The "Z" Channel

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***Abstract*— We consider a two transmitter two receiver channel where independent data is sent on each communication link of the system. We consider a three-link system, termed the "Z" channel, in which one transmitter is connected to both receivers while the other transmitter is only connected to one of the receivers. Thus, the "Z" channel has a three dimensional capacity region. We characterize the capacity region of a special class of *degraded* "Z" channels and establish an achievable region for the Gaussian "Z" channels. Finally, we use genie-aided techniques previously used for the interference and broadcast channels to obtain an outer bound for general "Z" channels.**

I. Introduction

Historically, the study of information theory has been primarily motivated by wireline systems and cellular systems. Since interference channels are a common occurrence in such systems, they have been the primary two transmit two receive systems investigated in the past [1], [4], [5]. As the importance of non-centralized ("ad-hoc") wireless networks increases, there are many new multiuser channel configurations that are of increasing importance. In this paper, we define the two transmitter, two-receiver "Z" channels that is relevant in the ad-hoc wireless network scenario. We obtain the capacity region of the "Z" channel for the degraded, discrete-memoryless "Z" channel and we establish an achievable region for a special case of the Gaussian version of this channel. Lastly, we find an outer bound on the capacity region of the general, non-degraded "Z" channel.

II. System Model

The "Z" channel has two transmitters, labeled T_1 and T_2, and two receivers R_1 and R_2. The channel is characterized by input alphabets $X_1 \times X_2$, channel p.d.f.s $p(y_1|x_1, x_2)$ and $p(y_2|x_2)$, and output alphabets $Y_1 \times Y_2$, where x_1 and x_2 are transmitted from T_1 and T_2 respectively, and y_1 and y_2 are received at the receivers R_1 and R_2.

An $((2^{nR_{11}}, 2^{nR_{21}}, 2^{nR_{22}}), n)$ code for a two user channel consists of three sets of message indices i, j, k, $i \in 1, 2, \ldots, 2^{nR_{11}}; j \in 1, 2, \ldots, 2^{nR_{21}}; k \in 1, 2, \ldots, 2^{nR_{22}}$, two encoding functions $x_1^n(i)$ and $x_2^n(j, k)$, and two decoding functions $(\hat{i}, \hat{j}) = g_1(y_1^n)$ and $\hat{k} = g_2(y_2^n)$. An error is made if any of $\hat{i} \neq i$, $\hat{j} \neq j$, or $\hat{k} \neq k$.

The capacity region for this "Z" channel is a three dimensional region, which contains the capacity region of the "Z" interference channel, the broadcast channel and the multiple access channel as some of its bounding planes. The "Z" channel is a special case of the "X" channel. In the "X" channel,

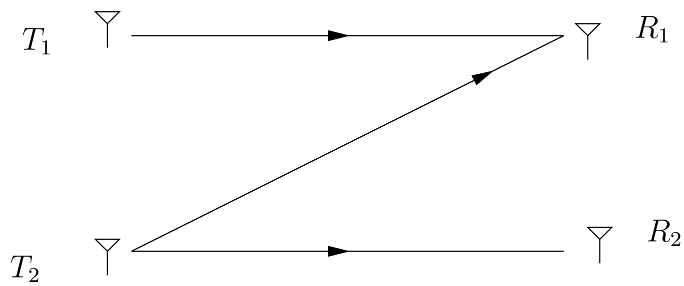

Fig. 1. The "Z" Channel

x_1 affects both y_1 and y_2, and there is a message from T_1 to R_2 corresponding to R_{12}. For simplicity, we consider only the "Z" channel in this paper. Note that similar to the broadcast channel, the capacity region of both the "X" and "Z" channels depends only on the marginals $p(y_1|x_1, x_2)$ and $p(y_2|x_2)$, and not on the joint channel distribution $p(y_1, y_2|x_1, x_2)$.

III. Degraded "Z" Channel

We consider a discrete memoryless "Z" channel in which the received signal at R_2 is a degraded version of the received signal at R_1. This degraded condition must be satisfied for *all* input distributions $p(x_1)$ at transmitter 1, i.e. for every $p(x_1)$, $p(y_2|x_2)$ equals $\sum p(y_1|x_2)p(y_2|y_1)$ for some $p(y_2|y_1)$. We term such a "Z" channel a *degraded* "Z" channel. For this case, the capacity region is the closure of the convex hull of all (R_{11}, R_{21}, R_{22}) satisfying

$$\begin{aligned} R_{11} &\leq I(X_1; Y_1|X_2) \\ R_{21} &\leq I(X_2; Y_1|U, X_1) \\ R_{11} + R_{21} &\leq I(X_1, X_2; Y_1|U) \\ R_{22} &\leq I(U; Y_2). \end{aligned} \quad (1)$$

for some distribution $p(x_1)p(u)p(x_2|u)$. Achievability of this region is described in Appendix *A* and the converse is presented in Appendix *B*.

The degraded nature of the channel allows R_1 to decode the signal intended for R_2. Thus, the rates of transmission to R_1 must lie in the multiple-access capacity region (given the message for R_2, which is the auxiliary random variable U) defined by $p(y_1|x_1, x_2)$. The structure of these equations is quite similar to the degraded broadcast channel capacity region, but the channel corresponding to the stronger user (R_1) is a multiple-access channel instead of a single-user channel.

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As a special case, we can also obtain the capacity region of the interference channel embedded in the degraded "Z" channel as

$$\begin{aligned} R_{11} &\leq I(X_1; Y_1 | X_2) \\ R_{22} &\leq I(X_2; Y_2). \end{aligned} \tag{2}$$

for some distribution $p(x_1)p(x_2)$. Again, the degraded nature of the channel allows R_1 to decode the intended for R_2. The two pairs T_1, R_1 and T_2, R_2 thus act as independent parallel channels, with the resulting capacity region given by (2).

IV. Gaussian "Z" Channel

The "Z" channel can be simplified in the Gaussian case and represented as in Figure 2 with power constraints P_1 and P_2 on the two transmitters. When $\alpha > 1 + P_1$, the

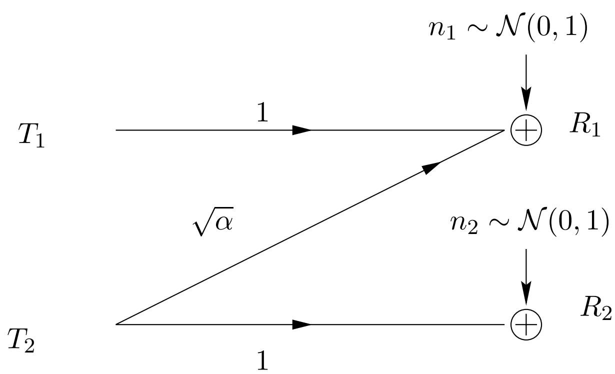

Fig. 2. The Gaussian "Z" Channel

channel is degraded for Gaussian inputs. In this special case, an achievable region can be obtained by using Gaussian inputs and successive decoding at R_2. Transmitter 2 generates two independent Gaussian codebooks, one intended for R_1 with average power βP_2 and one intended for R_2 with average power $(1-\beta)P_2$. Transmitter 2 chooses a codeword from each codebook and sends the sum of these codewords. Transmitter 1 generates one Gaussian codebook with average power P_1. Receiver 2 decodes its intended message while treating the codeword intended for R_1 as noise. Due to the degraded nature of the channel, R_1 can also decode the message intended for R_2 while treating all other signals as noise. Receiver 1 then subtracts this message off, leaving a Gaussian multiple-access channel from T_1 (with power P_1) and T_2 (with power βP_2). The corresponding achievable region is:

$$\begin{aligned} R_{11} &\leq C(P_1) \\ R_{21} &\leq C(\alpha\beta P_2) \\ R_{11} + R_{21} &\leq C(P_1 + \alpha\beta P_2) \\ R_{22} &\leq C\left(\frac{(1-\beta)P_2}{\beta P_2 + 1}\right), \end{aligned}$$

for β varying between 0 and 1.

Note that this region is simply that of (1) with Gaussian inputs. The achievable region is a combination of a Gaussian

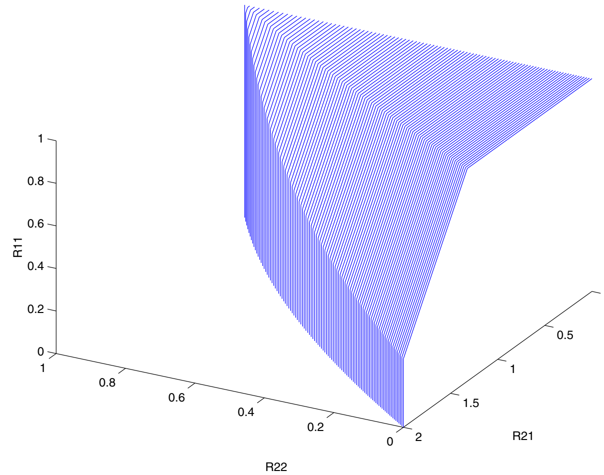

Fig. 3. Achievable Region for the Gaussian "Z" channel

broadcast channel (between R_{21} and R_{22}) and a Gaussian multiple-access channel (between R_{11} and R_{21}). When $R_{21} = 0$, the channel becomes a "Z" interference channel. The conditions $\alpha > 1 + P_1$ corresponds to the very strong interference case. This is why both R_{11} and R_{22} can simultaneously achieve their single-user capacities (when $\beta = 0$).

The achievable region for a Gaussian "Z" channel with $P_1 = P_2 = 1$ and $a = \sqrt{3}$ is illustrated in Figure 3. The optimality of Gaussian inputs is still an open problem. It can easily be seen that $\alpha > 1 + P_1$ is a necessary condition for the "Z" channel to be degraded, but a sufficient condition is not yet known.

V. Genie Aided Outer Bounds for the "Z" Channel

Along the lines of the outer bounds for the interference [8] and the broadcast channels [9], we can obtain genie aided bounds for the non-degraded "Z" channel. In this outer

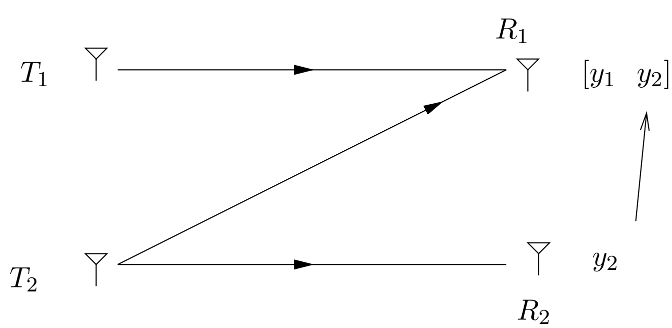

Fig. 4. The "Z_{upper}"" Channel

bound, the channel output of receiver 2 y_2 is made available to receiver 1, as seen in Figure 4. Thus, we obtain a new "Z" channel (which we call the "Z_{upper}" channel), where the received word at receiver 1 is a vector $[y_1 \; y_2]$ while that at receiver 2 is y_2. The "Z_{upper}" channel clearly is degraded for every choice of $p(x_1)$. Thus, from our earlier results we know that the capacity region of "Z_{upper}" is given by the closure of the convex hull of:

$$\begin{aligned}R_{11} &\leq I(X_1;Y_1|X_2)\\ R_{21} &\leq I(X_2;Y_1,Y_2|U,X_1)\\ R_{11}+R_{21} &\leq I(X_1,X_2;Y_1,Y_2|U)\\ R_{22} &\leq I(U;Y_2).\end{aligned}$$

over distributions $p(x_1)p(u)p(x_2|u)$.

As mentioned earlier, the capacity region of the "Z" channel depends only on the marginals and not on the joint distribution of the channel $p(y_1, y_2|x_1, x_2)$. However, the capacity region of the "Z_{upper}" channel *depends* on the joint distribution. Thus, we can tighten this upper bound by minimizing over all joint distributions while retaining the same marginals $p(y_1|x_1, x_2)$ and $p(y_2|x_2)$. An achievable region for the general "Z" channel can be constructed by extending the arguments of Marton [6], [7], but we defer this to a later paper.

VI. Conclusion

In this paper we defined a new two transmit, two receive channel - the "Z" channel. We find the capacity region of the degraded "Z" channel and we use this capacity region to construct an outer bound to the general "Z" channel. We also considered the Gaussian version of the Z channel and established an achievable region which is a combination of broadcast and multiple-access channel capacity regions.

Appendix

A. Achievability of (1)

Fix $p(u)p(x_2|u)$ and $p(x_1)$. Generate $2^{nR_{11}}$ independent codewords of length n, $x_1^n(w_{11})$, according to $p(x_1)$, and $2^{nR_{22}}$ independent codewords of length n, $u_1^n(w_{22})$, according to $p(u)$. For each codeword $u^n(w_{22})$, generate $2^{nR_{21}}$ independent codewords $x_2^n(w_{21}, w_{22})$ according to $p(x_2|u(w_{22}))$.

Decoding: Receiver 1 declares $(\hat{w}_{11}, \hat{w}_{21})$ to be the received message if it, along with some $\hat{\hat{w}}_{22}$ is the only set such that $(u^n(\hat{\hat{w}}_{22}), x_1^n(\hat{w}_{11}), x_2^n(\hat{w}_{21}, \hat{\hat{w}}_{22}), y_1^n)$ is jointly typical.

Receiver 2 declares $\hat{w}_{22}$ was sent if there exists a unique $\hat{w}_{22}$ such that $(u^n(\hat{w}_{22}), y_2^n)$ is jointly typical.

Let us define the events:

$$\begin{aligned}E_{2i} &= \{(U^n(i), Y_2^n) \in A_\epsilon^n\}\\ E_{1i} &= \{(U^n(i), Y_1^n) \in A_\epsilon^n\}\\ E_{1ijk} &= \{(U^n(i), X_2^n(j,i), X_1^n(k), Y_1^n) \in A_\epsilon^n\}.\end{aligned}$$

As usual, we assume $i = j = k = 1$ was sent. Similar to the degraded broadcast channel achievability, $R_{22} < I(U; Y_2)$ implies that the probability of error at receiver 2 goes to zero. Notice that

$$\begin{aligned}P_1^{(n)} \leq\ & P(E_{1111}^c) + \sum_{i\neq 1} P(E_{1i}) + \sum_{j\neq 1} P(E_{11j1})\\ &+ \sum_{k\neq 1} P(E_{111k}) + \sum_{j\neq 1, k\neq 1} P(E_{11jk})\end{aligned}$$

since E_{1ijk} implies E_{1i}. The first term goes to zero by the A.E.P. and from degradedness we have $I(U;Y_1) \geq I(U;Y_2)$ for all $p(x_1)$, and thus the second term goes to zero. Thus, we need to concentrate only on the events E_{111k}, E_{11j1}, and E_{11jk} for $j \neq 1$ and $k \neq 1$. Now,

$$\begin{aligned}P(E_{111k}) &= P\{(U^n(1), X_2^n(1,1), X_1^n(k), Y_1^n) \in A_\epsilon^n\\ &= \sum_{(u^n, x_1^n, x_2^n, y_1^n)\in A_\epsilon} p(x_1^n)p(u^n, x_2^n, y_1^n)\\ &\leq 2^{-n(H(X_1)+H(U,X_2,Y_1)-H(U,X_1,X_2,Y_1)-3\epsilon)}\\ &= 2^{-nI(X_1;Y_1,X_2,U)-3\epsilon}\\ &= 2^{-nI(X_1;Y_1|X_2,U)-3\epsilon}\end{aligned}$$

Thus if $R_{11} < I(X_1; Y_1|X_2, U)$, $\sum_{k\neq 1} P(E_{111k}) \to 0$.

Similarly,

$$\begin{aligned}P(E_{11j1}) &= P\{(U^n(1), X_2^n(j,1), X_1^n(1), Y_1^n) \in A_\epsilon^n\\ &= \sum_{(u^n, x_1^n, x_2^n, y_1^n)\in A_\epsilon} p(x_1^n, u^n, y_1^n)p(x_2^n|u^n)\\ &\leq 2^{-n(H(X_1,U,Y_1)+H(X_2|U)-H(U,X_1,X_2,Y_1)-5\epsilon)}\\ &= 2^{-n(H(Y_1|X_1,U)+H(X_2|X_1,U)-H(Y_1,X_2|X_1,U)-5\epsilon)}\\ &= 2^{-nI(X_2;Y_1|X_1,U)-5\epsilon}\end{aligned}$$

Thus if $R_{21} < I(X_2; Y_1|X_1, U)$, $\sum_{j\neq 1} P(E_{11j1}) \to 0$. Lastly, we have

$$\begin{aligned}P(E_{11jk}) &= P\{(U^n(1), X_2^n(j,1), X_1^n(k), Y_1^n) \in A_\epsilon^n\\ &= \sum_{(u^n, x_1^n, x_2^n, y_1^n)\in A_\epsilon^(n)} p(x_1^n)p(x_2^n|u^n)p(u^n, y_1^n)\\ &\leq 2^{-n(H(X_1)+H(X_2|U)+H(U,Y_1)-H(U,X_1,X_2,Y_1)-5\epsilon)}\\ &\leq 2^{-n(H(X_1,X_2|U)+H(Y_1|U)-H(X_1,X_2,Y_1|U)-5\epsilon)}\\ &= 2^{-nI(X_1,X_2;Y_1|U)-5\epsilon}\end{aligned}$$

Thus if $R_{11} + R_{21} < I(X_1, X_2; Y_1|U)$, $\sum_{j\neq 1, k\neq 1} P(E_{11jk}) \to 0$. As usual, time-sharing allows for achievability of the convex hull.

B. Converse for (1)

In this section we prove that the region given in (1) is the actual capacity region for the "Z" channel where for all input distributions $p(x_1)$ the output y_2 is a stochastically degraded version of the output y_1.

By Fano's inequality we clearly have the following:

$$\begin{aligned}H(W_{11}|Y_1^n) &\leq n\epsilon_{11}\\ H(W_{21}|Y_1^n) &\leq n\epsilon_{21}\\ H(W_{22}|Y_2^n) &\leq n\epsilon_{22}.\end{aligned}$$

We first bound R_{11} using the same argument as used in the MAC converse proof. Following the MAC converse proof in [2, p. 400] where we replace $X_2(W_2)$ with $X_2(W_{22}, W_{21})$, we get

$$nR_{11} \leq \sum_{i=1}^{n} I(X_{1i}; Y_{1i}|X_{2i}) + n\epsilon_{11}$$

We bound $R_{11}+R_{21}$ by

$$\begin{aligned}
n(R_{11} \quad + \quad R_{21}) &= H(W_{11},W_{21})\\
&= I(W_{11},W_{21};Y_1^n)+H(W_{11},W_{21}|Y_1^n)\\
&\leq I(W_{11},W_{21};Y_1^n)+n\epsilon_{21}\\
&\leq I(W_{11},W_{21};Y_1^n|W_{22})+n\epsilon_{21}\\
&= \sum_{i=1}^n I(W_{11},W_{21};Y_{1i}|W_{22},Y_1^{i-1})+n\epsilon_{21}\\
&= \sum_{i=1}^n I(W_{11},W_{21};Y_{1i}|U_i)+n\epsilon_{21}\\
&\leq \sum_{i=1}^n I(X_{1i},X_{2i},W_{11},W_{21};Y_{1i}|U_i)+n\epsilon_{21}\\
&= \sum_{i=1}^n I(X_{1i},X_{2i};Y_{1i}|U_i)+n\epsilon_{21} \qquad (3)
\end{aligned}$$

where $U_i=(W_{22},Y_1^{i-1})$, and we get (3) from the memoryless nature of the channel.

Now we bound R_{22} by the following:

$$\begin{aligned}
nR_{22} &= H(W_{22})\\
&= I(W_{22};Y_2^n)+H(W_{22}|Y_2^n)\\
&\leq I(W_{22};Y_2^n)+n\epsilon_{22}\\
&= \sum_{i=1}^n I(W_{22};Y_{2i}|Y_2^{i-1})+n\epsilon_{22}\\
&= \sum_{i=1}^n (H(Y_{2i}|Y_2^{i-1})-H(Y_{2i}|Y_2^{i-1},W_{22}))+n\epsilon_{22}\\
&\leq \sum_{i=1}^n (H(Y_{2i})-H(Y_{2i}|Y_1^{i-1},Y_2^{i-1},W_{22}))+n\epsilon_{22}\\
&= \sum_{i=1}^n (H(Y_{2i})-H(Y_{2i}|Y_1^{i-1},W_{22}))+n\epsilon_{22} \qquad (4)\\
&= \sum_{i=1}^n I(U_i;Y_{2i})+n\epsilon_{21}
\end{aligned}$$

where we used the degraded property of the channel to get (4).

We bound R_{21} by

$$\begin{aligned}
nR_{21} &= H(W_{21})\\
&= I(W_{21};Y_1^n)+H(W_{21}|Y_1^n)\\
&\leq I(W_{21};Y_1^n)+n\epsilon_{21}
\end{aligned}$$

Note that $I(W_{21};Y_1^n)$ can be bounded above by

$$\begin{aligned}
& I(W_{21};Y_1^n|W_{22})\\
\leq\ & I(X_2^n;Y_1^n|W_{22})\\
=\ & H(X_2^n|W_{22})-H(X_2^n|W_{22},Y_1^n)\\
\leq\ & H(X_2^n|W_{22},X_1^n)-H(X_2^n|W_{22},Y_1^n,X_1^n) \qquad (5)\\
=\ & I(X_2;Y_1^n|W_{22},X_1^n)\\
=\ & H(Y_1^n|X_1^n,W_{22})-H(Y_1^n|W_{22},X_1^n,X_2^n)\\
=\ & \sum_{i=1}^n H(Y_{1i}|Y_1^{i-1},W_{22},X_1^n)\\
& \quad -\sum_{i=1}^n H(Y_{1i}|Y_1^{i-1},W_{22},X_1^n,X_2^n)\\
=\ & \sum_{i=1}^n H(Y_{1i}|U_i,X_1^n)-\sum_{i=1}^n H(Y_1 i|U_i,X_1^n,X_2^n)\\
=\ & \sum_{i=1}^n H(Y_{1i}|U_i,X_1^n)-\sum_{i=1}^n H(Y_{1i}|U_i,X_{1i},X_{2i}) \qquad (6)\\
\leq\ & \sum_{i=1}^n H(Y_{1i}|U_i,X_{1i})-\sum_{i=1}^n H(Y_{1i}|U_i,X_{1i},X_{2i})\\
=\ & \sum_{i=1}^n I(X_{2i};Y_{1i}|U_i,X_{1i})
\end{aligned}$$

where (5) follows from the independence of X_2^n and X_1^n, and (6) follows from the memoryless nature of the channel.

Thus, we have

$$\begin{aligned}
nR_{11} &\leq \sum_{i=1}^n I(X_{1i};Y_{1i}|X_{2i})+n\epsilon_{11}\\
nR_{21} &\leq \sum_{i=1}^n I(X_{2i};Y_{1i}|U_i,X_{1i})+n\epsilon_{21}\\
n(R_{11}+R_{21}) &\leq \sum_{i=1}^n I(X_{1i},X_{2i};Y_{1i}|U_i)+n\epsilon_{21}\\
nR_{22} &\leq \sum_{i=1}^n I(U_i;Y_{2i})+n\epsilon_{21}
\end{aligned}$$

with $U_i=(W_{22},Y_1^{i-1})$, which satisfies the property that $U\to X_2\to(Y_1,Y_2)$ be a Markov chain. Using a time-sharing variable Q (similar to the MAC converse), we finally get

$$\begin{aligned}
R_{11} &\leq I(X_1;Y_1|X_2,Q)\\
R_{21} &\leq I(X_2;Y_1|U,X_1,Q)\\
R_{11}+R_{21} &\leq I(X_1,X_2;Y_1|U,Q)\\
R_{22} &\leq I(U;Y_2|Q)
\end{aligned}$$

for some $p(q)p(u|q)p(x_1|q)p(x_2|u,q)$.

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