There are two Problems and they are assigned 20 points each (for a total of 40/40)

Part #1 (20 points total):
Consider an inertial system modeled as a lumped scalar linear system

$$\frac{d^2x(t)}{dt^2} = u(t).$$

Here $u(t)$ represents a force applied to it by a controller, while $x(t)$ represents its position at time $t \in [0, \infty)$. The controller input is taken as the difference $v(t) = r(t) - x(t)$, between a reference signal $r(t)$ and the position $x(t)$. The controller output is the applied force $u(t)$.

Address the following three independent sub-parts to Part 1:
1. **(5 point)** Consider the control law $u(t) = Kv(t)$ and determine whether the feedback system is asymptotically stable for any choice of the gain K. (Recall that a system is asymptotically stable if all of its poles have negative real part.)

2. **(5 point)** Consider a dynamic control law where

$$u(t) = v(t) - \int_0^t v(\tau)e^{-2(t-\tau)}d\tau.$$

Determine whether the feedback system is asymptotically stable for this choice of control law. (Prove that it is, or prove that it is not.)

3. **(10 points)** Design a control law which stabilizes the feedback system and has the additional property that, at steady state, the position follows exactly a sinusoidal reference signal $r(t) = \sin(t)$.

Solution:

1. The transfer function from $u(t)$ to $x(t)$ is $P(s) = \frac{1}{s^2}$. Hence the characteristic equation for the closed loop system with unity negative feedback and a gain K is:

$$1 + \frac{K}{s^2} = 0.$$

This equation has roots on the imaginary axis for all values of $K \geq 0$, and has at least one root in the right half plane for $K < 0$. Hence, $\boxed{\text{the system is not asymptotically stable for any choice of } K.}$

2. The control law can also be written as:

$$u(t) = v(t) - v(t) \star e^{-2t},$$

where $\star$ denotes convolution. Evidently, its transfer function is

$$C(s) = 1 - \frac{1}{s+2}.$$

Hence, the characteristic equation of the closed loop system is

$$\begin{aligned} 0 &= 1 + (1 - \frac{1}{s+2})\frac{1}{s^2} \\ &= 1 + \frac{s+1}{s+2}\frac{1}{s^2} \\ &= \frac{s^2(s+2) + (s+1)}{(s+2)s^2} \end{aligned}$$

Thus, the characteristic polynomial is

$$s^2(s+2) + (s+1) = s^3 + 2s^2 + s + 1.$$

The characteristic polynomial has all its roots in the left half plane as it can be readily verified using Routh's test:

s^3	1	1
s^2	2	1
s	0.5	
1	1	

with the first column having all positive entries.

This proves that the feedback system is asymptotically stable.

3. The controller transfer function must have a pole at $s = 1$, so that the closed loop transfer function from $r(t)$ to $v(t)$, namely,

$$S(s) = \frac{1}{1 + P(s)C(s)}$$

realizes a transmission zero at the frequency of the sinusoid. Thus,

$$C(s) = \frac{1}{s^2+1} \times \text{ possibly more dynamics for stabilization.}$$

We notice that if we used a second order controller

$$C(s) = \frac{as^2 + bs + c}{s^2 + 1},$$

the characteristic polynomial for the feedback system would be

$$s^2(s^2+1) + as^2 + \cdots = s^4 + (1+a)s^2 + \ldots$$

with a zero coefficient for the term s^3. Therefore, the system would be unstable with such a controller. This suggests that we need a higher order controller. In this case there is a lot of flexibility on where for instance to place closed loop poles. Below is one particular solution.

We can choose a controller

$$C(s) = \frac{as^3 + bs^2 + cs + d}{(s^2+1)(s+f)}$$

in which case the characteristic polynomial is

$$s^5 + fs^4 + (1+a)s^3 + (f+b)s^2 + cs + d.$$

Clearly, we have enough authority to select the poles so that the characteristic polynomial is in fact equal to

$$(s+1)^5 = s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + 1.$$

This choice requires that we take $f = 5$, $a = 9$, $b = 5$, $c = 5$, $d = 1$.

Part #2 (20 points total):
Consider a delay system, described by the delay-differential equation

$$\frac{d}{dt}x(t) = u(t) - Kx(t-1) - Kx(t-2),$$

where $x(t)$ represents the state and $u(t)$ the input. Determine the maximal value of K for which the system is stable. (Hint: You may use Nyquist's stability criterion.)

`Solution:`
The characteristic equation is

$$s + Ke^{-s} + Ke^{-2s} = 0.$$

Equivalently we may consider

$$1 + K\frac{e^{-s}}{s} + K\frac{e^{-2s}}{s} = 0$$

and we may consider the Nyquist diagram for loop gain with transfer function

$$K\frac{e^{-j\omega}}{j\omega} + K\frac{e^{-2j\omega}}{j\omega}.$$

Since for both terms in the transfer function of the loop gain the phase decreases with increasing ω, we only need to find the phase crossover frequency (i.e., the frequency where the phase first becomes $-\pi$ radians). Equivalently, we need to determine the frequency where the phase of

$$e^{-j\omega} + e^{-2j\omega}$$

is $-\pi/2$ radians. At that frequency, the real part of $e^{-j\omega} + e^{-2j\omega}$ must be zero, therefore

$$\cos(\omega) + \cos(2\omega) = 0.$$

The smallest value for which this is true is $\omega = \frac{\pi}{3}$. At this frequency, the absolute value of the loop gain is

$$K\frac{(\sin(\pi/3) + \sin(2\pi/3))}{\pi/3} = K\frac{\sqrt{3}}{\pi/3} = K\frac{3^{3/2}}{\pi}$$

and for stability of the system, the absolute value of the gain at the phase crossover frequency needs to be less than one. Hence, the maximal value for K is $\frac{\pi}{3^{3/2}}$.