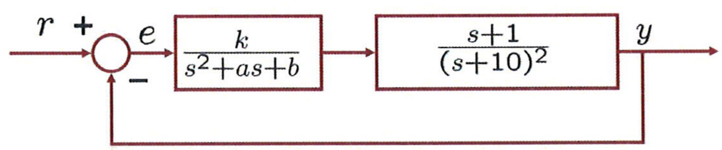
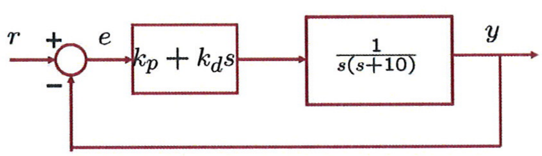
Q. 1



Consider the feedback system shown in the figure with transfer function $G(s) = \frac{s+1}{(s+10)^2}$ and a controller of the form $C(s) = \frac{k}{s^2+as+b}$, where $k > 0$.

1. Determine values of a, b and a range of values for k, so that the feedback system can track a ramp input $r(t) = t,\ t > 0$ with zero steady state error.
2. A block diagram of a servo system for motion control is shown in the Figure below:



The plant transfer function is given by $G(s) = \frac{1}{s(s+10)}$. For the unity feedback setup, design a Proportional-Derivative controller such that the resulting closed loop system has a damping $\zeta = \frac{1}{\sqrt{2}} = 0.707$ and a natural frequency $\omega_n = 8\ rad/s$. (Note that a second order system of the form $\frac{A}{s^2+2\zeta\omega_n+\omega_n^2}$ has a damping ζ and natural frequency ω_n.)

Q 1(a): The open loop gain is $L = GK = \frac{k(s+1)}{s^2+as+b)(s+10)^2}$. The transfer function with the error E as output and the input as R is

$$\frac{E(s)}{R(s)} = \frac{1}{1+L}.$$

If $R = \frac{1}{s^2}$ if r is a ramp input. If the ramp has to be tracked with zero steady state error then from the internal model principle, the controller K has to include two poles at zero. Setting $a = b = 0$ we get the controller to be k/s^2 which has two poles at zero. Now we have to determine k for stability. The characteristic polynomial in this case is

$$s^2(s+10)^2 + k(s+1) = s^2(s^2+20s+100) + ks + 1 = s^4 + 20s^3 + 100s^2 + ks + k$$

We use the Routh Hurwitz criterion

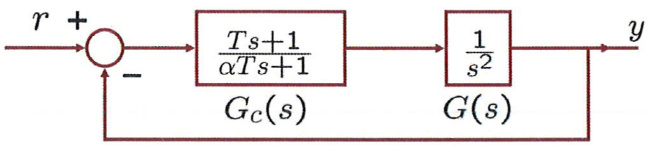
$$\begin{array}{lccc} s^4 : & 1 & 100 & k \\ s^3 : & 20 & k & \\ s^2 : & \frac{2000-k}{20} & \frac{20k}{20} & \\ s^1 : & \frac{(100-k/20)k-20k}{100-k/20} & & \\ s^0 : & k & & \end{array}$$

Thus we need $100 - k/20 > 0,\ k > 0$ and $\frac{(100-k/20)k-20k}{100-k/20} > 0$. Thus $0 < k < 2000$ and $100 - k/20 - 20 > 0$. Thus $0 < k < 1600$. Thus the controller can be of the form k/s^2 where $0 < k < 1600$.

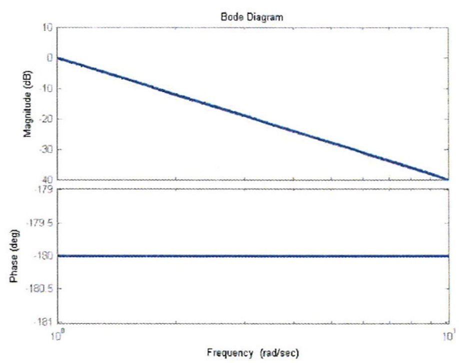
Q 1 (b): Note that $L = \frac{k_p+k_d s}{s(s+10)}$ and the closed-loop transfer function is $\frac{L}{1+L} = k_p + k_d s s^2 + (k_d + 10)s + k_p$. Thus we need $k_p = \omega_n^2 = 8^2 = 64$ and $k_d + 10 = 2\zeta\omega_n = 16/\sqrt{2} = 8\sqrt{2}$. Thus $k_p = 64$ and $k_d = 8\sqrt{2} - 10 = 1.313$.

Q. 2

Consider the figure given below. Design a lead controller $G_c(s) = \frac{Ts+1}{\alpha Ts+1}$ for the plant $G(s) = \frac{1}{s^2}$ such that



the phase margin is 45^o. The bode plot of $G(s)$ is given below. *Hint: The frequency ω_m at which the phase of $G_c(j\omega)$ is a maximum is $\omega_m = \frac{1}{T\sqrt{\alpha}}$..*



Solutions to Q. 2: The phase of the plant is constant at -180. Thus to have a phase margin of 45^o the controller needs to have a phase of 45^o at gain crossover. We will choose the gain crossover frequency $\omega_c = \omega_m$ where $\omega_m = \frac{1}{T\sqrt{\alpha}}$ where the lead controller has the maximum phase. Thus we need $\phi_m := \angle\{G_c(j\omega_m\} = \pi/4$. This implies that

$$\sin\phi_m = \frac{1-\alpha}{1+\alpha} \Rightarrow \frac{1}{\sqrt{2}} = \frac{1-\alpha}{1+\alpha} \Rightarrow \alpha = 0.1716$$

We also need ω_m to be the gain crossover and therefore

$$\begin{aligned} &|G_c(j\omega_m)||G(j\omega_m)| = 1 \\ \Rightarrow\ &\frac{\sqrt{1+(\omega_m T)^2}}{\sqrt{1+\alpha^2(\omega_m T)^2}}|G(j\omega_m)| = 1 \\ \Rightarrow\ &\frac{\sqrt{1+\frac{1}{\alpha}}}{\sqrt{1+\alpha^2\frac{1}{\alpha}}}|\frac{1}{\omega_m^2}| = 1 \\ \Rightarrow\ &\frac{1}{\sqrt{\alpha}} = \omega_m^2 \end{aligned}$$

Thus $\omega_m = 1.5537$ and $T = \frac{1}{\omega_m\sqrt{\alpha}} = 1.5537$ and $G_c = \frac{1.5537s+1}{0.2666s+1}$

Q. 3

Consider the systems with the following transfer functions:

$G_A(s) = \frac{1}{s^2+0.5s+1}$

$G_B(s) = \frac{4}{s^2+1s+4}$

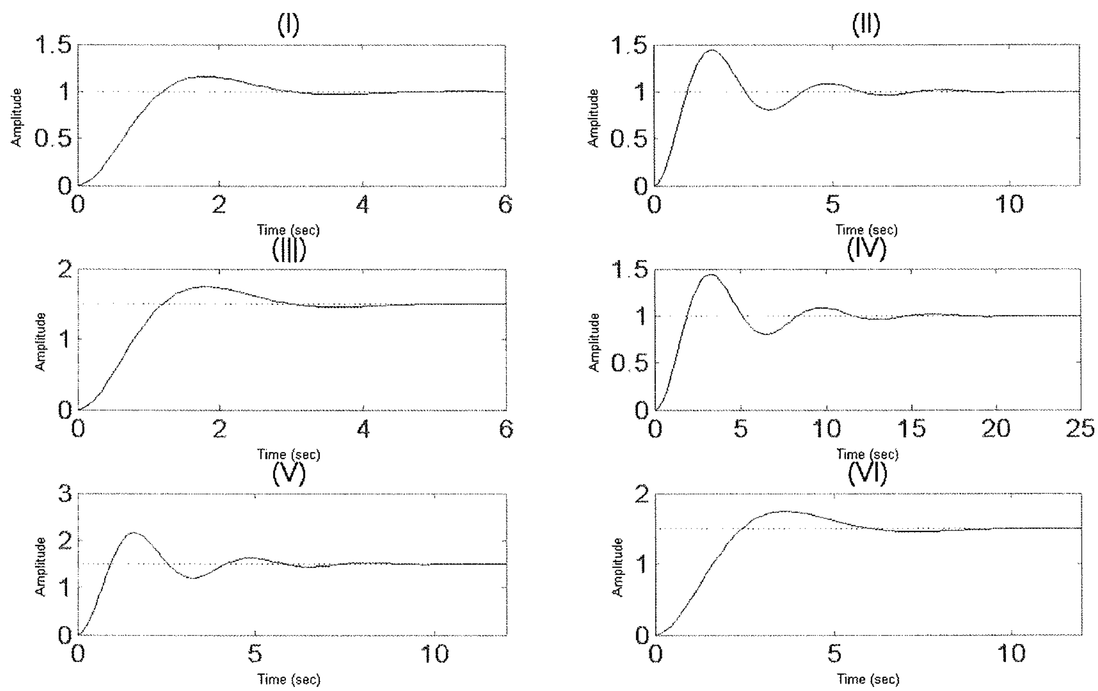
$G_C(s) = \frac{3}{2s^2+2s+2}$

$G_D(s) = \frac{6}{s^2+2s+4}$

$G_E(s) = \frac{4}{s^2+2s+4}$

$G_F(s) = \frac{12}{2s^2+2s+8}$

You are required to match these with the unit step responses shown below (*Hint: calculate the damping* ζ, *the natural frequency* ω_n *for each system and the corresponding steady state output values*).



Solutions to Q. 30)

$G_A(s) = \frac{1}{s^2+0.5s+1}$, $\omega_n = 1,\ \zeta = 0.25,\ x_\infty = 1$

$G_B(s) = \frac{4}{s^2+1s+4}$, $\omega_n = 2,\ \zeta = 0.25,\ x_\infty = 1$

$G_C(s) = \frac{3}{2s^2+2s+2}$, $\omega_n = 1,\ \zeta = 0.5,\ x_\infty = 1.5$

$G_D(s) = \frac{6}{s^2+2s+4}$, $\omega_n = 2,\ \zeta = 0.5,\ x_\infty = 1.5$

$G_E(s) = \frac{4}{s^2+2s+4}$, $\omega_n = 2,\ \zeta = 0.5,\ x_\infty = 1$

$G_F(s) = \frac{12}{2s^2+2s+8}$, $\omega_n = 2$, $\zeta = 0.25$, $x_\infty = 1.5$

(I), (II), (IV) have $x_\infty = 1$. Thus these have to be matched with G_A, G_B and G_E. I has the least peak (indicates greatest damping and thus G_E corresponds to I. Also (IV) settles slower than II and thus G_A corresponds to IV and G_B to II.

Of the cases with $x_\infty = 1.5$, V is the least damped and thus G_F corresponds to V. III settles faster than VI and thus G_D corresponds to III and G_C to VI.