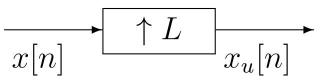
Instructions:

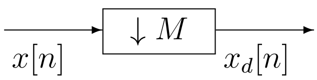
1. Answer all problems.
2. There are 2 problems with a total of 40 points (15 points for Problem 1 and 25 points for Problem 2).
3. Upsampling and downsampling operations are defined mathematically in the sequence domain (see below). You do not have to derive the relationships in the frequency domain. For example, if you know the relationship between $X_u\left(e^{j\omega}\right)$ and $X\left(e^{j\omega}\right)$, then simply use it if it is pertinent to your answer.
4. Show work for partial credit.

Upsampling/Downsampling:



(Upsampling)

$$x_u[n] = \begin{cases} x[n/L] & \text{for } n = 0, \pm L, \pm 2L, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

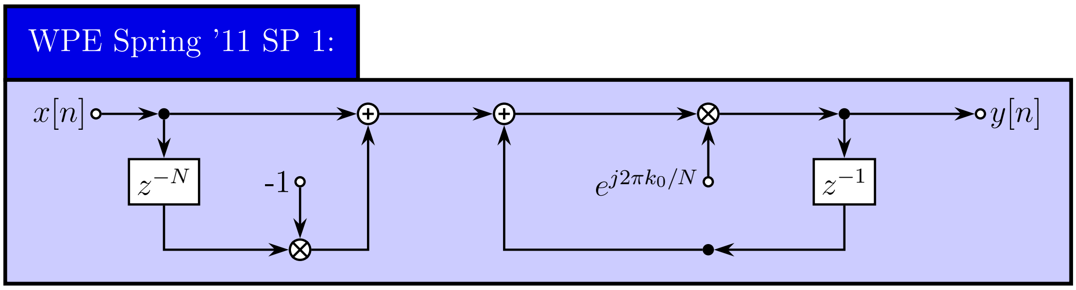


(Downsampling)

$$\begin{aligned} x_d[n] &= x[Mn] \\ &= p[Mn]x[Mn]. \end{aligned}$$

Where $p[n]$ is a periodic unit pulse with period M.

1. Consider the following system:



Assume the input to be a finite duration sequence of length $L \le N$ and $0 \le k_0 < N$ (assume integer, for simplicity). Answer the following questions:

(a) Derive the difference equation for the output, $y[n]$. Does this equation represent an FIR or IIR filter? Explain your answer and comment on the stability of the filter. [6 points]

Solution.

$$y[n] = e^{j2\pi k_0/N}(x[n]-x[n-N]+y[n-1]) = -e^{j2\pi k/N}y[n-1]+e^{j2\pi k_0/N}(x[n]-x[n-N])$$

The filter equation takes a recursive form, but it is suspiciously similar to the MA filter. We need the transfer function to gain more insight into the workings of this system:

$$H(z) = \frac{Y(z)}{X(z)} = e^{j2\pi k_0/N}\frac{1-z^{-N}}{1+e^{j2\pi k_0/N}z^{-1}}$$

The transfer function has non-trivial pole at $z = -e^{-j2\pi k_0/N}$ (US) which indicates possible instability. However, this pole is canceled by one of the roots of the numerator distributed uniformly around the UC. Therefore, the filter is an FIR filter, not an IIR, even if implemented in recursive form. The filter is stable when implemented with full precision.

(b) Derive the relationship between $Y(e^{j\omega})$ and $X(e^{j\omega})$ and sketch the magnitude response of the filter. Assume $N = 8$ and $k_0 = 2$. [6 points]

Solution.

$$Y(e^{j\omega}) = e^{j2\pi k_0/N}\frac{1-e^{-j\omega N}}{1+e^{-j(\omega-2\pi k_0/N)}}X(e^{j\omega})$$

(c) This filter is sometimes referred to as a sliding DFT. Explain this statement based on your derivations above. [3 points]

QUESTION 1 CONTINUES OVER THE PAGE

1 (Continued)

Solution. Evaluating $Y\left(e^{j\omega}\right)$ at $\omega_k = \frac{2\pi k}{N}, k = 0, 1, ..., N-1$ yields $Y[k] = \delta(k - k_0)X[k]$, which results from the most recent N values of $x[n]$.

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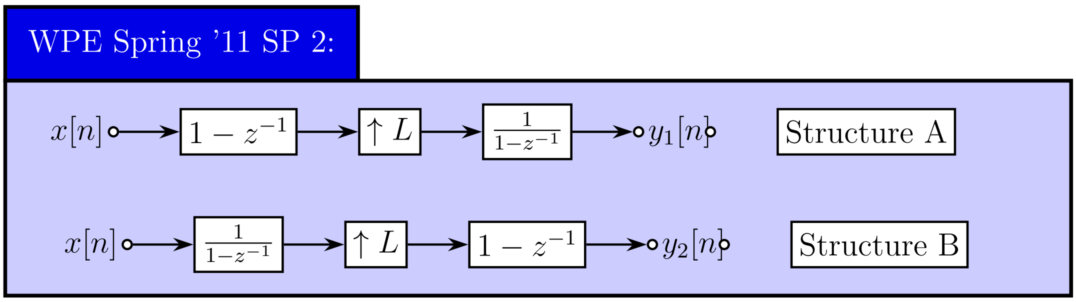
2. John Smith is currently taking a DSP class, where he is learning introductory material on multirate systems. He learned that an ideal interpolator can be implemented by an upsampler followed by an ideal LPF. While surfing the Internet, he learned that multiplier-free DSP structure (labeled Structure A in the figure below) can be used as a basic block in a practical interpolator. John did not understand how this structure works and came to you for help:



(a) Derive expression for $Y_1\left(e^{j\omega}\right)$ and explain how Structure A approximates an interpolator. [10 points]

Solution. In Structure A, the upsampler operates on $X\left(e^{j\omega}\right)\left(1 - e^{-j\omega}\right)$, producing $X\left(e^{j\omega L}\right)\left(1 - e^{-j\omega L}\right)$. It follows:

$$Y_1\left(e^{j\omega}\right) = \underbrace{X\left(e^{j\omega L}\right)}_{Upsampler} \underbrace{\frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}}_{LPF}$$

This is an upsampler followed by a LPF. It is not an ideal interpolator due to imperfect suppresion of the spectral images created by the upsampler at multiples of $\frac{2\pi}{L}$.

(b) Thinking about cascade relationships from system theory, John wondered if Structure B is equivalent to Structure A. Explain how the two structures are different and why Structure B does not function as an interpolator. [10 points]

QUESTION 2(b) CONTINUES OVER THE PAGE

2(b) (Continued)

Solution. Applying the upsampling identity to Structure A, we obtain an upsampler followed by a MA filter of order L. The MA approximates a LPF with cutoff at $\omega_c = \frac{2\pi}{L}$ (1st zero crossing in the magnitude response). Therefore, the overall structure approximates an interpolation. On the other hand, applying the upsampling identity to Structure B, we end up with an upsampler followed with an inverse MA filter. This is opposite to what we need to achieve by using the filter. Specifically, attenuating or eliminating the extra spectral images created by the upsampler. Therefore, Structure B does not approximate an interpolator.

(c) John thought that, if there is an interpolation filter implemented with one or both of the structures in the figure, there exists a similar structure for implementing a downsampler. What do you think of this assertion? If you agree, sketch the appropriate structure. [5 points]

Solution. He is right, we can use Structure B with the upsampler replaced by a downsampler.