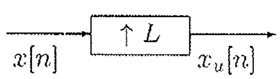
PhD Written Preliminary Exam: Problem 3 (Signal Processing)
November 13, 2010

Instructions:

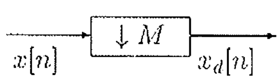
1. Answer all problems.
2. There are 2 problems with a total of 40 points (15 points for Problem 1 and 25 points for Problem 2).
3. Upsampling and downsampling operations are defined mathematically in the sequence domain (see below). You do not have to derive the relationships in the frequency domain. For example, if you know the relationship between $X_u\left(e^{j\omega}\right)$ and $X\left(e^{j\omega}\right)$, then simply use it if it is pertinent to your answer.
4. Show work for partial credit.

Upsampling/Downsampling:



(Upsampling)

$$x_u[n] = \begin{cases} x[n/L] & \text{for } n = 0, \pm L, \pm 2L, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

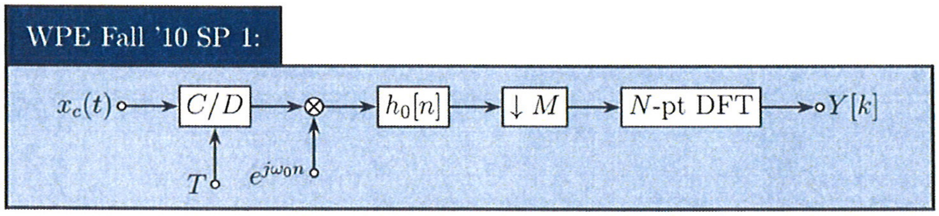


(Downsampling)

$$\begin{aligned} x_d[n] &= x[Mn] \\ &= p[Mn]x[Mn]. \end{aligned}$$

Where $p[n]$ is a periodic unit pulse with period M.

1. Consider the following system:



The input signal is given by

$$x_c(t) = \sum_{k=0}^{K-1} \left(1 - \frac{k}{K}\right) \cos\left(\frac{2\pi k}{T_0} t\right),$$

and the ideally sampled signal is

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

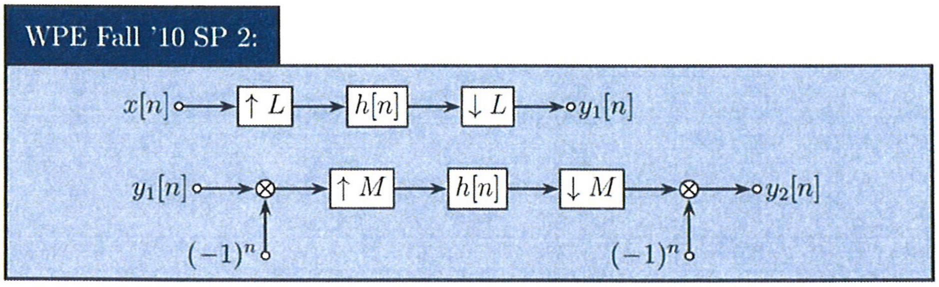
where $K = 5$ and $T_0 = 10^{-3}$. The impulse response, $h_0[n]$, is an ideal lowpass filter with cutoff at ω_c. In the above implementation, the sampling frequency was set at 20 kHz with $\omega_0 = \omega_c = \pi/4$, $M = 4$, and (for convenience) $N = 1000$.

(a) Sketch $X_c(j\Omega)$, $X_s(j\Omega)$, and $X(e^{j\omega})$ (the DTFT of $x[n] = x_c(nT)$). Carefully label your axes and the amplitudes of all frequency components. [6 points]

(b) Sketch $Y[k]$ for $k = 0, 1, \ldots, N-1$. Indicate the analog frequency range covered by the values of the DFT index values. [6 points]

(c) How are the values $Y[k]$ related to the input signal spectrum, $X_c(j\Omega)$? Comment on your result. [3 points]

2. Consider the following system



The digital filter prototype, with impulse response $h[n]$, was designed to meet the following specifications

$$0.99 \leq \left|H\left(e^{j\omega}\right)\right| \leq 1.01, \qquad 0 \leq |\omega| \leq 0.25\pi$$
$$\left|H\left(e^{j\omega}\right)\right| \leq 0.005, \qquad 0.3\pi \leq |\omega| \leq \pi.$$

The filter is FIR with linear phase characteristics, $\theta(\omega) = -n_d\omega + \beta$, where $n_d = 42.5$ and β is either 0 or π. Assume $L = 2$ and $M = 3$ and answer the following questions:

(a) What is the order of $H(z)$? Briefly justify your answer. [2 points]

(b) For $x[n] = A\cos(0.1\pi n)$, is it possible to write $y_1[n] = A_1\cos(0.1\pi n + \theta_2)$ and $y_2[n] = A_2\cos(0.1\pi n + \theta_2)$? If so, specify the range of values of $A_1, A_2, \theta_1, \& \theta_2$. If not, explain why not. [5 points]

(c) For $x[n] = A\cos(0.2\pi n)$, is it possible to write $y_1[n] = A_1\cos(0.2\pi n + \theta_2)$ and $y_2[n] = A_2\cos(0.2\pi n + \theta_2)$? If so, specify the range of values of $A_1, A_2, \theta_1, \& \theta_2$. If not, explain why not. [5 points]

(d) For $x[n] = A\cos(0.4\pi n)$, is it possible to write $y_1[n] = A_1\cos(0.4\pi n + \theta_2)$ and $y_2[n] = A_2\cos(0.4\pi n + \theta_2)$? If so, specify the range of values of $A_1, A_2, \theta_1, \& \theta_2$. If not, explain why not. [5 points]

(e) Based on your answers above, describe the filtering operation performed by the cascade. Specifically, is the overall system LPF, BPF, BSF, HPF, or all-pass filter? If such characterization is possible, give the filter tolerances for the frequency range $[-\pi, \pi)$. If not, succinctly explain why not. [8 points]
[*Hint: For Parts (b) - (d), your answers should be based on the tolerances of the filter prototype and the cascade architecture of the system.*]