1) **[15 points]** Consider discrete-time LTI system depicted below.

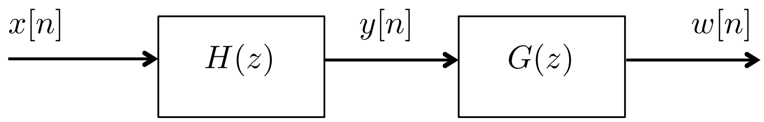

Figure 1: Discrete-time LTI system.

Suppose that $H(z) = (1 - 2z^{-1})(1 - \frac{1}{2}z^{-1})$ with ROC $|z| > 0$. Find the impulse response $g[n]$ of a *stable* system G such that $w[n] = x[n-2]$.

Hint: Recall the following z-Transform relations, which you may find useful:

$$\begin{aligned} x[n] = \alpha^n u[n] \longleftrightarrow X(z) &= \frac{1}{1 - az^{-1}}, \quad \textit{ROC: } |z| > |a| \\ x[n] = -\alpha^n u[-n-1] \longleftrightarrow X(z) &= \frac{1}{1 - az^{-1}}, \quad \textit{ROC: } |z| < |a|. \end{aligned}$$

SOLUTION:

The problem asks for $G(z)$ such that

$$H(z)G(z) = z^{-2},$$

or

$$G(z) = \frac{z^{-2}}{(1 - 2z^{-1})\left(1 - \frac{1}{2}z^{-1}\right)},$$

where the constraint that $G(z)$ be stable implies that the unit circle $|z| = 1$ must be in the ROC of $G(z)$.

To find the impulse response $g[n]$, we expand $G(z)$ using partial fraction expansion. Note that an equivalent expression for $G(z)$ is

$$G(z) = \frac{1}{(z-2)\left(z - \frac{1}{2}\right)},$$

and to factor $G(z)$, we seek A, B such that

$$G(z) = \frac{1}{(z-2)(z-\frac{1}{2})} = \frac{A}{z-2} + \frac{B}{z - \frac{1}{2}}.$$

The solutions are obtained by

$$A = (z-2)G(z)\Big|_{z=2}, \quad B = \left(z - \frac{1}{2}\right)G(z)\Big|_{z=\frac{1}{2}},$$

giving $A = 2/3$, and $B = -2/3$.

Thus,

$$\begin{aligned} G(z) = \frac{1}{(z-2)(z-\frac{1}{2})} &= \frac{(2/3)}{z-2} - \frac{(2/3)}{z-\frac{1}{2}} \\ &= \frac{(2/3)z^{-1}}{1-2z^{-1}} - \frac{(2/3)z^{-1}}{1-\frac{1}{2}z^{-1}}. \end{aligned}$$

Now, since the ROC of $G(z)$ must contain the unit circle ($|z| = 1$), we must choose the ROC of the first term to include the region $|z| < 2$, and the ROC for the second term to include $|z| > 1/2$. Using the expressions given in the hint, and the fact that the z^{-1} term simply induces a delay of one unit, we obtain

$$\begin{aligned} g[n] &= -(2/3)2^{n-1}u[-n] - (2/3)\left(\frac{1}{2}\right)^{n-1} u[n-1] \\ &= -(2/3)\left(2^{n-1}u[-n] + \left(\frac{1}{2}\right)^{n-1} u[n-1]\right) \end{aligned}$$

2) **[10 points]** Recall that the discrete-time convolution operation satisfies several standard arithmetic properties. Namely, for all functions $f[n], g[n]$, and $h[n]$, we have:

- Commutativity: $f[n] * g[n] = g[n] * f[n]$
- Associativity: $f[n] * (g[n] * h[n]) = (f[n] * g[n]) * h[n]$
- Distributivity: $f[n] * (g[n] + h[n]) = (f[n] + g[n]) + (f[n] + h[n])$

Note, however, that for arbitrary functions $g[n], h[n]$, the relationship

$$f[n]\,(g[n] * h[n]) = (f[n]g[n]) * (f[n]h[n]) \tag{1}$$

does *not* hold for all $f[n]$, though it does hold for the trivial cases $f[n] = 1$ and $f[n] = 0$.

Besides the trivial examples given above, find another function $f[n]$ for which (1) holds for all $g[n], h[n]$.

SOLUTION:

Recall the definition of the discrete-time convolution sum: for discrete time functions $g[n], h[n]$,

$$g[n] * h[n] = \sum_{k=-\infty}^{\infty} g[k]h[n-k].$$

Using this, we can express the relationship (1) as

$$f[n] \sum_{k=-\infty}^{\infty} g[k]h[n-k] = \sum_{k=-\infty}^{\infty} f[k]g[k]f[n-k]h[n-k],$$

or

$$\sum_{k=-\infty}^{\infty} f[n]g[k]h[n-k] = \sum_{k=-\infty}^{\infty} f[k]f[n-k]g[k]h[n-k],$$

from which we see that the relationship (1) is satisfied provided $f[n]$ satisfies

$$f[n] = f[k]f[n-k]$$

for all k. This condition is satisfied if $f[n]$ is any exponential function of the form

$$f[n] = \alpha^n,$$

where $\alpha \in \mathbb{C}$. Choosing $\alpha \neq \{0, 1\}$ gives a nontrivial function $f[n]$ for which (1) holds.

3) Consider the system shown below.

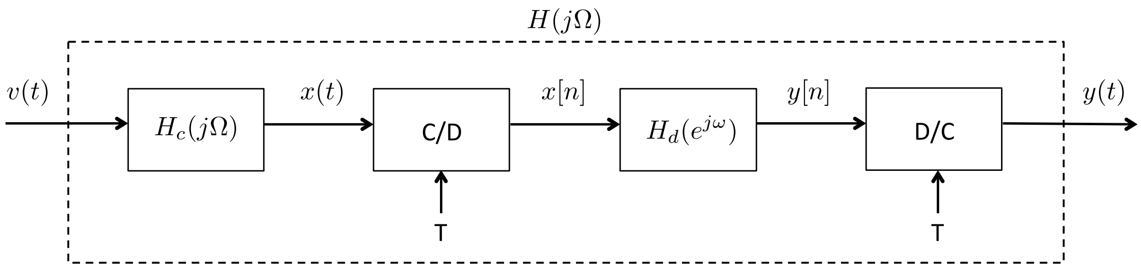

Figure 2: System for digital processing of analog signals.

The anti-aliasing filter $H_c(j\Omega)$ is an ideal low-pass continuous-time filter with unit gain and cutoff frequency π/T. The frequency response of the LTI discrete time system between the converters is given by:

$$H_d(e^{j\omega}) = j\omega e^{-j\omega/3}, \quad |\omega| < \pi.$$

Assume that the C/D and D/C converters are ideal.

a) **[5 points]** Find the impulse response $h_d[n]$ of the discrete-time LTI system.

SOLUTION:

Recall the inversion formula for discrete time Fourier transforms:

$$h_d[n] = \frac{1}{2\pi}\int_{2\pi} H_d(e^{j\omega})e^{j\omega n}d\omega,$$

where the integral is over any interval of length 2π. Choosing the interval of integration to be $[-\pi, \pi]$, we have

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} j\omega e^{-j\omega/3}e^{j\omega n}d\omega \\ &= \frac{j}{2\pi}\int_{-\pi}^{\pi} \omega e^{j\omega(n-1/3)}d\omega. \end{aligned}$$

Perform integration by parts: let $u = \omega$ and $dv = e^{j\omega(n-1/3)}d\omega$. Then $du = d\omega$ and

$$v = \frac{e^{j\omega(n-1/3)}}{j(n-1/3)}.$$

Now,

$$\begin{aligned} h_d[n] &= \frac{j}{2\pi}\int_{-\pi}^{\pi} \omega e^{j\omega(n-1/3)}d\omega \\ &= \frac{j}{2\pi}\left[\omega\frac{e^{j\omega(n-1/3)}}{j(n-1/3)}\right]_{-\pi}^{\pi} - \frac{j}{2\pi}\int_{-\pi}^{\pi}\frac{e^{j\omega(n-1/3)}}{j(n-1/3)}d\omega. \end{aligned}$$

For the first term, note that

$$\begin{aligned} \frac{j}{2\pi}\left[\omega\frac{e^{j\omega(n-1/3)}}{j(n-1/3)}\right]_{-\pi}^{\pi} &= \frac{j}{2\pi}\left[\pi\frac{e^{j\pi(n-1/3)}}{j(n-1/3)} + \pi\frac{e^{j\pi(n-1/3)}}{j(n-1/3)}\right] \\ &= \frac{\cos\left[\pi(n-1/3)\right]}{(n-1/3)} \end{aligned}$$

For the second, we have

$$
\begin{aligned}
-\frac{j}{2\pi}\int_{-\pi}^{\pi}\frac{e^{j\omega(n-1/3)}}{j(n-1/3)}d\omega &= -\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{j\omega(n-1/3)}}{(n-1/3)}d\omega \\
&= -\frac{1}{2\pi}\left[\frac{e^{j\omega(n-1/3)}}{j(n-1/3)^2}\right]_{-\pi}^{\pi} \\
&= -\frac{1}{\pi(n-1/3)^2}\left[\frac{e^{j\pi(n-1/3)}-e^{-j\pi(n-1/3)}}{2j}\right] \\
&= -\frac{\sin\left[\pi(n-1/3)\right]}{\pi(n-1/3)^2}.
\end{aligned}
$$

Combining the results, we have

$$
h_d[n] = \frac{\cos\left[\pi(n-1/3)\right]}{(n-1/3)} - \frac{\sin\left[\pi(n-1/3)\right]}{\pi(n-1/3)^2}.
$$

b) **[5 points]** What is the effective continuous-time frequency response of the overall system, $H(j\Omega)$?

SOLUTION:

Note that because of the filter $H_c(j\Omega)$, $x(t)$ is bandlimited ($X(j\Omega) = 0$ for $|\Omega| > \pi/T$). Thus, the continuous-time input and continuous-time output of the subsystem shown in Figure 3 are related by

$$Y(j\Omega) = X(j\Omega) \cdot H_d(e^{j\omega})\Big|_{\omega=\Omega T}.$$

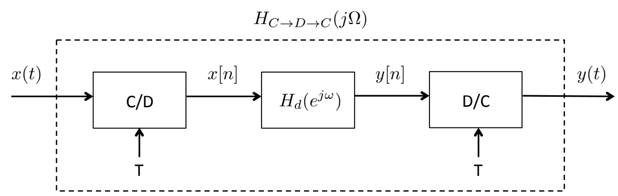

Figure 3: Subsystem for digital processing of analog signals.

This implies $H_{C\to D\to C}(j\Omega) = H_d(e^{j\Omega T}) = j\Omega T e^{-j\Omega T/3}$. Overall, the frequency response of the entire system is given by

$$H(j\Omega) = \begin{cases} j\Omega T e^{-j\Omega T/3}, & |\Omega| < \pi/T \\ 0, & \text{otherwise} \end{cases}$$

Given this overall frequency response, we see that for *bandlimited inputs* the net effect of this system would be amplification by the factor T, differentiation, and time delay by $T/3$.

c) **[5 points]** Choose the most accurate statement:

(i) $y(t) = x(t - 1/3)$

(ii) $y(t) = \frac{d}{dt}x(3t)$

(iii) $y(t) = T\frac{d}{dt}x(t - 1/3)$

(iv) $y(t) = T\frac{d}{dt}x(t - T/3)$ $\Leftarrow$ **by the above argument**

(v) $y(t) = \frac{d}{dt}x(t - 1/3)$

(vi) $y(t) = \frac{d}{dt}x(t - T/3)$