
There are four Parts; assigned 1 point each, for a total of 4 points.

Part I (1 point):
Heat conduction across a slab of metal obeys a partial differential equation where the heat flow per unit time depends on the temperature gradient and the thermal conductivity. A system which consists of such a slab of metal, a heating element on one end, and a temperature sensor at some fixed point is considered. This is a distributed-parameter system with a transfer function which is not a rational function of s. The transfer function between the rate of heat $u(t)$ provided at one end as the input and the temperature $y(t)$ being recorded at the sensor location (all in appropriate units) is given and it is

$$G(s) = e^{-\sqrt{s}}.$$

Suppose that $u(t)$ varies periodically and is

$$u(t) = \sin(\frac{2\pi}{T}t)$$

(where t denotes time and T the period of oscillation) and suppose that the oscillation recorded at the sensor location trails that of the input by a quarter of the period, i.e., that

$$y(t) = A\sin(\frac{2\pi}{T}(t - \frac{T}{4})).$$

Determine the amplitude A of the temperature oscillations as well as the period T.

Solution:

Evaluate the transfer function at $s = \frac{2\pi}{T}j$ and identify the amplitude and phase

$$\begin{aligned} e^{-\sqrt{\frac{2\pi}{T}j}} &= e^{-\sqrt{\frac{2\pi}{T}}\sqrt{j}} \\ &= e^{-\sqrt{\frac{2\pi}{T}}(\frac{\sqrt{2}}{2}+j\frac{\sqrt{2}}{2})} \\ &= e^{-\sqrt{\frac{\pi}{T}}-j\sqrt{\frac{\pi}{T}}}. \end{aligned}$$

Then

$$y(t) = e^{-\sqrt{\frac{\pi}{T}}}\sin(\frac{2\pi}{T}t - \sqrt{\frac{\pi}{T}}).$$

Therefore

$$\sqrt{\frac{\pi}{T}} = \frac{\pi}{2}$$

which leads us to the values:

$$\boxed{T = \frac{4}{\pi}}$$

$$\boxed{A = e^{-\frac{\pi}{2}}}$$

Part II (1 point):
Consider a dynamical system modeled as a lumped scalar linear system

$$\frac{d^3x(t)}{dt^3} = u(t).$$

Here $u(t)$ represents an input to the system, while $x(t)$ represents its position at time $t \in [0, \infty)$. Determine a second-order stabilizing controller which measures and processes $x(t)$ as its input, and determines the value for the input $u(t)$ to the given system. Explain why your design works.

Solution:

It is clear by using Root-locus or Nyquist arguments that we need lead to stabilize the system. In particular, two stable zeros are needed to provide the necessary lead. Thus, we choose as controller

$$C(s) = \frac{(s+1)^2}{(.1s+1)^2}$$

We verify that the characteristic polynomial is

$$\begin{aligned} s^3(.1s+1)^2 + (s+1)^2 &= .01s^5 + .2s^4 + s^3 + s^2 + 2s + 1 \\ &= .01 \times (s^5 + 20s^4 + 100s^3 + 100s^2 + 200s + 100). \end{aligned}$$

Then, applying the Routh test we can readily verify that this polynomial has its roots in the left half of the complex plane.

This problem has many solution – you need to justify your answer.

Part III (1 point):

A physical system is modeled by a second order differential equation

$$\ddot{y}(t) = -\dot{y}(t) + u(t-\tau)$$

where $y(t)$ is the output, $u(t)$ is the input, and τ represents a time-delay measured in seconds. Thus, the transfer function from

$$u(t-\tau) \text{ to } y(t)$$

is

$$G(s) = \frac{1}{s(s+1)}.$$

frequency	magnitude	phase in degrees
1.1716	0.5541	-139.5176
1.4921	0.3731	-146.1702
1.8126	0.2665	-151.1152
2.1332	0.1990	-154.8834
2.4537	0.1538	-157.8266
2.7742	0.1222	-160.1776
3.0947	0.0994	-162.0928
3.4153	0.0823	-163.6798
3.7358	0.0692	-165.0143
4.0563	0.0590	-166.1511
4.3768	0.0509	-167.1302
4.6974	0.0443	-167.9820
5.0179	0.0389	-168.7294

You are given the magnitude and the phase of $G(j\omega)$ for a range of frequencies between .1716 and 5.0179 [rad/sec] in the table above (and to the right). The system is controlled using negative feedback; that is, the control input is proportional to the difference of an external reference signal $r(t)$ and the output $y(t)$, namely

$$u(t) = K\left(r(t) - y(t)\right),$$

with K a gain factor which is now taken to be $K = 10$. Determine the range of values for the time delay for which the closed loop system is stable.

Solution:

Clearly, when $\tau = 0$ the closed loop system is stable: it is a second order system with characteristic equation $1 + \frac{10}{s(s+1)} = 0$, and this has no roots in the right half plane. This is easy to see. For instance, the characteristic polynomial is $s^2 + s + 10$ and has all coefficients positive, and since it is of degree 2 the roots are in the left half of the complex plane. You can of course compute the roots, or use the Routh test. Either is straightforward.

Now, from the table we see that at $\omega = 3.0947$ [rad/sec] the gain of $1/(s(s+1))$ is 0.0994 which is approximately equal to $1/K$ for $K = 10$. Therefore, for $K = 10$, the gain cross-over frequency is $\omega = 3.0947$ [rad/sec] and the phase is -162.0928 in degrees. Thus, the phase margin for the closed loop system is about $180^o - 162^o = 18^o$. Therefore, the system can tolerate a lag from the time-delay that does not exceed this margin. Thus, the maximal time-delay is approximately,

$$\boxed{\tau_{\max} = \frac{18\pi}{180}\frac{1}{3.0947} = 0.1 \text{ [sec]}}$$

Part IV (1 point):
Consider four systems with the following transfer functions:

$$
\begin{aligned}
G_A(s) &= \frac{20s + 400}{s^2 + 20s + 400}, \\
G_B(s) &= \frac{200s - 400}{s^2 + 8s + 400}, \\
G_C(s) &= \frac{2s + 4}{s^2 + 2s + 4}, \\
G_D(s) &= \frac{-2s + 4}{s^2 + 0.8s + 4}.
\end{aligned}
$$

You are required to match these with corresponding pole/zero plots, step responses, Bode plots, and Nyquist plots shown in the next four pages. (E.g., $G_A(s)$ corresponds to the first pole-zero plot. Then we should mark zp1 in the corresponding position as shown, etc.)

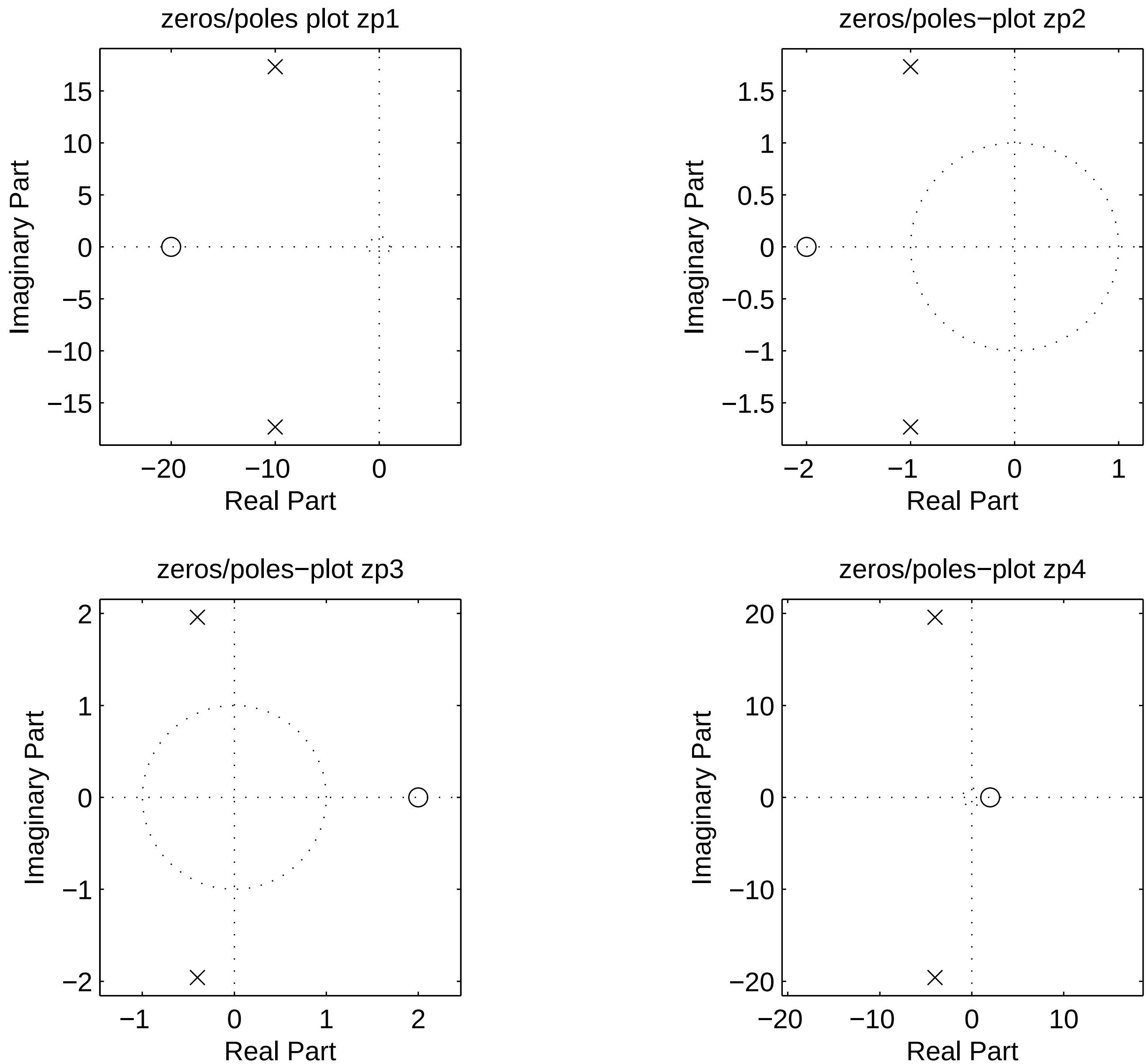
Transfer fn	Pole-zero plot	Step response	Bode plot	Nyquist plot
A	zp1			
B				
C				
D				

Note that the Nyquist plots $nq2$ and $nq4$ are identical. In this case mark the possibilities in the above table, and explain why these two plots are identical.

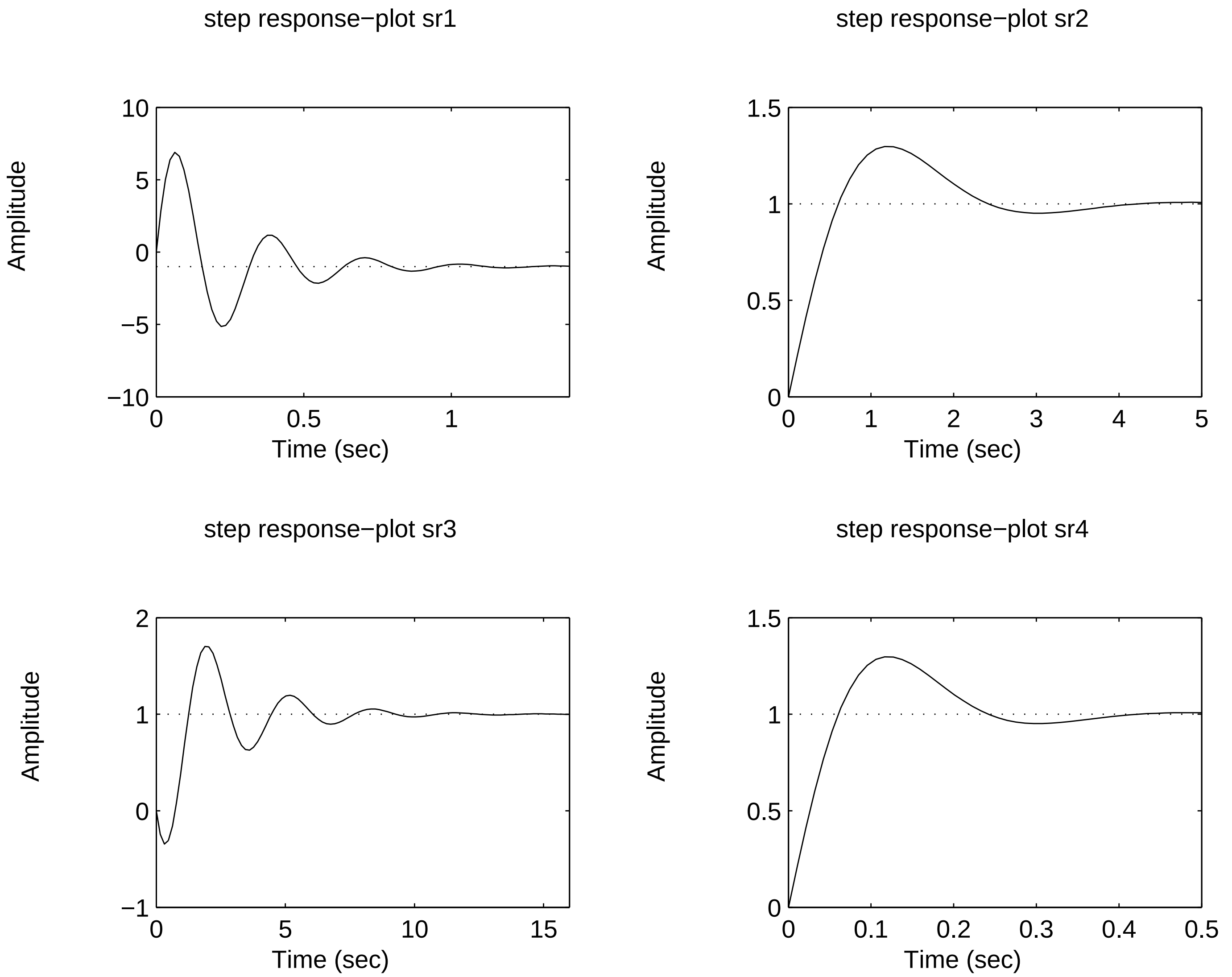
Solution:

Transfer fn	Pole-zero plot	Step response	Bode plot	Nyquist plot
A	zp1	sr4	bd3	nq2 or nq3
B	zp4	sr1	bd4	nq4
C	zp2	sr2	bd2	nq2 or nq3
D	zp3	sr3	bd1	nq1

The reason for the Nyquist plots being identical is that $G_A(s) = G_C(s/10)$.

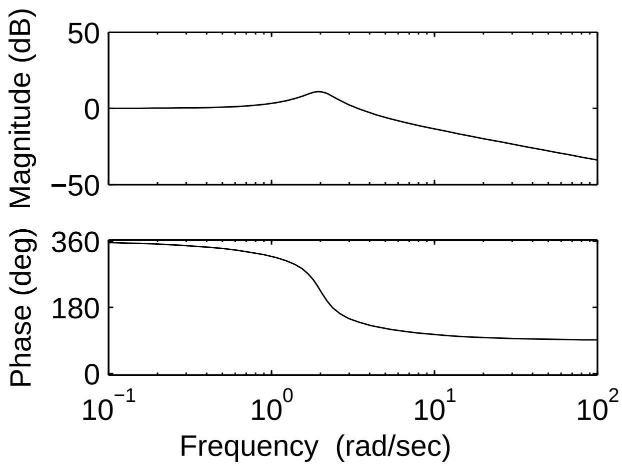


Pole-zero plots

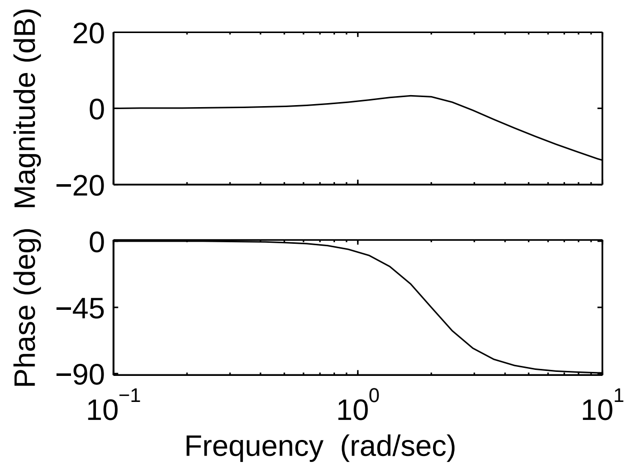


Step responses

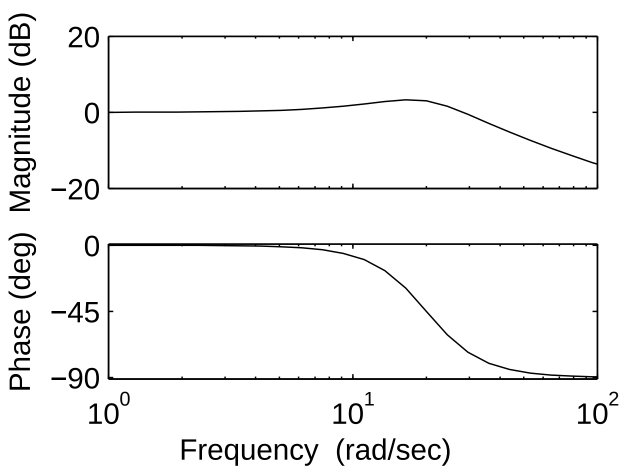
Bode plot bd1



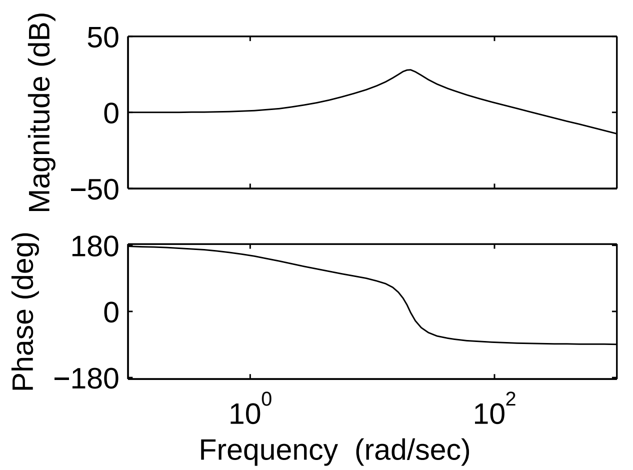
Bode plot bd2



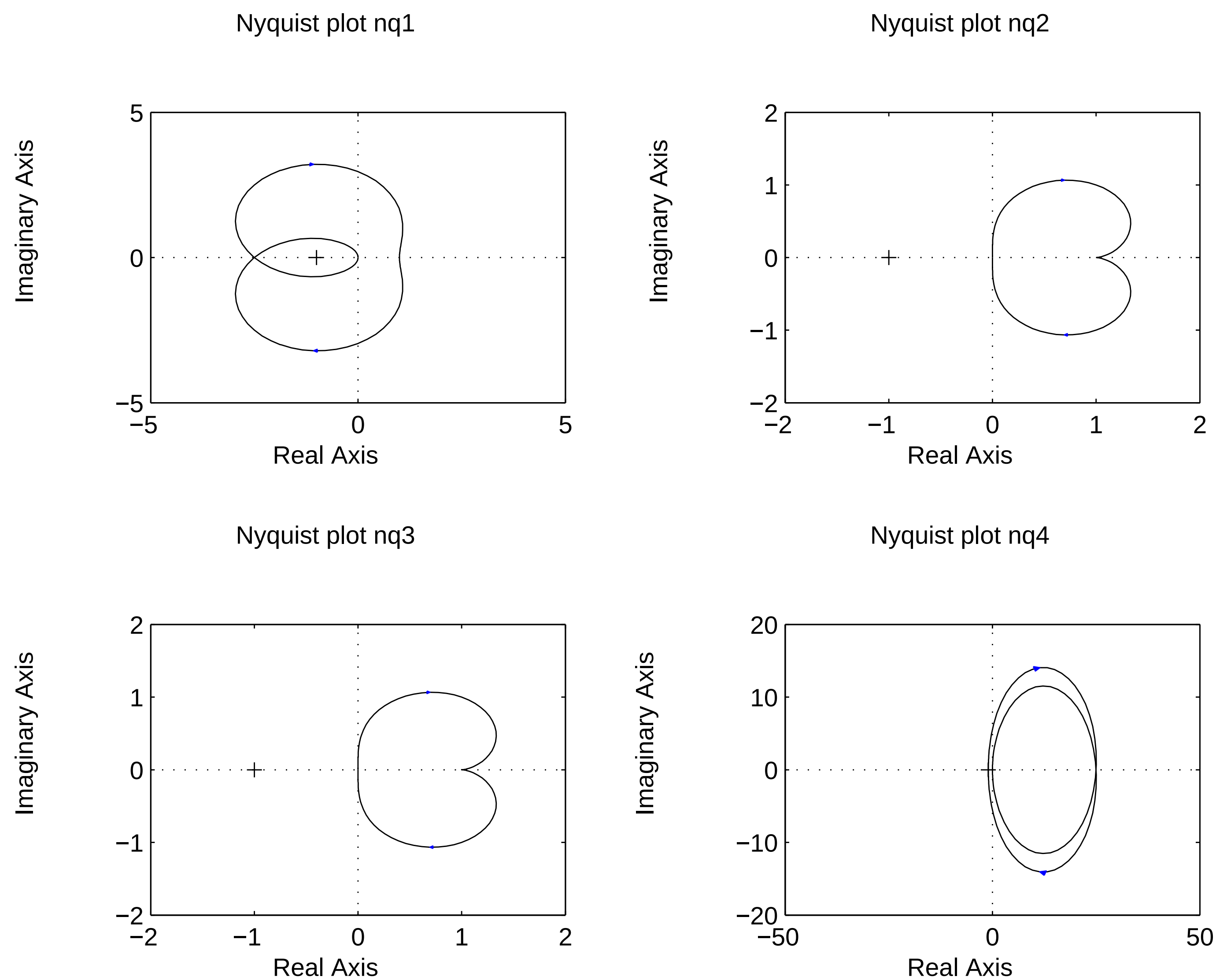
Bode plot bd3



Bode plot bd4



Bode plots



Nyquist plots