Solution

- Problem 1.

(a) Nyquist rate : $f_s \geq 2B$, where B is the bandwidth of $s(t)$

$$S(f) = \text{Fourier}(s(t)) = \frac{1}{i}\left[\delta(f-\tfrac{1}{2}) - \delta(f+\tfrac{1}{2})\right]$$

$$\Rightarrow B = \tfrac{1}{2}$$

$$\Rightarrow f_s \geq 2 \times \tfrac{1}{2} = 1 \ (Hz)$$

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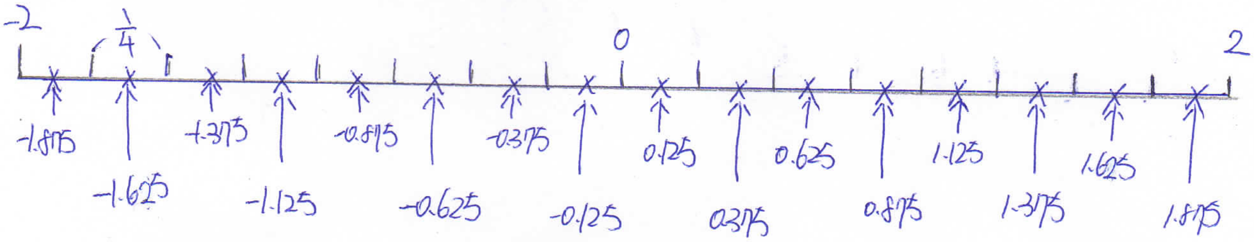
(b)

Use 4 bit PCM $\Rightarrow$ 16 quantization levels

$$\text{quantization step size} = \frac{\max_t s(t) - \min_t s(t)}{16} = \frac{2-(-2)}{16} = \frac{1}{4}$$

The 16 quantization levels are show below (x)



The quantization process is simply round the input to nearest level.

Suppose we sample at $T_s = 0.9$ (s), there are 6 samples between 0 and 5 sec.

$$s(0 \times T_s) = s(0) = 0 \xrightarrow{\text{Quantization}} -0.125$$

$$s(1 \times T_s) = s(0.9) = 0.618 \longrightarrow 0.625$$

$$s(2 \times T_s) = s(1.8) = -1.1756 \longrightarrow -1.125$$

$$s(3 \times T_s) = s(2.7) = 1.618 \longrightarrow 1.625$$

$$s(4 \times T_s) = s(3.6) = -1.9021 \longrightarrow -1.875$$

$$s(5 \times T_s) = s(4.5) = 2 \longrightarrow 1.875$$

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(c)

The mean square quantization error between $0 \sim 5$ sec

$$= \frac{1}{5}\Big[(0-(-0.125))^2 + (0.618-0.625)^2 + (-1.1756-(-1.125))^2 + (1.618-1.625)^2 + (-1.9021-(-1.875))^2 + (2-1.875)^2 \Big]$$

$$= \frac{1}{5}[0.0346]$$

$$= 0.0069$$

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Problem 2:

(a) Let $b_n = \begin{cases} -1, & \text{if } n=0 \text{ (symble 0 is tx)} \\ 1, & \text{if } n=1, \text{ (symble 1 is tx)} \end{cases}$

let $r(t) = b_n s(t) + n(t)$ be the received signal.

let V be the output of the integrator at $t = T_b$

$\Rightarrow V = \int_0^{T_b} r(t)\, s(t)\, dt$

$= \int_0^{T_b} [b_n + n(t)]\, s(t)\, dt$

$= b_n E_s + \int_0^{T_b} n(t)\, s(t)\, dt$

$= b_n E_s + n$, where $n \sim \mathcal{N}(0, \frac{\eta}{2} E_s)$

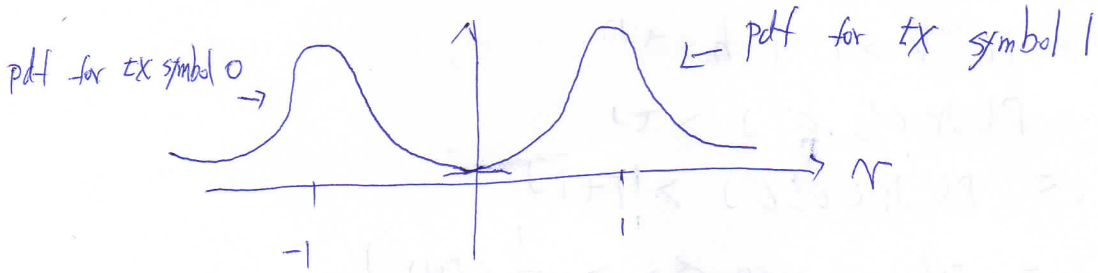
$\therefore$ Noise power $= \sigma^2 = \frac{\eta}{2} \times E_s = \frac{2\times 10^{-3}}{2} \times 1 = 10^{-3}$ #

(b) When tx symbol 0

$V = -1 \times E_s + n \sim \mathcal{N}(-1, 10^{-3}) = \mathcal{N}(-1, \sigma^2)$

when tx symbol 1

$V = 1 \times E_s + n \sim \mathcal{N}(1, 10^{-3}) = \mathcal{N}(1, \sigma^2)$



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(c)

Optimal Rule = MAP:

$$\log[p(v|b_n=-1)] + \log[p(b_n=-1)] \underset{1}{\overset{0}{\gtrless}} \log[p(v|b_n=1)] + \log[p(b_n=1)]$$

Define $\sigma^2 = 10^{-3}$

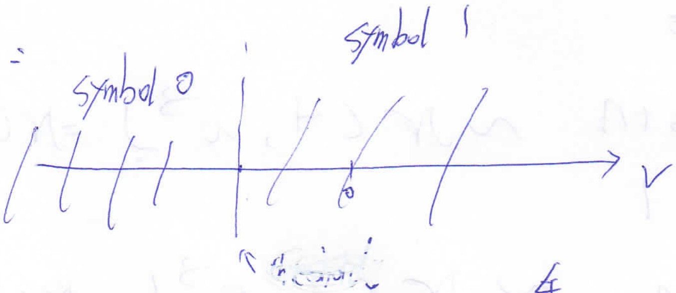
$$\Rightarrow -\frac{1}{2\sigma^2}|v-(-1)|^2 + \log\left(\frac{1}{5}\right) \underset{1}{\overset{0}{\gtrless}} -\frac{1}{2\sigma^2}|v-1|^2 + \log\left(\frac{4}{5}\right)$$

$$\Rightarrow \frac{1}{2}|v-1|^2 - \frac{1}{2}|v+1|^2 \underset{1}{\overset{0}{\gtrless}} \sigma^2\log\left(\frac{4}{5}\right) - \sigma^2\log\left(\frac{1}{5}\right)$$

$$\Rightarrow -v \underset{1}{\overset{0}{\gtrless}} \sigma^2 \log(2)$$

$$\Rightarrow v \underset{0}{\overset{1}{\gtrless}} -\sigma^2\log(2) = -10^{-3}\log(2) = \underline{-6.931\times10^{-4}} \ \#$$

Decision Rule:



$T = -6.931\times10^{-4}$

(d)

$$P_{err|b_n=-1} = P(v > T \mid b_n = -1)$$
$$= P(\mathcal{N}(-1,\sigma^2) > T)$$
$$= P(\mathcal{N}(0,\sigma^2) > T+1)$$
$$= P\left(\frac{1}{\sigma}\mathcal{N}(0,\sigma^2) > \frac{1}{\sigma}(T+1)\right)$$
$$= P\left(\mathcal{N}(0,1) > \frac{1}{\sigma}(T+1)\right)$$
$$= \Phi_c\left(\frac{1}{\sigma}(T+1)\right) = \Phi_c\left(\frac{1}{\sqrt{10^{-3}}}(-6.931\times10^{-4}+1)\right)$$
$$= \Phi_c(31.6009)$$

, where $\Phi_c(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt = P(\mathcal{N}(0,1) \geq x)$

c()

$$P_{err|b_n=1} = P(V < T \mid b_n = 1)$$
$$= P(\mathcal{N}(1, \sigma^2) < T)$$
$$= P(\mathcal{N}(0, \sigma^2) < T-1)$$
$$= P(\tfrac{1}{\sigma}\mathcal{N}(0, \sigma^2) < \tfrac{1}{\sigma}(T-1))$$
$$= P(\mathcal{N}(0,1) < \tfrac{1}{\sigma}(T-1))$$
$$= P(\mathcal{N}(0,1) > \tfrac{-1}{\sigma}(T-1))$$
$$= \Phi_c(\tfrac{1}{\sigma}(1-T))$$
$$= \Phi_c(31.6447)$$

$$\therefore \quad P_{err} = P(b_n = -1) P_{err|b_n=-1} + P(b_n = 1) P_{err|b_n=1}$$
$$= \frac{1}{5}\Phi_c(31.6009) + \frac{4}{5}\Phi_c(31.6447)$$

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