PhD Preliminary Written Exam   Problem 2   solutions
April 16, 2011   Controls

Solutions for Question #1:

i) We compute $\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\left(\frac{1}{s}P(s)\right) = 1.$ $\boxed{\text{Steady state value} = 1}$

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ii) Since $s^2+s+1 = s^2+2\zeta\omega s+\omega^2$, we read off the damping ratio $\zeta = \frac{1}{2}$ and the undamped natural frequency $\omega_n = 1$. Hence, the attenuation rate is

$$\boxed{\sigma = -\zeta\omega_n = -\tfrac{1}{2}}$$

and the frequency of damped oscillations is

$$\boxed{\omega_d = \omega\sqrt{1-\zeta^2} = \tfrac{\sqrt{3}}{2}}$$

or, the approximate period

$$\boxed{T_d = \tfrac{4\pi}{\sqrt{3}}}$$

Since the denominator of the Laplace transform of $Y(s)$ is $s(s^2+s+1)^2$, for $t > 0$, the response will be of the form

$$\boxed{A + e^{\sigma t}\left(B\sin(\omega_d t + \phi_1) + Dt\sin(\omega_d t + \phi_2)\right)}$$

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iii) The limit of the derivative of $y(t)$ at 0 (''initial value theorem'') is

$$\boxed{\lim_{t\to 0^+} \tfrac{dy(t)}{dt} = \lim_{s\to\infty} sP(s) = -1}$$

Hence, $y(t)$ turns negative before it turns positive towards the steady state limit of 1.

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iv) In this case

$$\boxed{\lim_{t\to 0^+} \tfrac{dy(t)}{dt} = \lim_{s\to\infty} sP(s) = b_{n-1}}$$

We may write the transfer function

$$P(s) = \frac{b_0(1 + s/z_1)(1 + s/z_2)\ldots(1 + s/z_{n-1})}{a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + s^n},$$

where all the coefficients in the denominator are positive (since the system is stable). We need to show that an odd number of right half plane zeros turn b_{n-1} negative. Complex roots come in pairs and the product of two such roots is $z\bar{z} = |z|^2 > 0$. Hence if there is an odd number of roots among $\{z_1,\ldots,z_{n-1}\}$ in the right half plane, there will be an odd number of negative ones. Then
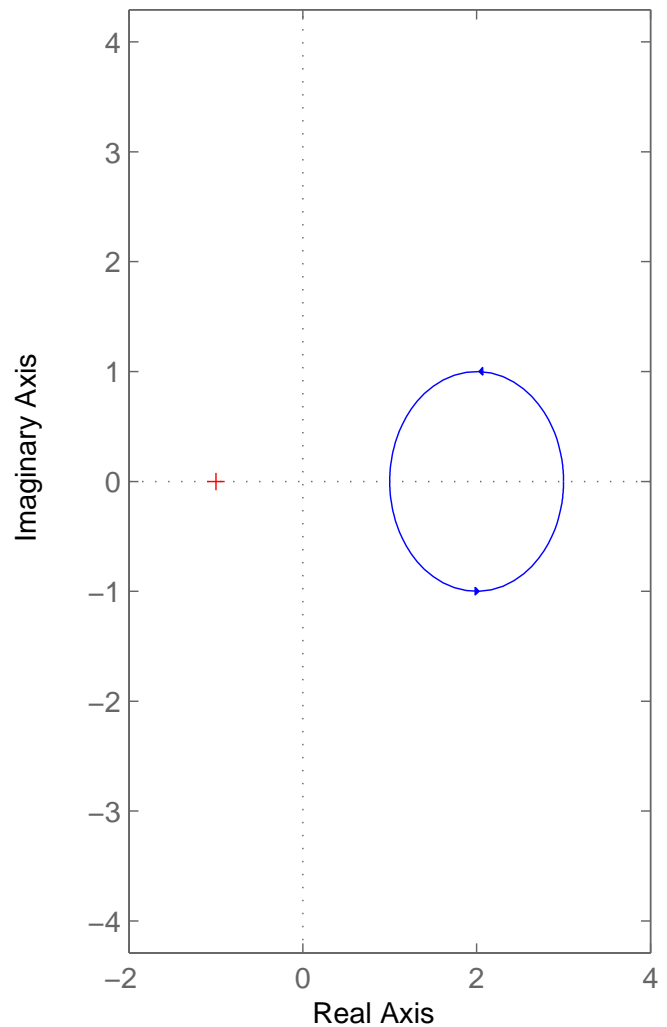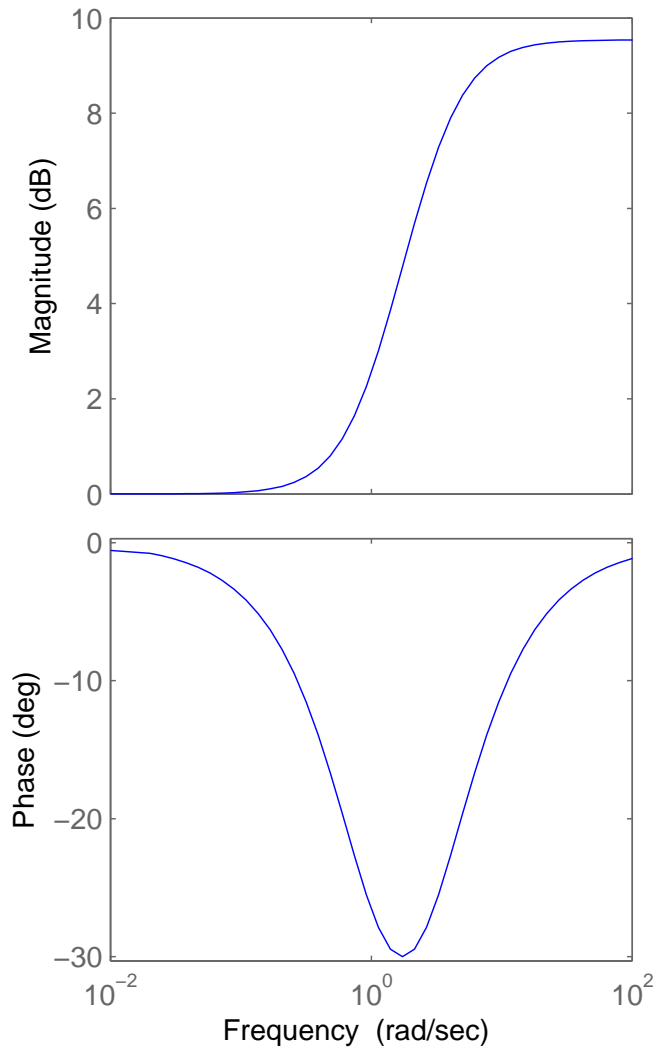
$$b_{n-1} = b_0 \prod_{i=1}^{n-1} \frac{1}{z_i}$$

must be negative. Therefore $y'(0^+) < 0$ and $y(t)$ turns negative before it settles to the steady-state value $\lim_{s\to 0} sP(s) = b_0/a_0 > 0$.

Solutions for Question #2:

i)

Bode Diagram

Nyquist Diagram

ii) Consider the characteristic equation $1 + kP(s) = 0$. For $k = -2/3$,
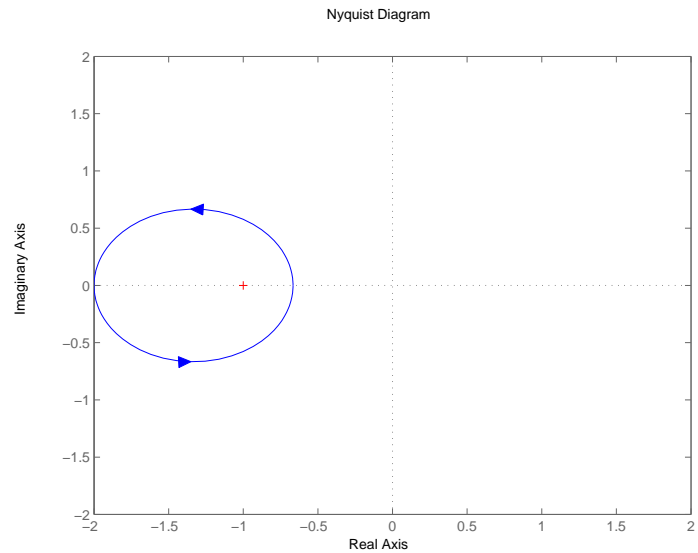
$$1 - \frac{2}{3} \times \frac{1-s}{1-s/3} = 0$$

has a single root at $s = -1$ which is in the left half of the complex plane.
Hence, $\boxed{k = -2/3}$ works.

Nyquist Diagram

iii) We see from the Nyquist plot that the encirclement count is $N = -1$. We know that

$$N = Z - P$$

where Z is the number of closed-loop poles in the right half plane and $P = 1$ is the number of open loop poles.

Therefore we have that $\boxed{Z = 0}$ and the feedback system has no RHP pole.

Nyquist Diagram

iv) We see from the Nyquist plot that now the plot no longer closes in around the point -1. This is due to the fact that while the loop gain $\frac{2}{3} \times \frac{1-s}{1-s/3}$ at $s = j\infty$ becomes equal to -2, when a delay is present, even a small one, the loop gain is now

$$e^{-\tau s}\frac{2}{3} \times \frac{1 - s}{1 - s/3}$$

and the phase due to the delay causes the plot to rotate clockwise. Hence N is no longer negative, and the system is unstable.
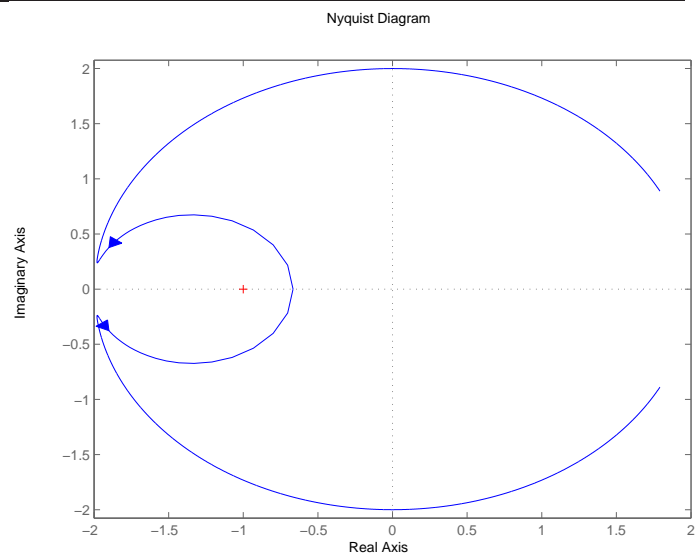
v) You may choose $C(s) = K\frac{1}{s/3-1}$ and the characteristic polynomial becomes

$$(s/3 - 1)^2 + K(s - 1) = \frac{s^2}{9} - \frac{2s}{3} + 1 + K(s - 1).$$

Now choose $K = \frac{2}{3} + \epsilon$. The characteristic polynomial is now

$$\frac{s^2}{9} + \epsilon s + 1/3 - \epsilon.$$

We need to take $0 < \epsilon < 1/3$ for stability, since then, all coefficients of the characteristic polynomial will be positive and its roots will be in the left half plane. It is insightful and instructive to consider the root locus which in this case is given on the right.

Root Locus