SOLUTIONS of the Ph.D. Qualifying Exam on Communications – Fall 2010

Problem 1. The minimum average number of bits per symbol is given by the entropy of the source, which is

$$H(X) = \frac{1}{2}\log(2) + \frac{1}{4}\log(4) + \frac{1}{8}\log(8) + \frac{1}{8}\log(8) = \frac{7}{4} .$$

Problem 2. The transmitted waveform of the given antipodal PAM system at baseband is: $s(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT_s)$, where I_n is the i.i.d. stream of ± 1s; $g(t)$ denotes the given pulse; and T_s stands for the symbol period. It follows readily that I_n has zero mean and unit variance. With $G(f)$ denoting the Fourier transform of $g(t)$, the wanted power spectral density of this linearly modulated signal is

$$\Phi(f) = \frac{\sigma^2}{T_s}|G(f)|^2 = \frac{1}{T_s}|G(f)|^2 = \frac{4\sin^4(2\pi f \tau)}{\pi^2 T_s f^2}\Big|_{\tau=T_s/4} = \frac{4\sin^4(\pi T_s f/2)}{\pi^2 T_s f^2}$$

where it was used that the Fourier transform of a rectangle is a sinc function.

Problem 3. (a) It follows by inspection that: $f_1(t) = \sqrt{1/T_s}\, s_1(t)$, $f_2(t) = \sqrt{2/T_s}\, s_2(t)$, and $f_3(t) = \sqrt{2/T_s}\, s_4(t)$. Clearly, $s_1(t) = \sqrt{T_s}\, f_1(t)$, $s_2(t) = \sqrt{T_s/2}\, f_2(t)$, $s_3(t) = -\sqrt{T_s/2}\, f_2(t)$, $s_4(t) = \sqrt{T_s/2}\, f_3(t)$, $s_5(t) = -\sqrt{T_s/2}\, f_3(t)$,

(b) Since the basis contains three orthogonal functions, the space dimensionality is 3.

(c) The energy of $s_3(t)$ can be readily obtained from the norm of the corresponding signal vector $[0, -\sqrt{T_s/2}, 0]$, which equals $T_s/2$.

(d) The distance between $s_2(t)$ and $s_4(t)$ equals the one between the corresponding signal vectors, $[0, \sqrt{T_s/2}, 0]$ and $[0, 0, \sqrt{T_s/2}]$; that is, $\sqrt{(0-0)^2 + (\sqrt{T_s/2} - 0)^2 + (0 - \sqrt{T_s/2})^2} = \sqrt{T_s}$.

(e) Since $d_{\min} = \sqrt{2E_s}$ and $E_s = E_b \log_2 M$ for any M-ary orthogonal modulation, it follows readily that $d_{\min} \to \infty$ as $M \to \infty$. Regarding bandwidth W, recall that the signal space dimensionality is M and the uncertainty principle dictates that $2T_s W \geq M$, where T_s denotes symbol duration. Using that $T_s = T_b \log_2 M$, it follows that $W \geq M/(2T_b \log_2 M)$; and thus, for T_b fixed, one has that $W \to \infty$, as $M \to \infty$.

Problem 4. (a) Using the given independence assumptions, the wanted joint conditional pdf can be obtained as

$$\begin{aligned} f(r_1, r_2|s) &= f(s + n_1, n_1 + n_2|s) \\ &= J\, f_{n_1,n_2}(r_1 - s, r_2 - r_1 + s) \\ &= f_{n_1}(r_1 - s) f_{n_2}(r_2 - r_1 + s) \\ &= \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(r_1 - s)^2 + (r_2 - r_1 + s)^2}{2\sigma^2}\right) \end{aligned} \tag{1}$$

where for the third equality it was used that the Jacobian is given by $J = \left|\det\left(\frac{\partial(r_1,r_2)}{\partial(n_1,n_2)}\right)\right|^{-1} = 1$.

(b) With equal prior probabilities, the maximum a posteriori probability (MAP) optimal decision rule reduces to the maximum likelihod one, which is

$$\text{Decide} \begin{cases} m_0, & \text{if } f(r_1, r_2|s_0) > f(r_1, r_2|s_1) \\ m_1, & \text{if } f(r_1, r_2|s_0) < f(r_1, r_2|s_1) \end{cases}$$

and after using (1), it can be readily simplified to

$$\text{Decide} \begin{cases} m_0, & \text{if } 2r_1 - r_2 < 0 \\ m_1, & \text{if } 2r_1 - r_2 > 0 \end{cases}$$

(c) The wanted probability of error is

$$\begin{aligned} P_e &= P(2r_1 - r_2 < 0|s_1) = P(2r_1 - r_2 > 0|s_0) \\ &= P(-2\sqrt{E_b} + n_1 - n_2 > 0) = Q\left(\frac{2\sqrt{E_b}}{\sqrt{\sigma^2 + \sigma^2}}\right) = Q\left(\sqrt{\frac{2E_b}{\sigma^2}}\right). \end{aligned}$$