(1) (20 points) Suppose, you want to transmit over a channel characterized by an additive noise Z, such that the probability density function of Z is given by

$$f_Z(z) = \begin{cases} (2-|z|)/4, & -2 \le z \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

You may transmit either a $+1$ or a -1 over this channel, and these two options are equally likely. This means, if you transmit $X \in \{+1, -1\}$, then you receive $Y = X + Z$.

a. What is the optimal strategy to recover an estimate of X from Y? 3

b. What is the probability of error in the above estimate? 3

c. Suppose you send the following three: X_1, X_2 and X_1X_2, where $X_1, X_2 \in \{+1, -1\}$. What is the probability that the errors remain unnoticed (Hint: Any one error will be noticed)? 4

Consider the following protocol to recover X:

$$\hat{X} = \begin{cases} -1 \text{ if } Y \le -0.5 \\ +1 \text{ if } Y \ge 0.5 \\ \text{Retransmit otherwise} \end{cases}$$

d. What is the average number of transmission that has to be performed then to transmit a vector of $+1$ and -1 of length 100? 5

e. What is the probability for any symbol (+1 or -1) to be wrongly estimated? 5

Solution

a. The optimal strategy is the maximum a-posteriori or MAP, where you simply check: given Y, which is more likely transmission, $+1$ or -1. Note that, $+1$ an -1 are equally likely, unconditionally. When X is -1, the probability that Y is positive is equal to $P(Z > 1) < 1/2$. Hence the optimal strategy is: when Y is positive declare $X = +1$, when Y is

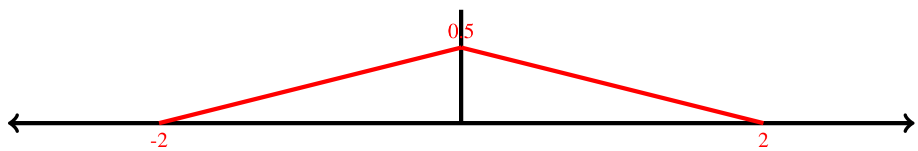

FIGURE 1. $f_Z(z)$

negative declare $X = -1$ and declare either if $Y = 0$.

b. From the symmetry of the distribution of Z, we can say: Probability of error $P(X = -1)P(Z > 1) + P(X = 1)P(Z < -1) = P(Z > 1) = 0.125$.

c. The errors will remain unnoticed only when exactly two of the three transmission (X_1, X_2, X_1X_2) are wrongly recovered. The probability of that happening is $\binom{3}{2} \cdot 0.125^2 \cdot 0.875 = 0.041$.

d. Note that, the probabilities of $X = +1$ and $X = -1$ are equal and the distribution of Z is symmetric. When $X = -1$, probability of retransmission for a symbol is $P(0.5 <$

$Z < 1.5) = 0.25$. Similarly, when $X = +1$, probability of retransmission for a symbol is $P(-1.5 < Z < -0.5) = 0.25$. Hence, the probability that a symbol is recovered (possibly erroneously) after i transmissions is $0.25^{i-1} \cdot 0.75$. Hence, average number of transmissions for each symbol is $\sum_{i=1}^{\infty} i \cdot 0.25^{i-1} \cdot 0.75 = 4/3$. Hence $100 * 4/3 = 133.33$ transmissions on average will be required.

3. When $X = -1$, probability that a symbol is incorrectly recovered in one transmission is $P(Z > 1.5) = 0.03125$. Same is true when $X = +1$. Hence, probability of error $= \sum_{i=1}^{\infty} 0.25^{i-1} \cdot 0.03125 = 0.04167$.

(2) (20 points) Suppose the signal $f(t)$ is going to be transmitted with double-sideband (DSB-SC) amplitude modulation (AM). That is, the signal $f(t)\cos(2.4\pi \times 10^6 t)$ is transmitted.

a. What can be the maximum bandwidth of $f(t)$ for distortionless reception? 3

b. If you are allowed to transmit only within the band of 1 MHz to 1.4MHz, what is the maximum bandwidth of $f(t)$ that you can support? 2

At the receiver end, you receive a phase-shifted version because of asynchronous communication:

$$X(t) = f(t)\cos(2.4\pi \times 10^6 t + \Theta),$$

where Θ is a random phase sampled from the uniform distribution in $[0, 2\pi]$.

c. Find expected value and autocorrelation function of $X(t)$. 3+4

d. Is $X(t)$ a stationary process? 1

Suppose, $f(t)$ is band-limited according to part **b.** above. Let us sample $f(t)$ at rate 25% above the Nyquist rate and use PCM (pulse-coded modulation) to transmit this signal.

e. What is the sampling rate? 1

f. Let $|f(t)| \leq 100$ and each sample drawn above is quantized into levels of size 0.25. Determine the number of binary pulses required to encode each sample? 3

g. Determine the bits per second transmission rate and the minimum bandwidth required to transmit the signal. 3

Solution

a. Carrier frequency $f_c = 1.2MHz$. If $f(t)$ is band-limited in $[-B, B]$ then, $B < 1.2MHz$.

b. Clearly $1.2 - 1 = 1.4 - 1.2 = 0.2$ MHz $= 200$ KHz.

c. $\mathbb{E}X(t) = f(t)\frac{1}{2\pi}\int_0^{2\pi}\cos(2\pi f_c t + \theta)d\theta = 0.$

$$\begin{aligned}
\mathbb{E}[X(t_1)X(t_2)] &= f(t_1)f(t_2)\int_0^{2\pi}\cos(2\pi f_c t_1 + \theta)\cos(2\pi f_c t_2 + \theta)d\theta \\
&= \frac{f(t_1)f(t_2)}{2}\int_0^{2\pi}\Big(\cos(2\pi f_c(t_1 + t_2) + 2\theta) + \cos(2\pi f_c(t_1 - t_2))\Big)d\theta \\
&= \frac{f(t_1)f(t_2)}{2}\cos(2\pi f_c(t_1 - t_2)) \cdot 2\pi \\
&= f(t_1)f(t_2)\pi\cos(2\pi f_c(t_1 - t_2)).
\end{aligned}$$

d. $X(t)$ is not a stationary process, unless $f(t_1)f(t_2)$ is only a function of $t_1 - t_2$.

e. Nyquist rate = 400 KHz. Hence sampling rate =500KHz.

f. Total number of levels = $200/0.25 = 800$. As, $\log_2 800 = 9.64$, we will need 10 pulses for each sample.

g. Transmission rate = $500000 \times 10 = 5$Megabits per second. As a unit bandwidth channel can transmit 2 bits of information per second, the required bandwidth is at least 2.5 MHz.