1) **[15 points total; 5 points each part]**

a) **[5 points]** By the periodicity of the complex exponential, we have that $e^{j(2\pi/8)k\cdot 9} = e^{j(2\pi/8)k}$ for all integers k. Thus,

$$\begin{aligned}\left(\frac{1}{8}\sum_{k=0}^{7} X[k]e^{j(2\pi/8)kn}\right)\Bigg|_{n=9} &= \frac{1}{8}\sum_{k=0}^{7} X[k]e^{j(2\pi/8)k\cdot 9} \\ &= \frac{1}{8}\sum_{k=0}^{7} X[k]e^{j(2\pi/8)k} \\ &= \left(\frac{1}{8}\sum_{k=0}^{7} X[k]e^{j(2\pi/8)kn}\right)\Bigg|_{n=1} = x[1]\end{aligned}$$

b) **[5 points]** Using the synthesis equation, we have that $y[n] = \frac{1}{4}\sum_{k=0}^{3} Y[k]W_4^{-kn}$. Now,

$$\begin{aligned}\frac{1}{4}\sum_{k=0}^{3} Y[k]W_4^{-kn} &= \frac{1}{4}\sum_{k=0}^{3}(X[k]+X[k+4])W_4^{-kn} \\ &= \frac{1}{4}\sum_{k=0}^{3} X[k]W_4^{-kn} + \frac{1}{4}\sum_{k=0}^{3} X[k+4]W_4^{-kn} \\ &= \frac{1}{4}\sum_{k=0}^{3} X[k]W_4^{-kn} + \frac{1}{4}\sum_{k=4}^{7} X[k]W_4^{-(k+4)n} \quad \textit{(by change of index)} \\ &= \frac{1}{4}\sum_{k=0}^{3} X[k]W_4^{-kn} + \frac{1}{4}\sum_{k=4}^{7} X[k]W_4^{-kn}\underbrace{W_4^{-4n}}_{1} \\ &= \frac{1}{4}\sum_{k=0}^{7} X[k]W_4^{-kn} \\ &= \frac{1}{4}\sum_{k=0}^{7} X[k]e^{j(2\pi/4)kn} \\ &= \frac{1}{4}\sum_{k=0}^{7} X[k]e^{j(2\pi/8)k(2n)} \\ &= 2\cdot x[2n]\end{aligned}$$

c) **[5 points]** Writing $Z[k] = X[k] + (-1)^k X[k] = X[k] + W_8^{4k}X[k]$, and recalling the time shift property, we have directly that $z[n] = x[n] + x[((n-4))_8]$.

2) **[10 points total]**

Using the convolution property, we can easily identify the transfer function $H(e^{j\omega})$ of the system by the relation

$$G(e^{j\omega}) = H(e^{j\omega}) \cdot \left(\frac{1}{1 - (1/3)e^{-j\omega}} \right).$$

Solving for $H(e^{j\omega})$, we obtain that

$$H(e^{j\omega}) = G(e^{j\omega}) \cdot \left(1 - (1/3)e^{-j\omega}\right) = G(e^{j\omega}) - (1/3)e^{-j\omega} G(e^{j\omega}).$$

A simple application of the time-shift property for DTFT's gives that the impulse response $h[n]$ of the system may be expressed as

$$h[n] = g[n] - \left(\frac{1}{3}\right) g[n-1].$$

Thus, the response $y[n]$ of the system to an arbitrary input $x[n]$ is given by convolution:

$$y[n] = x[n] * h[n] = x[n] * g[n] - \left(\frac{1}{3}\right) x[n] * g[n-1].$$

3) **[15 points total]**

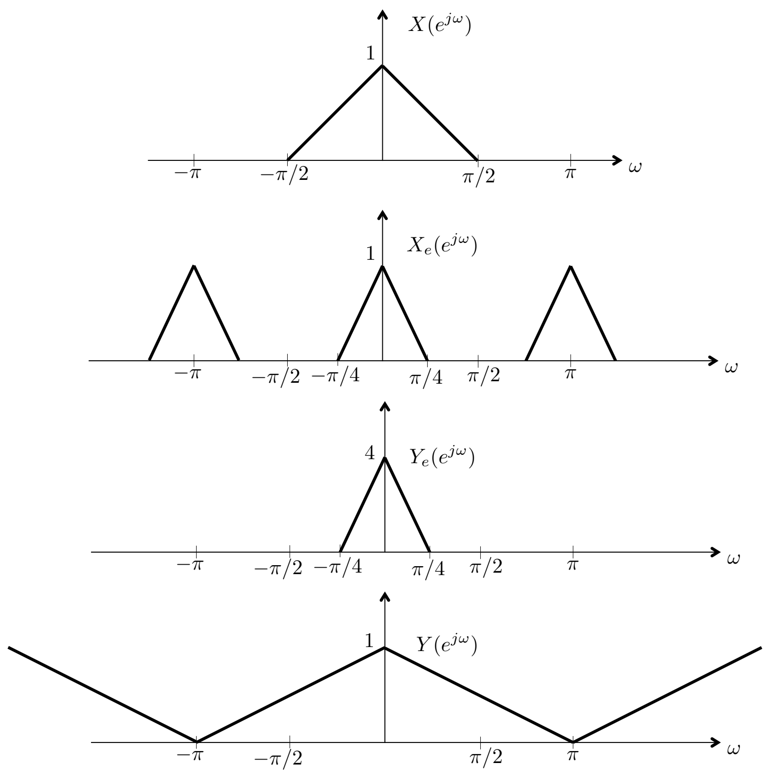
a) **[10 points]** Since

$$X_e(e^{j\omega}) = X(e^{j\omega L})$$

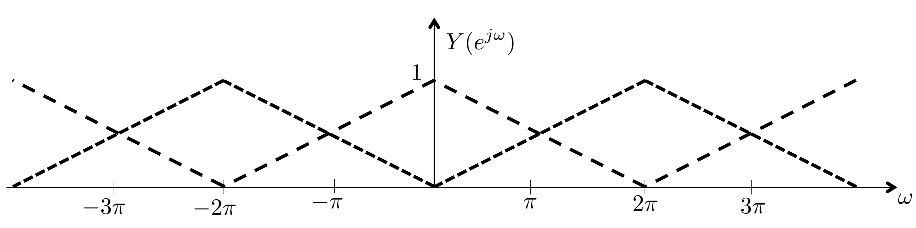
and $y[n]$ is related to $y_e[n]$ via

$$Y(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} Y_e(e^{j(\omega/M - 2\pi i/M)})$$

we have for $L = 2$ and $M = 4$ that



b) **[5 points]** For $L = 2$ and $M = 8$, we can extend the results above to see that the DTFT of $y[n]$, $Y(e^{j\omega})$ will be comprised of the sum of the spectra shown in dotted lines below:



It is easy to see that the resulting signal has DTFT that takes the *constant* value 1 at all frequencies. Thus, we conclude $y[n] = \delta[n]$ here.