This exam consists of **two problems**, each having multiple parts. Problem 1 is worth 25 points, and problem 2 is worth 15 points, for a total of 40 points.

Below are a few preliminaries and definitions that you may find useful for the exam.

- For a continuous time aperiodic signal $x(t)$ we write its Fourier Transform as $X(j\Omega)$, so that

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega \qquad X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt,$$

  are a Fourier transform pair.

- For a continuous time aperiodic signal $x(t)$, Parseval's equation states that

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega.$$

- For a discrete time aperiodic signal $x[n]$, we write its Fourier Transform as $X(e^{j\omega})$, so that

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},$$

  are a Fourier transform pair.

1) **[25 points total]** For this problem, your goal is to implement a linear system that takes as its input a continuous-time signal $x(t)$ and outputs $y(t) = x(t - t_0)$, for a fixed, specified value $t_0 \in \mathbb{R}$. You choose to implement this using a system of the following form:

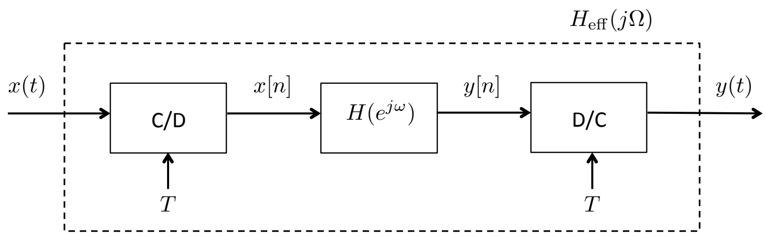


Figure 1: System for digital processing of analog signals.

where $T > 0$ is the sampling period, and the C/D and D/C converters are assumed ideal. For the following questions, assume that the input $x(t)$ is band limited, so that $X(j\Omega) = 0$ for $|\Omega| \geq \pi/T$.

a) **[10 points]** Find an expression for the Fourier Transform $H(e^{j\omega})$ of the intermediate discrete-time system, so that the overall system implements the specified time shift operation (i.e., so that $y(t) = x(t - t_0)$ for a fixed, specified $t_0$).

b) **[6 points]** Find a simple, closed-form expression for the impulse response $h[n]$ corresponding to the system you obtained in part (a).

c) **[3 points]** For what values of $t_0$ can the intermediate discrete-time system $h[n]$ be implemented using an **FIR** filter?

d) **[3 points]** For what values of $t_0$ can the intermediate discrete-time system $h[n]$ be implemented as a **causal** filter?

e) **[3 points]** Is the intermediate discrete-time system $h[n]$ **bounded input, bounded output (bibo) stable** for *all* shifts $t_0 \in \mathbb{R}$? Justify your answer.

2) **[15 points total]** Suppose that your implementation in the previous problem is imperfect, so that instead of $y(t) = x(t - t_0)$, your system gives the output $\widetilde{y}(t) = x(t - \widetilde{t}_0)$, where $\widetilde{t}_0 \neq t_0$. Suppose, further, that the input signal $x(t)$ is band limited, aperiodic, and has Fourier Transform $X(j\Omega)$ satisfying

$$|X(j\Omega)| = \begin{cases} 1, & |\Omega| < \Omega_c \\ 0, & \text{otherwise} \end{cases},$$

for some fixed $\Omega_c > 0$.

a) **[10 points]** Find a simple closed-form expression for the error

$$\int_{-\infty}^{\infty} |\widetilde{y}(t) - y(t)|^2 \, dt.$$

**Express your answer** as a function of the time shift error, $\delta \triangleq \widetilde{t}_0 - t_0$ (and the cutoff frequency $\Omega_c$).

b) **[3 points]** Find a simple expression for the limit

$$\lim_{\delta \to \infty} \mathcal{E}(\delta, \Omega_c),$$

where

$$\mathcal{E}(\delta, \Omega_c) \triangleq \int_{-\infty}^{\infty} |\widetilde{y}(t) - y(t)|^2 \, dt$$

is the time shift error you obtained as your answer to part (a).

c) **[2 points]** Fix $\Omega_c = \pi$, and sketch the error $\mathcal{E}(\delta, \Omega_c)$ as a function of $\delta$.