)$ for *any* deterministic (or non-deterministic) probability transition matrix. Thus, the region shown in Fig. 2 is an upper bound to *any* BC capacity region with the given alphabet sizes.

Consider the BC channel function such that *every* row and *every* column has exactly a entries in it. If we map inputs $\{1, \ldots, a\}$ to $y_1 = 1$ and $y_2 = \{1, \ldots, a\}$, inputs $\{a+1, \ldots, 2a\}$ to $y_1 = 2$ and $y_2 = \{2, \ldots, a+1\}$ (where the column y_2 is assumed to wrap-around for values larger than N), then this condition is satisfied. For the $M = 6, N = 3$ channel, this corresponds to the following channel function:

$x \mapsto (y_1, y_2)$:

y_1, y_2	1	2	3
1	1	2	
2		3	4
3	6		5

(8)

If $p(x)$ is chosen to be the uniform distribution over the input alphabet, then we have $H(X) = \log(M) = \log(6)$ and $H(Y_1) = H(Y_2) = \log(N) = \log(3)$. Thus, the upper bound is achievable. For arbitrary N and a, the region

$$\mathcal{R}_{pentagon} = \{(R_1, R_2) : \; 0 \leq R_1 \leq \log(N), \\ 0 \leq R_2 \leq \log(N), R_1 + R_2 \leq \log(aN)\} \tag{9}$$

is achievable if the channel matrix has exactly a entries in every row and column and the input x is chosen equiprobably on $\{1, \ldots, M\}$. This region coincides exactly with the upper bound on the capacity region of any BC with input alphabet M and output alphabet N. Thus, the capacity region of the deterministic BC with the channel function satisfying the above condition is equal to $\mathcal{R}_{pentagon}$. Since $\mathcal{R}_{pentagon}$ is an upper bound to any capacity region for the given input/output alphabets, the union of capacity regions over *all* channel functions is equal to $\mathcal{R}_{pentagon}$.

For the dual MAC, it is again easy to see that R_1 and R_2 are bounded by $\log(N)$ and $R_1 + R_2 \leq \log(M) = \log(aN)$. Consider the channel function defined by:

$$y = \begin{cases} a(x_1 - 1) + x_2 & x_2 \leq a \\ 1 & x_2 > a \end{cases}. \tag{10}$$

For the $M = 6, N = 3$ channel, this corresponds to the following channel function:

$(x_1, x_2) \mapsto y$:

x_1, x_2	1	2	3
1	1	2	1
2	3	4	1
3	5	6	1

(11)

If x_1 is chosen uniformly on $\{1, \ldots, N\}$ and x_2 is chosen uniformly on $\{1, \ldots, a\}$ (notice x_2 doesn't use all possible inputs), then the receiver will always be able to determine which symbols were sent by both users. Thus, the rate vector $(R_1 = \log(N), R_2 = \log(a))$ can be achieved. The capacity region of this channel is denoted by $C_{MAC,1}$ in Fig. 2. By reversing the roles of Users 1 and 2 in the channel function, the rate vector $(R_1 = \log(a), R_2 = \log(N))$ can also be achieved. The corresponding capacity region is denoted by $C_{MAC,2}$ in Fig. 2. It follows then that the convex hull of the capacity regions corresponding to these two channels equals the region $\mathcal{R}_{pentagon}$. Thus, $\mathcal{R}_{pentagon}$ is equal to the convex hull of the union of MAC capacity regions over all transition matrices.

Thus, for $M = aN$, we have that the convex hull of the union of BC capacity regions over *all* deterministic mappings equals the convex hull of the union of MAC capacity regions over all deterministic mappings.

$$Co\left(\bigcup_{f(x)} \mathcal{C}_{BC}(f(x))\right) = Co\left(\bigcup_{f(x_1,x_2)} \mathcal{C}_{MAC}(f(x_1,x_2))\right) \tag{12}$$

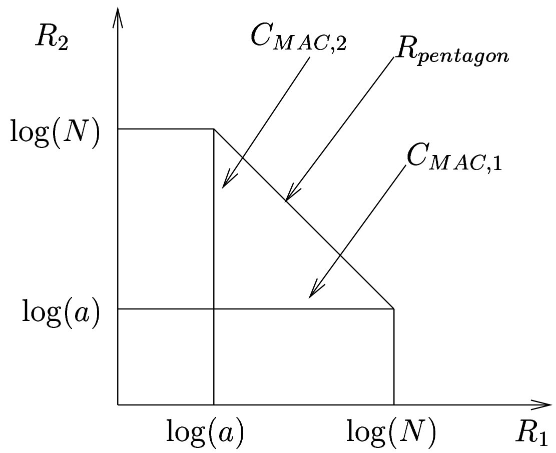


Fig. 2. Capacity region of MAC and BC for $M = aN$

where $f(x)$ is any map from $\mathcal{X}$ to $(\mathcal{Y}_1 \times \mathcal{Y}_2)$ and $f(x_1, x_2)$ is any map from $(\mathcal{X}_1 \times \mathcal{X}_2)$ to $\mathcal{Y}$. We use $Co(\cdot)$ to denote the convex hull operation. The equivalence in (12) can in fact be strengthened to hold when the union is taken over deterministic and non-deterministic channels (i.e. over all $p(y_1, y_2|x)$ for the BC and all $p(y|x_1, x_2)$ for the MAC) as well, because the region $\mathcal{R}_{pentagon}$ is an upper bound to the capacity region of the BC or MAC for *any* probability transition function, deterministic or not. This relationship then leads to a few interesting questions:

1. Does the relationship in (12) hold for arbitrary M and N?
2. Similarly, does (12) hold for arbitrary M and N when the union is taken over deterministic and non-deterministic channels?
3. Can a deterministic channel achieve any rate vector achievable by a non-deterministic channel?

We address Question 1 in the next two sections. We first establish a limited version of this relationship for $N < M < 2N$, but then we provide a counter-example for which this duality does not hold. In Section VI, we answer Question 3 by relating the capacity regions of the deterministic and non-deterministic BC (and of the MAC) and we find that Questions 1 and 2 are entirely equivalent because non-deterministic channels can never have a larger capacity region than deterministic channels for the same input/output alphabets.

IV. Extension to $N < M < 2N$

In this section we show a limited duality between deterministic broadcast and multiple-access channels for $N \leq M \leq 2N$. More explicitly, we show that the convex hull of the union of BC capacity regions, where the union is over a specific class of BC channel functions, is equal to the convex hull of the union of MAC capacity regions, where the union is over a related class of MAC channel functions.

We consider the class of *balanced* BC channel functions, by which we mean channel functions such that no output in $\mathcal{Y}_1$ or in $\mathcal{Y}_2$ has more than 2 channel inputs mapping to it. In terms of the channel function, this corresponds to having no row or column with more than 2 channel inputs in it. Of course, as before, we require that every channel input goes to a different channel output. Since $M \leq 2N$, it is clear that with such a channel function, $M - N$ of the outputs of each user will have 2 inputs mapping to it, while the remaining $2N - M$ outputs of each user will have only a single input mapping to it. For the $M = 5, N = 3$ channel, the following channel function is *balanced*:

$x \mapsto (y_1, y_2)$: (13)

y_1, y_2	1	2	3
1	1	2	
2		3	4
3			5

Notice that no row or column has more than 2 inputs in it. A similar channel function can be defined for other values of M and N by using the same procedure of placing inputs along the main diagonal and the upper off-diagonal. If we consider the rate achieved by User 1 in a balanced BC, at one corner point of the pentagon region (for a fixed $p(x)$) we have $R_1 = H(Y_1)$ and $R_2 = H(Y_1, Y_2) - H(Y_1) = H(Y_2|Y_1)$. Thus the rate of User 1 is equal to the received entropy of User 1, while the rate of User 2 is equal to the conditional entropy of each *row* of the channel matrix, which can be no larger than one since each row has no more than 2 inputs in it.

Analogously, we consider *balanced* MAC channel functions in which exactly $M - N$ inputs of User 1 have two possible outputs and $2N - M$ inputs of User 1 have only one output. For the $M - N$ inputs of User 1 which have two possible outputs, the $x_2 = 1$ output is assumed to be the smaller of the two outputs, while all other values of x_2 correspond to the larger of the two outputs. For the $M = 5, N = 3$ channel, the following channel function is *balanced*:

$(x_1, x_2) \mapsto y$: (14)

x_1, x_2	1	2	3
1	1	2	2
2	3	4	4
3	5	5	5

We also consider the transposes of such matrices, or the situation where the roles of Users 1 and 2 are reversed. It can be shown that the rate vector $(R_1 = H(Y|X_2), R_2 = H(Y|X_1))$ can be achieved for any input product distribution. For the channel described above, $R_1 = H(Y|X_2) = H(X_1)$ is the input entropy of User 1 and $R_2 = H(Y|X_1)$ is the conditional entropy of each row of the channel matrix.

By choosing the MAC input X_1 to have the same distribution as Y_1 in the BC and choosing X_2 to equiprobably be 1 or 2, it can be shown that any BC rate vector can also be achieved in the balanced MAC. Similarly, any rate vector achievable in the MAC can be achieved in the BC by choosing the BC input distribution $p(x)$ such that $H(Y_1) = H(X_1)$ and such that the conditional entropy $H(Y_2|Y_1) = H(Y|X_1)$. The capacity region of the balanced deterministic BC is completely symmetric in R_1 and R_2, but for the MAC we must consider the channel function where the roles of Users 1 and 2 are reversed (i.e. the transpose of the channel matrix). Using this second MAC channel, we can show any BC rate vector of the form $(H(Y_1|Y_2), H(Y_2))$ is achievable in the MAC, and vice versa. Thus, the convex hull of the union of the capacity regions over all balanced deterministic BC's equals the convex hull of the union of capacity regions of all balanced MAC's (including channels where the roles of Users 1 and 2 are reversed).

Fig. 3 shows the capacity region of the balanced deterministic broadcast channel for $M = 5, N = 3$ along with the capacity region of the two balanced MAC's. A closer examination reveals that the convex hull of the MAC regions is equal to the BC capacity region.

It still remains to be shown, however, that the convex hull of the balanced deterministic channels contains the convex hull of all deterministic channels. We conjecture that this is true, but have been unable to show this as of yet.

V. Duality Counter-example

In this section we provide a simple counter-example which shows that the duality between the deterministic MAC and BC

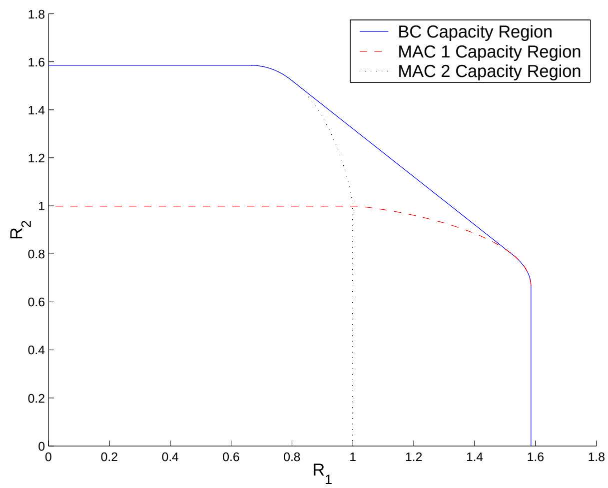


Fig. 3. Deterministic BC and MAC capacity regions for $M = 5,\ N = 3$

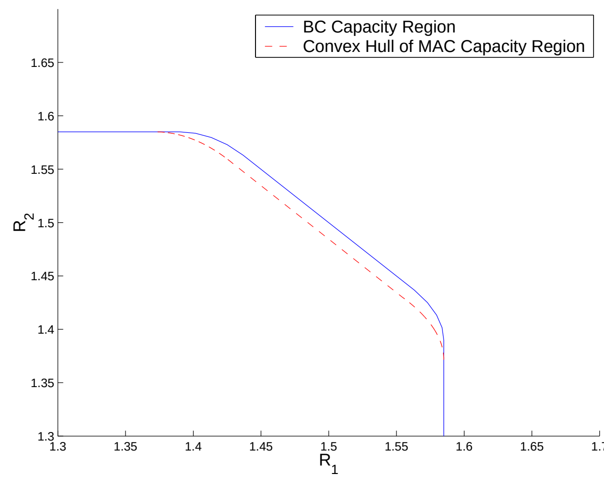


Fig. 4. Closeup of deterministic BC and MAC capacity regions for $M = 8,\ N = 3$

does not always exist. Consider a deterministic BC with $M = 8$ and $N = 3$, with each input going to a unique output (Y_1, Y_2) pair. (Any such deterministic BC mapping is equivalent for this choice of M and N) Clearly, by choosing x uniformly over the input alphabet, we can achieve $H(Y_1, Y_2) = \log(8) = 3$. Thus, the maximum sum rate for the deterministic BC is 3 bits/use. In the dual MAC, with $M = 8$ and $N = 3$, we will show that for any choice of deterministic MAC, the output entropy (i.e. $H(Y)$) is *strictly* less than 3. Therefore, the deterministic BC capacity region cannot be written in terms of dual MAC capacity regions.

We now prove that the output entropy $H(Y)$ for a deterministic MAC is strictly upper bounded by $\log(8)$ by contradiction. In order for $H(Y) = \log(8)$, we require the outputs of the channel to be equiprobable for some given input product distributions $p(x_1)$ and $p(x_2)$ and some channel function. Since every MAC output (i.e. $1, \ldots, 8$) must correspond to some input pair, without loss of generality we assume the first 8 outputs are mapped as follows:

$$(x_1, x_2) \mapsto y: \quad \begin{array}{c||c|c|c} & 1 & 2 & 3 \\ \hline 1 & 1 & 2 & 3 \\ \hline 2 & 4 & 5 & 6 \\ \hline 3 & 7 & 8 & ? \end{array} \tag{15}$$

and the bottom right output is not yet chosen. We are free to choose any output for $(x_1 = 3, x_2 = 3)$. Since numbering of inputs and outputs is arbitrary, rows and columns can be arbitrarily interchanged as well as output numbers. Thus, there are only two choices for the output in question: we can either place an output that also appears in the same row or column as the bottom right input pair (i.e. 3,6,7, or 8), or an output that is not in the same row or column (i.e. 1). Assume we choose to map $(x_1 = 3, x_2 = 3)$ to 3. Since outputs 4,5, and 6 appear only once, in order for $P(y = 4) = P(y = 5) = P(y = 6) = 1/8$ we need $P(x_2 = 1) = P(x_2 = 2) = P(x_2 = 3) = 1/3$. This implies $P(x_1 = 2) = 3/8$. Similarly, to have $P(y = 1) = P(y = 7) = 1/8$, we need $P(x_1 = 1) = P(x_1 = 3) = 3/8$, which is a contradiction.

If we choose to map $(x_1 = 3, x_2 = 3)$ to 1, then by the same argument as above we need $P(x_2 = 1) = P(x_2 = 2) = P(x_2 = 3) = 1/3$. This again implies $P(x_1 = 1) = P(x_1 = 2) = P(x_1 = 3) = 3/8$, which is a contradiction. Thus, we see that $H(Y) < \log(8)$ for any channel map[1].

In Fig. 4, a closeup of the deterministic BC capacity region and the convex hull of the MAC capacity regions for $M = 8, N = 3$ is shown. For this case, there are essentially only 2 different deterministic MAC channels (corresponding to the two possible choices for the $(x_1 = 3, x_2 = 3)$ output), so the convex hull of the regions is in fact easy to compute numerically. We see that there is a slight gap between the sum rate achievable in the MAC and BC, but a picture of the entire capacity regions would show that the capacity regions differ only very slightly. Using the same methodology, we believe that the deterministic BC is in general larger than any deterministic MAC channel for $M > 2N$, but this claim has yet to be proven.

VI. Deterministic vs. Non-deterministic channels

In this section we explore the relationship between the capacity region of the deterministic and non-deterministic broadcast channel, for fixed alphabets. As before, the transmitter alphabet is assumed to have cardinality M and both receivers are assumed to have alphabets of size N. The capacity region of the general broadcast channel is not known, but we still are able to relate the capacity region of the non-deterministic channel to that of the deterministic channel. We decompose this channel into a *finite-state broadcast channel*, where in each state the channel acts as a *deterministic* broadcast channel. The probabilistic nature of the channel is captured by the state probabilities.

We first perform this decomposition on a single-user channel to illustrate the idea. Consider a finite-state channel defined by $p(y|x, s)$ where s is an auxiliary random variable unknown to either the transmitter or the receiver and independent of the channel inputs and outputs which is chosen according to some distribution $p(s)$ defined on $\{1, \ldots, k\}$. If the channel state is not known by the transmitter nor receiver, the channel probability transition function can be written as $p(y|x) = \sum_{i=1}^{k} p(y|x, s)p(s = i)$. Thus the capacity of any channel with probability transition function $p(y|x)$ is equal to the capacity of a finite-state channel defined by $p(y|x, s)$ in which neither the transmitter nor the receiver know the state and $p(y|x) = \sum_{i=1}^{k} p(y|x, s)p(s = i)$. Clearly, the capacity of the finite-state channel where neither the transmitter nor receiver knows the variable s is upper-bounded by the capacity of the finite-state channel where both the transmitter and receiver

[1] Since the MAC capacity region is defined as the *closure* of achievable rates, this condition does not necessarily imply that the MAC capacity region does not contain a sum-rate point of $\log(8)$ bits/use. However, we can show the tighter result that $H(Y) < 2.99$ which guarantees that the MAC region is *strictly* smaller than the dual BC capacity region.

know s. If the transmitter and receiver both have knowledge of the state, then the capacity is equal to the statistical average over all states of the corresponding capacity in each state [6].

Let us now decompose the binary symmetric channel (BSC) into a finite-state channel, where the channel is strictly deterministic in each state. Consider a standard BSC with crossover probability p, i.e. $p(y=1|x=0)=p(y=0|x=1)=p$. The BSC can be decomposed into two states: an error-free state($s=1$, w.p. $1-p$), where $y=x$, and an error state ($s=2$, w.p. p) where y equals the compliment of x. Clearly, the capacity in either state is 1, so the upper bound on capacity is 1. An alternative method of decomposing the channel is the following. When $s=1$ (w.p. p), $p(y=0|x=0)=1$ and $p(y=0|x=1)=1$. When $s=2$ (w.p. p), $p(y=1|x=0)=1$ and $p(y=1|x=1)=1$. Finally, when $s=3$ (w.p. $1-2p$), $y=x$. This decomposition is illustrated in Fig. 5. The capacity of the channel when $s=1$ or $s=2$ is clearly 0 and $C(s=3)=1$. Thus the upper bound to capacity is $p(s=1)C(s=1)+p(s=2)C(s=2)+p(s=3)C(s=3)=1-2p$, which is strictly greater than the actual capacity $1-H(p)$ except when $p=0$ or $p=.5$.

We can similarly decompose any DM broadcast channel into a finite-state broadcast channel, where in each state the broadcast channel is *deterministic*[2]. We will refer to such a channel as a finite-state deterministic BC. It can be shown that the capacity region of a finite-state deterministic BC where the transmitter and receivers know the state is equal to the weighted "sum" of the capacity region of each state, where the weights are equal to the probability of each state. Below we more precisely define the notion of a sum of regions. Again, this is clearly an upper bound to the channel where neither the transmitter nor receivers know the state. As we saw with the BSC, there are many different ways of decomposing a channel into a finite-state deterministic channel. Each of these decompositions may yield a different upper bound to the capacity of the original channel. Thus, we take the intersection of all of these upper bounds to get the following theorem:

Theorem 1: The capacity region of a DM broadcast channel $(\mathcal{X}, p(y_1|x), p(y_2|x), \mathcal{Y}_1, \mathcal{Y}_2)$ is upper bounded by the intersection of capacity regions of finite-state deterministic BC's where the transmitter and receivers know the state:

$$\mathcal{C}_{BC} \subseteq \bigcap_{p(s),\ p(y_1,y_2|x,s)} \mathcal{C}_{BC}(p(s), p(y_1,y_2|s)) \qquad (16)$$

where the intersection is taken over deterministic $p(y_1,y_2|x,s)$ that satisfy $\sum_{i=1}^{k} p(y_1,y_2|x,s=i)p_i = p(y_1,y_2|x)$ where k is assumed to be the cardinality of the state variable s and $p_i = P(s=i)$. Here we use $\mathcal{C}_{BC}(p(s), p(y_1,y_2|x,s))$ to indicate the capacity region of the finite-state deterministic channel with transmitter and receiver knowledge of the state, given by:

$$\mathcal{C}_{BC}(p(s), p(y_1,y_2|x,s)) = p_1 C(s=1) + \cdots + p_k C(s=k) \qquad (17)$$

where $C(s=k)$ is the capacity region of the deterministic broadcast channel when $s=k$ and the sum is defined as element-by-element addition of rate vectors(i.e. a sum of sets).

Proof: The bound given in (16) follows obviously from the fact that both the transmitter and receiver can ignore information about state s and achieve any rate vector that can be achieved without knowledge of s. The capacity region of a finite-state deterministic BC can be shown to equal the expression given in (17) by bounding $H(Y_1), H(Y_2)$ and $H(Y_1, Y_2)$ in the probabilistic channel by considering the entropies while in each channel state. The full proof is omitted for brevity. ■

[2]It is easy to show that a decomposition into a finite-state deterministic channel is always possible for a single or multi user channel

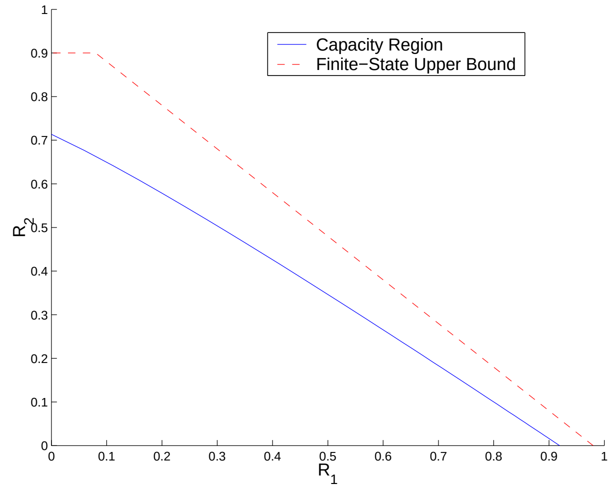


Fig. 6. Binary symmetric BC capacity region and Theorem 1 upper bound

From the definition of the convex hull, it follows that $\mathcal{C}_{BC}(p(s), p(y_1,y_2|x,s))$ lies in the convex hull of the union of $C(s=1), \ldots, C(s=k)$. Thus, any rate vector in the capacity region of a non-deterministic broadcast channel lies in the convex hull of capacity regions of deterministic channels with the same input/output alphabets. Thus, we conclude that, similar to single-user channels, *randomness* never helps in broadcast channels in the sense that any rate vector achievable with a random channel is also achievable (at least in the convex hull sense) by deterministic channels. Theorem 1 can also be extended to multiple-access channels.

In Figure 6, this bound is shown for a binary symmetric broadcast channel (which is a degraded broadcast channel) with crossover probabilities $p_1=.01$ and $p_2=.05$.

Notice that since the broadcast channel capacity region depends only on the *marginal* transition probabilities $p(y_1|x)$ and $p(y_2|x)$ [7], any decomposition into a finite-state channel with the correct marginals and a different joint distribution $p(y_1,y_2|x)$ can be added to the upper bound. Furthermore, we conjecture that this upper bound can be tightened by allowing decomposition into not only deterministic broadcast channels, but also into semi-deterministic broadcast channels, in which the output of only one of the two receivers must be a deterministic function of the input. The capacity of such a channel is known [8], and it appears that extending this result to a finite-state broadcast channel is not too difficult. The bound can further be tightened by allowing only the receiver (and not the transmitter) to have state information. Lack of receiver knowledge would force the input distribution to be independent of the channel state.

We can use Theorem 1 applied to the MAC to strengthen the duality counter-example presented in Section V. In the counter-example, we found that the convex hull of the capacity regions of all deterministic MAC's is strictly smaller than the capacity region of the deterministic BC. Since Theorem 1 implies that non-deterministic channels are not better than deterministic channels, we can now state that for the $M=8, N=3$ case, no MAC, deterministic *or* non-deterministic, can achieve the capacity region of the deterministic BC described in Section V.

VII. Conclusion

In this paper we discussed the existence of a duality between the discrete memoryless broadcast channel and multiple-access channel. We focused mainly on deterministic channels and we showed a duality exists between the capacity regions of the MAC and BC when $M=aN$, where M is the size of the in-

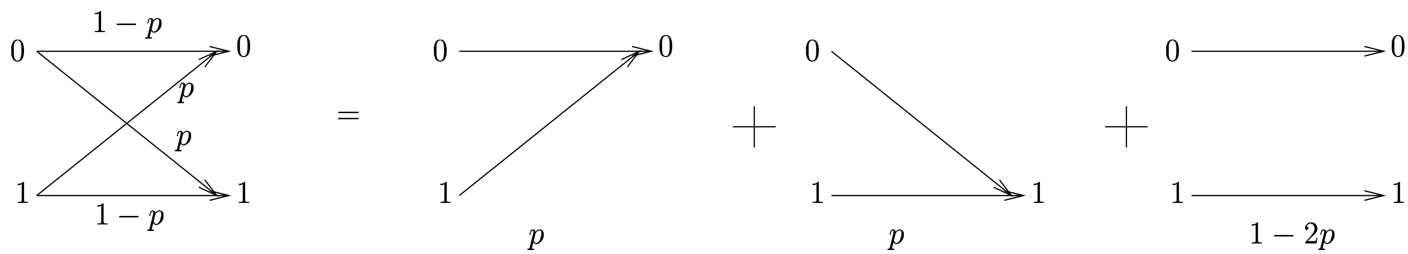


Fig. 5. BSC and an equivalent finite-state channel

put/output alphabet in the MAC/BC (respectively), N is the size of the output/input alphabet of the MAC/BC (respectively) and a is an integer. We conjectured that this holds for $N < M < 2N$ as well, but we also showed that the duality does not hold for $M = 8, N = 3$. We believe the deterministic BC capacity region is larger than the deterministic MAC capacity region for $M > 2N$. This would imply that the deterministic BC capacity region contains the capacity regions of the deterministic MAC for all M and N, but this claim remains unproven.

We also showed that the capacity region of any MAC/BC is bounded above by the convex hull of related deterministic MAC/BC capacity regions. Though this upper bound is not tight in general, it is a new (and easily computable) method of obtaining upper bounds on channel capacity. This upper bound allowed us to find an example where the capacity region of a deterministic BC is strictly larger than any deterministic or non-deterministic dual MAC capacity region.

Lack of knowledge of the capacity region of general broadcast channels limits our study of the MAC-BC relationship. In this paper we have chosen to focus on deterministic channels, but we believe it may be helpful to study other classes of channels for which the broadcast channel capacity region is known, such as degraded channels.

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