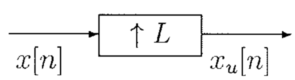
Solutions

PhD Written Preliminary Exam: Problem 3 (Signal Processing)
November 13, 2010

Instructions:

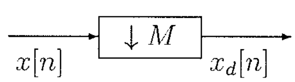
1. Answer all problems.
2. There are 2 problems with a total of 40 points (15 points for Problem 1 and 25 points for Problem 2).
3. Upsampling and downsampling operations are defined mathematically in the sequence domain (see below). You do not have to derive the relationships in the frequency domain. For example, if you know the relationship between $X_u(e^{j\omega})$ and $X(e^{j\omega})$, then simply use it if it is pertinent to your answer.
4. Show work for partial credit.

Upsampling/Downsampling:



(Upsampling)

$$x_u[n] = \begin{cases} x[n/L] & \text{for } n = 0, \pm L, \pm 2L, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$



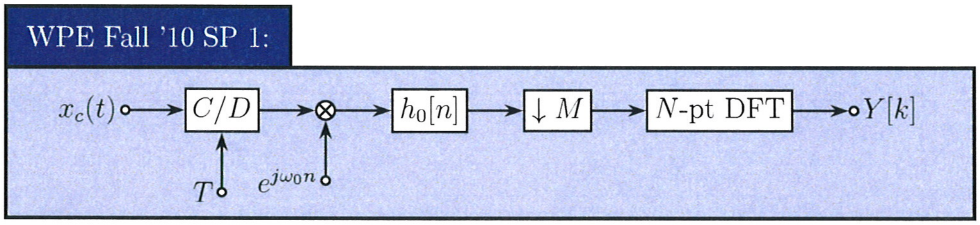
(Downsampling)

$$\begin{aligned} x_d[n] &= x[Mn] \\ &= p[Mn]x[Mn]. \end{aligned}$$

Where $p[n]$ is a periodic unit pulse with period M.

Solutions

1. Consider the following system:



The input signal is given by

$$x_c(t) = \sum_{k=0}^{K-1} \left(1 - \frac{k}{K}\right) \cos\left(\frac{2\pi k}{T_0} t\right),$$

and the ideally sampled signal is

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

where $K = 5$ and $T_0 = 10^{-3}$. The impulse response, $h_0[n]$, is an ideal lowpass filter with cutoff at ω_c. In the above implementation, the sampling frequency was set at 20 kHz with $\omega_0 = \omega_c = \pi/4$, $M = 4$, and (for convenience) $N = 1000$.

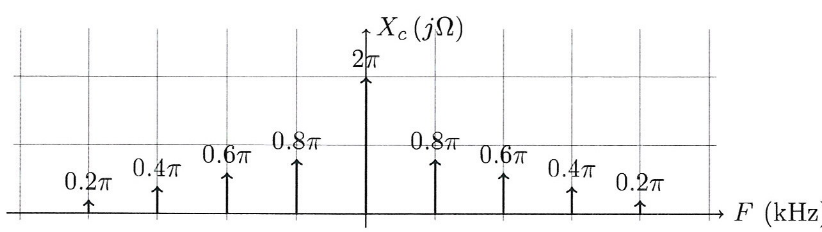
(a) Sketch $X_c(j\Omega)$, $X_s(j\Omega)$, and $X(e^{j\omega})$ (the DTFT of $x[n] = x_c(nT)$). Carefully label your axes and the amplitudes of all frequency components. [6 points]

Solution. The Fourier transform of $x_c(t)$ can be obtained analytically

$$X_c(j\Omega) = 2\pi\delta(\Omega) + \pi \sum_{k=1}^{4} \left(1 - \frac{k}{5}\right) \left[\delta(\Omega + \frac{2\pi k}{T_0}) + \delta(\Omega - \frac{2\pi k}{T_0})\right]$$

and $X_s(j\Omega)$ is given by

$$X_s(j\Omega) = \frac{\pi}{T} \sum_{l=-\infty}^{\infty} \left[2\delta\left(\Omega - \frac{2\pi l}{T}\right) + \sum_{k=1}^{4}\left(1 - \frac{k}{5}\right)\left[\delta\left(\Omega + \frac{2\pi k}{T_0}\right) + \delta\left(\Omega - \frac{2\pi k}{T_0} - \frac{2\pi l}{T}\right)\right]\right]$$



QUESTION 1(a) CONTINUES OVER THE PAGE

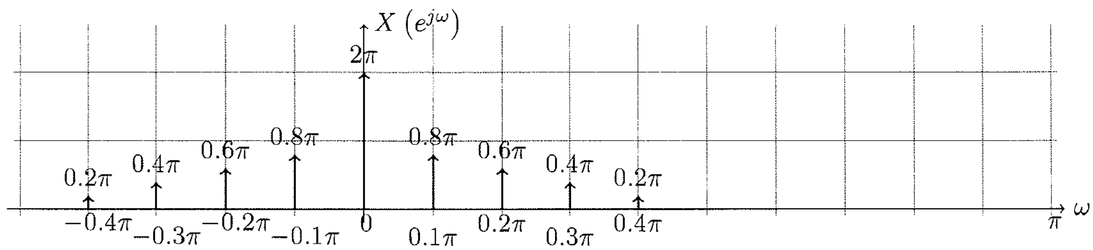
Solutions

1(a) (Continued)

$X_s(j\Omega)$ is a scaled version of $X_c(j\Omega)$ (scaling by $\frac{1}{T} = 20000$) and periodic with a period of 20 kHz.

The DTFT of the sampled sequence

$$X\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$



Remarks:

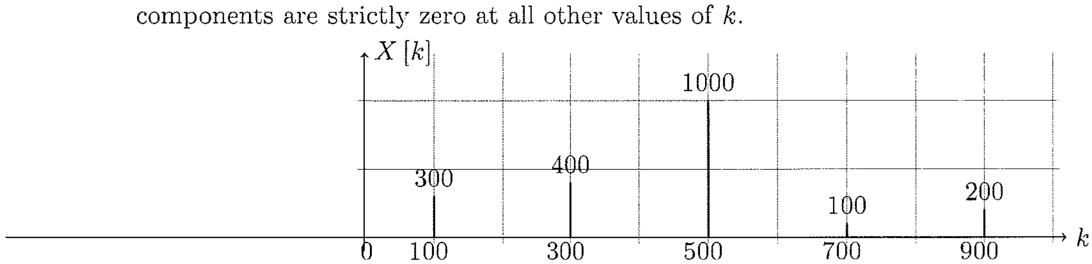
- $X\left(e^{j\omega}\right)$ is periodic with a period of 2π.
- We have used the scaling property of the δ function ($\delta(\alpha t) = \delta(t)/|\alpha|$), which explains the amplitude scaling of $X\left(e^{j\omega}\right)$.

(b) Sketch $Y[k]$ for $k = 0, 1, \ldots, N-1$. Indicate the analog frequency range covered by the values of the DFT index values. [6 points]

Solution. After the modulator,

$$X_m\left(e^{j\omega}\right) = X\left(e^{j\left(\omega - \frac{\pi}{4}\right)}\right) \qquad \text{periodic shift right}$$

The five frequency components originally at $\omega = -0.4\pi, -0.3\pi, -0.2\pi, -0.1\pi$, and 0 will pass through the filter appearing at $\omega = -0.15\pi, -0.05\pi, 0.05\pi, 0.15\pi$, and 0.25π, respectively. After the compressor, these frequency components will appear at $\omega = -0.6\pi, -0.2\pi, 0.2\pi, 0.6\pi$, and π, respectively. For a 1000-pt DFT, the DFT index k corresponds to a normalized frequency component $\omega_k = \frac{\pi k}{500}$. Therefore, the five frequency components will appear at $k = 1000 - 300, 1000 - 100, 100, 300$, and 500, respectively. Since the input frequency components coincide with whole values of k, spectral leakage components are strictly zero at all other values of k.



1 (Continued)

(c) How are the values $Y[k]$ related to the input signal spectrum, $X_c(j\Omega)$? Comment on your result. [3 points]

Solution. The system implements a 1000-point DFT on the normalized frequency range $[-\pi/2, 0]$, which correspond to the frequency range of -5000 Hz to DC. $k = 100$ corresponds to $F = -2000$ Hz, $k = 300$ corresponds to $F = -1000$ Hz, $k = 500$ corresponds to DC, $k = 700$ corresponds to $F = -4000$ Hz, and $k = 900$ corresponds to $F = -3000$ Hz.

This system is an example of a *zoom DFT* in a limited frequency band starting at $-\frac{\omega_0}{T}$ with a width determined by the filter bandwidth $(\frac{2\omega_c}{T})$.

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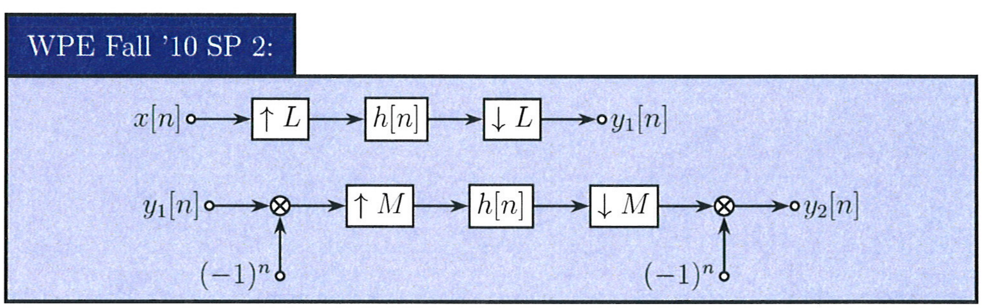
2. Consider the following system



The digital filter prototype, with impulse response $h[n]$, was designed to meet the following specifications

$$0.99 \le |H(e^{j\omega})| \le 1.01, \qquad 0 \le |\omega| \le 0.25\pi$$
$$|H(e^{j\omega})| \le 0.005, \qquad 0.3\pi \le |\omega| \le \pi.$$

The filter is FIR with linear phase characteristics, $\theta(\omega) = -n_d\omega + \beta$, where $n_d = 42.5$ and β is either 0 or π. Assume $L = 2$ and $M = 3$ and answer the following questions:

QUESTION 2 CONTINUES OVER THE PAGE

2 (Continued)

(a) What is the order of $H(z)$? Briefly justify your answer. [2 points]

Solution. The FIR filter has generalized linear phase characteristics. The possible values of β (and the fact that the filter is LP) suggest Type I or Type II symmetry. The slope of the phase curve is equal to $-M/2$, where M is the order. For this filter, $M = 85$.

(b) For $x[n] = A\cos(0.1\pi n)$, is it possible to write $y_1[n] = A_1\cos(0.1\pi n + \theta_2)$ and $y_2[n] = A_2\cos(0.1\pi n + \theta_2)$? If so, specify the range of values of $A_1, A_2, \theta_1, \& \theta_2$. If not, explain why not. [5 points]

Solution. While the individual operations like modulation, upsampling and downsampling are time-varying, their balanced use results in time-invariant effective filters that can be characterized as follows:

The filter $H_1(e^{j\omega}) = Y_1(e^{j\omega})/X(e^{j\omega})$ is related to $H(e^{j\omega})$ as follows:

- Its edge frequencies are scaled by L, i.e., $\omega_p \to L\omega_p$ and $\omega_s \to L\omega_s$.
- Its gain is scaled by $1/L$.

Therefore, for $L = 2$, $H_1(e^{j\omega})$ is a LPF with tolerances:

$$\begin{aligned} 0.99/2 \le \left|H_1\left(e^{j\omega}\right)\right| &\le 1.01/2, && 0 \le |\omega| \le 0.5\pi \\ \left|H_1\left(e^{j\omega}\right)\right| &\le 0.005/2, && 0.6\pi \le |\omega| \le \pi. \end{aligned}$$

The second filter in the cascade, $H_2(e^{j\omega}) = Y(e^{j\omega})/Y_1(e^{j\omega})$ is related to $H(e^{j\omega})$ as follows:

- The edge frequencies of the LP prototype are scaled by M, i.e., $\omega_p \to M\omega_p$ and $\omega_s \to M\omega_s$.
- Its gain is scaled by $1/M$.
- The use of the modulators shifts the center frequency of the modified filter by π.

Therefore, for $M = 3$, $H_2(e^{j\omega})$ is a HPF with tolerances:

$$\begin{aligned} 0.99/3 \le \left|H_2\left(e^{j\omega}\right)\right| &\le 1.01/3, && 0.25\pi \le |\omega| \le \pi \\ \left|H_2\left(e^{j\omega}\right)\right| &\le 0.005/3, && 0 \le |\omega| \le 0.1\pi. \end{aligned}$$

The effective filter $H_{eff}(e^{j\omega}) = Y(e^{j\omega})/X(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega})$ and it meets the following tolerances

$$\begin{aligned} 0.99^2/6 \le \left|H_{eff}\left(e^{j\omega}\right)\right| &\le 1.01^2/6, && 0.25\pi \le |\omega| \le 0.5\pi \\ \left|H_{eff}\left(e^{j\omega}\right)\right| &\le 1.01 * 0.005/6, && 0.6\pi \le |\omega| \le \pi. \\ \left|H_{eff}\left(e^{j\omega}\right)\right| &\le 1.01 * 0.005/6, && 0 \le |\omega| \le 0.1\pi. \end{aligned}$$

QUESTION 2(b) CONTINUES OVER THE PAGE

Solutions

2(b) (Continued)

The filters $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ retain their linear phase characteristics and so does the effective filter.

For $x[n] = A\cos(0.1\pi n)$, $y_1[n] \approx (1 \pm 0.01) * A/2\cos(0.1\pi(n - n_d))$ and $y_2[n] \approx 0.005 * A/6\cos(0.1\pi(n - 2n_d))$ (input signal in the passband of the first filter, but in the stop band of the second filter in the cascade).

(c) For $x[n] = A\cos(0.2\pi n)$, is it possible to write $y_1[n] = A_1\cos(0.2\pi n + \theta_2)$ and $y_2[n] = A_2\cos(0.2\pi n + \theta_2)$? If so, specify the range of values of A_1, A_2, θ_1, & θ_2. If not, explain why not. [5 points]

Solution. For $x[n] = A\cos(0.2\pi n)$, $y_1[n] \approx (1 \pm 0.01) * A/2\cos(0.2\pi(n - n_d))$ and $y_2[n] \approx 0.67 * A/6\cos(0.1\pi(n - 2n_d))$ (we approximated the filter response in the transmission band by a linear roll off with 0.2π being 0.05π from the passband edge and 0.1π from the stopband edge.

(d) For $x[n] = A\cos(0.4\pi n)$, is it possible to write $y_1[n] = A_1\cos(0.4\pi n + \theta_2)$ and $y_2[n] = A_2\cos(0.4\pi n + \theta_2)$? If so, specify the range of values of A_1, A_2, θ_1, & θ_2. If not, explain why not. [5 points]

Solution. For $x[n] = A\cos(0.4\pi n)$, $y_1[n] \approx (1 \pm 0.01) * A/2\cos(0.4\pi(n - n_d))$ and $y_2[n] \approx (1 \pm 0.02) * A/6\cos(0.4\pi(n - 2n_d))$ (input signal is in the passband of both filters in the cascade)

(e) Based on your answers above, describe the filtering operation performed by the cascade. Specifically, is the overall system LPF, BPF, BSF, HPF, or all-pass filter? If such characterization is possible, give the filter tolerances for the frequency range $[-\pi, \pi)$. If not, succinctly explain why not. [8 points]

Solution. The cascade is a bandpass filter with tolerances as given above. Apart from the scaling by 1/6, which can be easily compensated for, the performance of the filter in both the passband and stop bands is worse than the LPF prototype.

[*Hint: For Parts (b) - (d), your answers should be based on the tolerances of the filter prototype and the cascade architecture of the system.*]