There are four Parts, assigned 1 point each, for a total of 4 points.

Part I (1 point):
A feedback system is shown in Figure 1 where $P(s)$ is a system model and $e^{-\tau s}$ represents transmission delay in the feedback path. The system transfer function can be fairly accurately

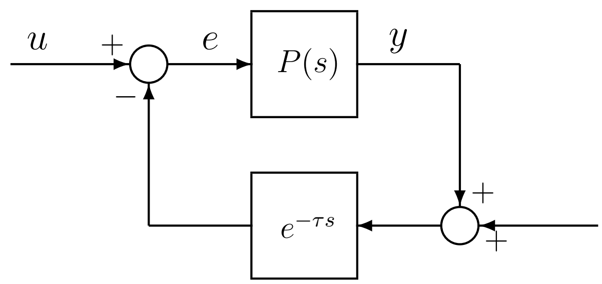

Figure 1: Feedback system.

modeled over the range of frequencies that are relevant to stability analysis by

$$P(s) = \frac{2}{\sqrt{s}}$$

for $s = j\omega$ with ω measured in radians/sec. It is also given that when the transmission delay τ is sufficiently small or zero, the feedback system is bounded-input/bounded-output stable.
Determine the maximal amount of time delay that the feedback system can tolerate before it becomes unstable.

Space for your work:

Solution:
The frequency where the gain is 1, is $\omega_{\text{gc}} = 4$ [rad/sec]. The phase, for all frequencies, is

$$\boxed{\angle P(j\omega) = \angle \frac{2}{\sqrt{j} \times \sqrt{\omega}} = -\frac{\pi}{4}}$$

Thus, the phase margin is $3\pi/4$ and the maximal amount of time-delay must be such that

$$\boxed{\omega_{\text{gc}}\tau = 3\pi/4}$$

hence

$$\boxed{\boxed{\tau_{\text{maximal}} = \frac{3\pi}{16} \simeq 0.5890}}$$

Part II (1 point):
Determine the maximal interval $[0, K_{\mathrm{maximal}})$ for the gain K for which the delay-differential equation

$$\dot{y}(t) = u(t) - Ky(t) - 2Ky(t-1) - Ky(t-2)$$

is stable. Here, $u(t)$ is thought of as the input to the system and $y(t)$ as the output.

Space for your work:

Solution:
The characteristic equation is

$$s + K(1 + 2e^{-s} + e^{-2s}) = 0.$$

Equivalently, we may consider

$$1 + K\frac{(1 + e^{-s})^2}{s} = 0,$$

and therefore, the Nyquist diagram for loop gain

$$K\frac{(1 + e^{-j\omega})^2}{j\omega}.$$

Thus, we need to determine the frequency where the

$$\angle\frac{(1 + e^{-j\omega})^2}{j\omega} = -\pi,$$

that is, the frequency where

$$\angle 1 + e^{-j\omega} = -\frac{\pi}{4},$$

or, equivalently, the smallest frequency for which

$$1 + \cos(\omega) = \sin(\omega).$$

Clearly, the smallest such frequency is $\boxed{\omega = \frac{\pi}{2}}$ and at that frequency the critical value for K would be such that

$$\boxed{K\frac{|1+e^{-j\omega}|^2}{\omega} = 1}$$

But at $\omega = \pi/2$, $|1 + e^{-j\pi/2}| = \sqrt{2}$. It follows that

$$\boxed{\boxed{K_{\text{maximal}} = \frac{\pi}{4} \simeq 0.7854.}}$$

Part III (1 point):
Consider the feedback system in Figure 2. Do the following:

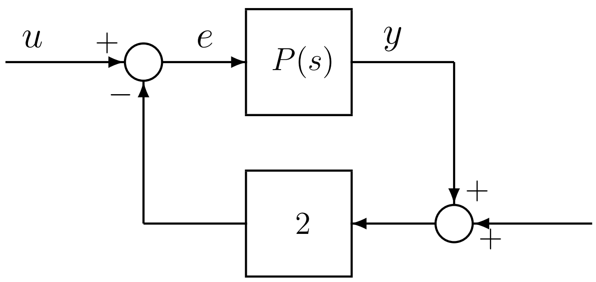

Figure 2: Feedback system.

i) Determine whether, for $P(s) = \frac{s}{1-s}$, the feedback loop is stable or not.
ii) Determine whether, for $P(s) = \frac{s}{(1-s)(1+\epsilon s)}$ and sufficiently small ϵ, the feedback loop is stable or not.
iii) Determine whether, for $P(s) = \frac{se^{-\tau s}}{1-s}$ and sufficiently small τ, the feedback loop is stable or not.
iv) What conlcusions do you reach regarding the theoretical answer to part i)?

Space for your work:

Solution:
i) The characteristic equation is $1 + \frac{2s}{1-s} = 0$ which has a single root at $s = -1$. Hence, in "theory" the system is $\boxed{\text{stable}}$.

ii) Since the loop gain has a zero at the origin and a pole on either side, root locus considerations tell us that there is a branch starting from the pole in the right half plane that stays in the right half plane for all values of K. Hence the system is $\boxed{\text{unstable}}$. Alternatively, compute the roots of $1 + \frac{2s}{(1-s)(1+\epsilon s)} = 0$. These are

$$\frac{\epsilon + 1 \pm \sqrt{\epsilon^2 + 6\epsilon + 1}}{2\epsilon}$$

and one of the two is always in the right half of the complex plane.

iii) The Nyquist diagram for $2s/(1-s)$ begins at the origin for $\omega \simeq 0$ and circles the point -1 once in the counterclock-wise sense. It reaches the negative part of the real axis as $\omega \to \infty$. The counterclock-wise encirclement is completed by the negative portion of the Nyquist diagram. It is now clear that, even with an infinitessimal time-delay present, the phase

$$\angle e^{-j\tau\omega} \frac{j\omega}{1 - j\omega}$$

fails to cross π in the counterclock-wise direction. Hence, the encirclement count cannot be -1 which is needed to ensure stability. Therefore the system is $\boxed{\text{unstable}}$.

iv) The conclusion in part i) is not relevant in practice where $s/(1-s)$ is only an ideal model. High frequency attenuation and/or infinitesimal time-delays render the conclusion in part i) invalid.

Part IV (1 point):
Consider the Nyquist plot computed for a transfer function $G(s)$ and shown in Figure 3. Determine whether $G(s)$ is stable or not, and explain your reasoning.

[Warning: the above question is <u>not</u> about the stability of the feedback system with $G(s)$ in the forward path and, perhaps, negative unity feedback. The question is about whether $G(s)$ itself is stable or not.]

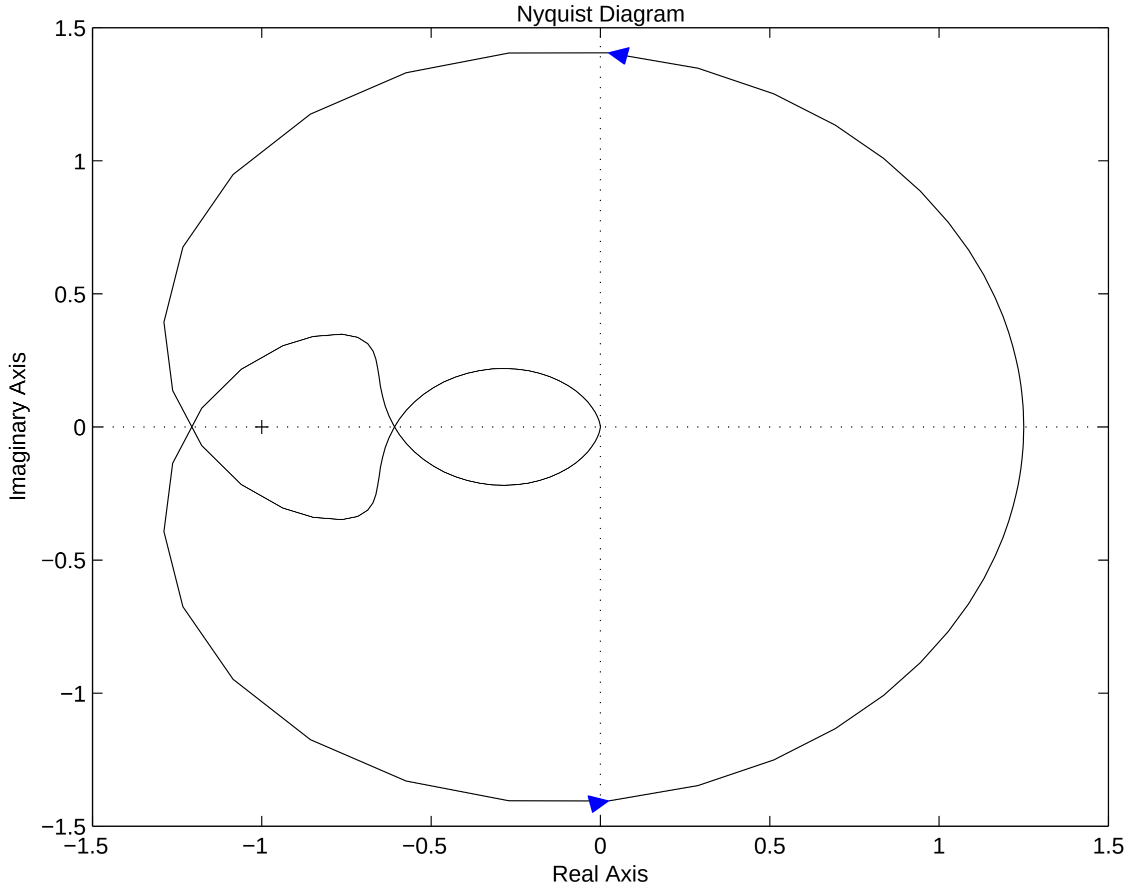

Figure 3: Nyquist plots

Space for your work:

Solution:
Consider the system $G(s)$ in a negative unity feedback loop anyway. Nyquist stability theory tells us that the encirclement count, N, which in this case is -2, must be equal to $Z - P$ where Z is the number or roots of the characteristic equation that lie in the right half of the complex plane (i.e., the unstable poles) and P is the number of unstable poles of the loop gain which in this case is $G(s)$. Clearly, $Z \geq 0$ and since $N = -2$ we conclude that $P \geq 2$ which means that $\boxed{G(s) \text{ has at least two unstable poles}}$.