This exam consists of **three problems**, worth a total of 40 points.

Problem 1 has three parts, and is worth 15 points.
Problem 2 has one part and is worth 10 points.
Problem 3 has two parts and is worth 15 points.

Below are a few preliminaries and definitions that you may find useful for the exam.

- For an N-point discrete-time sequence $x[n]$ with $x[n] = 0$ for $n < 0$ and $n > N-1$, we denote its N-point discrete Fourier Transform (DFT) by $X[k]$, where $X[k] = 0$ for $k < 0$ and $k > N-1$. The analysis and synthesis equations are:

$$\text{Analysis equation:} \quad X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad 0 \le k \le N-1,$$

$$\text{Synthesis equation:} \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \qquad 0 \le n \le N-1,$$

  where $W_N \triangleq e^{-j(2\pi/N)}$.

- Recall the DFT time shift property: if $x[n]$ is an N-point discrete sequence with N-point DFT $X[k]$, then for W_N as above, we have that the N-point sequence with DFT given by $W_N^{km} X[k]$ is given by $x[(n-m) \text{ modulo } N] \triangleq x[((n-m))_N]$.

- For a general discrete time sequence $x[n]$, we denote its discrete-time Fourier Transform (DTFT), when it exists, by $X(e^{j\omega})$. The analysis and synthesis equations are:

$$\text{Analysis equation:} \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\text{Synthesis equation:} \quad x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega.$$

- For the DTFT, Parseval's theorem states that

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega.$$

- You may wish to know that the discrete time sequence $x[n] = a^n u[n]$, where $|a| < 1$ and $u[n]$ is the discrete-time unit step function, has DTFT $X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$.

1) **[15 points total; 5 points each part]** Let $x[n]$ be a real eight-point sequence, with $x[n] = 0$ for all $n < 0$ and all $n > 7$. Let $X[k]$ denote the eight-point discrete Fourier Transform (DFT) of $x[n]$.

a) **[5 points]** Evaluate

$$\left(\frac{1}{8} \sum_{k=0}^{7} X[k] e^{j(2\pi/8)kn} \right) \Bigg|_{n=9}$$

in terms of $x[n]$.

b) **[5 points]** Let $y[n]$ be a four-point sequence, with $y[n] = 0$ for all $n < 0$ and all $n > 3$. Let $Y[k]$ be the four-point DFT of $y[n]$.

If $Y[k] = X[k] + X[k+4]$, express $y[n]$ in terms of $x[n]$.

c) **[5 points]** Let $z[n]$ be an eight-point sequence, with $z[n] = 0$ for all $n < 0$ and all $n > 7$. Let $Z[k]$ be the eight-point DFT of $Z[n]$.

If

$$Z[k] = \begin{cases} 2X[k], & k = 0, 2, 4, 6 \\ 0, & k = 1, 3, 5, 7, \end{cases} ,$$

express $z[n]$ in terms of $x[n]$.

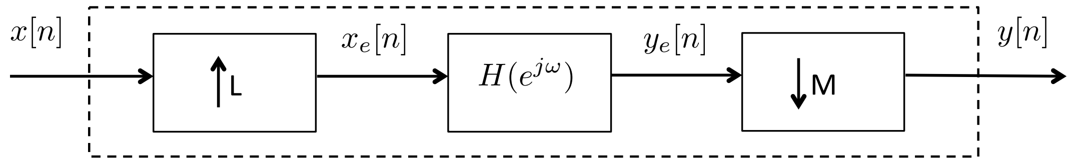
2) **[10 points total]** Let the operator $\mathcal{H}\{\cdot\}$ describe the action of a discrete-time LTI system, and suppose that we characterize the system in terms of its response to an input of the form $(1/3)^n u[n]$, where $u[n]$ denotes the discrete time unit step function. That is, we know that

$$\mathcal{H}\left\{\left(\frac{1}{3}\right)^n u[n]\right\} = g[n],$$

for some (known) $g[n]$.

Find a simple expression for the output $y[n]$ of the system to a general input $x[n]$. Express your answer in terms of the input $x[n]$ and the known $g[n]$.

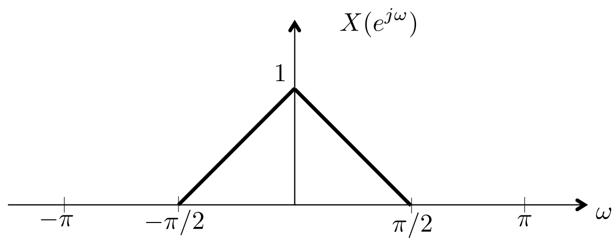
3) **[15 points total]** Consider the discrete-time system shown here



where

- L and M are positive integers
- $x_e[n] = \begin{cases} x[n/L], & n = kL, \quad (k \text{ is any integer}) \\ 0, & \text{otherwise} \end{cases}$
- $y[n] = y_e[nM]$, and
- $H(e^{j\omega}) = \begin{cases} M, & |\omega| \le \pi/4 \\ 0, & \pi/4 < |\omega| \le \pi \end{cases}$.

Suppose that $X(e^{j\omega})$, the DTFT of $x[n]$, is real and as is shown below



a) **[10 points]** For $L = 2$ and $M = 4$, draw **and label** $X_e(e^{j\omega})$, $Y_e(e^{j\omega})$, and $Y(e^{j\omega})$, the DTFT's of $x_e[n]$, $y_e[n]$, and $y[n]$, respectively.

b) **[5 points]** Determine a simple expression for $y[n]$ when $L = 2$ and $M = 8$.