DSP WPE Solution, Spring 2013

a)

$$x_3(n+1) = x_2(n) - b x_3(n)$$

$$x_2(n+1) = x_1(n)$$

$$x_1(n+1) = u(n) - a\left[x_3(n) + b x_3(n+1)\right]$$

$$= u(n) - a\left[x_3(n) + b\left(x_2(n) - b x_3(n)\right)\right]$$

$$= u(n) - ab\, x_2(n) - a(1-b^2)\, x_3(n)$$

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \\ x_3(n+1) \end{bmatrix} = \begin{bmatrix} 0 & -ab & -a(1-b^2) \\ 1 & 0 & 0 \\ 0 & 1 & -b \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \\ x_3(n) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(n)$$

$$y(n) = a\left[u(n) - a\left(b x_3(n+1) + x_3(n)\right)\right]$$

$$+ b x_3(n+1) + x_3(n)$$

$$= a u(n) + (1-a^2)\left(b x_2(n) - b^2 x_3(n) + x_3(n)\right)$$

$$y(n) = \begin{bmatrix} 0 & b(1-a^2) & (1-b^2)(1-a^2) \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \\ x_3(n) \end{bmatrix} + a u(n)$$

2) Take Z transform

b)

$$z X(z) - A X(z) = b \cancel{X}(z)$$

$$\frac{X(z)}{\cancel{X}(z)} = (zI - A)^{-1} b$$

$$Y(z) = c^T X(z) + d\,U(z)$$

$$\frac{Y(z)}{U(z)} = c^T (zI - A)^{-1} b + d$$

$$zI - A = \begin{bmatrix} z & ab & a(1-b^2) \\ -1 & z & 0 \\ 0 & -1 & z+b \end{bmatrix}$$

$$D = |zI - A| = z(z)(z+b) + 1(ab(z+b) + a(1-b^2))$$

$$= z^3 + bz^2 + abz + a$$

$$(zI - A)^{-1} = \frac{1}{D} \begin{bmatrix} z^2+bz & & \\ z+b & & \\ 1 & & \end{bmatrix}$$

no need to compute these as $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$H(z) = \frac{b(1-a^2)(z+b) + (1-a^2)(1-b^2)}{z^3 + bz^2 + abz + a} + a$$

$$= \frac{b z + \cancel{b^2} - \cancel{a^2 b z} - \cancel{a^2 b^2} + 1 - \cancel{a^2} - \cancel{b^2} + \cancel{a^2 b^2} + a z^3 + a b z^2 + \cancel{a^2 b z} + \cancel{a^2}}{z^3 + b z^2 + a b z + a}$$

$$= \frac{a z^3 + a b z^2 + b z + 1}{z^3 + b z^2 + a b z + a}$$

$$= \frac{a + a b z^{-1} + b z^{-2} + z^{-3}}{1 + b z^{-1} + a b z^{-2} + a z^{-3}}$$

c)

$$z \phi_1(n-1) = \phi_1(n) - k \phi_2(n-1)$$
$$\phi_2(n) = \phi_2(n-1) + k z \phi_1(n-1)$$

$$\begin{cases} \phi_1(n) = z \phi_1(n-1) + k \phi_2(n-1) \\ \phi_2(n) = \phi_2(n-1) + k z \phi_1(n-1) \end{cases}$$

d)

$$\phi_1(0) = \phi_2(0) = 1, \quad k_3 = a, \ k_2 = 0, \ k_1 = b$$

$$\phi_1(1) = z \phi_1(0) + b \phi_2(0) = z + b$$
$$\phi_2(1) = \phi_2(0) + b z \phi_1(0) = 1 + b z$$

$$\phi_1(2) = z \phi_1(1) = z^2 + b z$$
$$\phi_2(2) = \phi_2(1) = 1 + b z$$
$$\phi_1(3) = z \phi_1(2) + a \phi_2(2) = z^3 + b z^2 + a b z + a$$
$$\phi_2(3) = \phi_2(2) + a z \phi_1(2) = a z^3 + a b z^2 + b z + 1$$

$$H(z) = \frac{V(z)}{U(z)} = \frac{Y_2(z)}{Y_1(z)}$$

$$= \frac{a z^3 + ab z^2 + b z + 1}{z^3 + b z^2 + ab z + a}$$

(same as in (b))

e) Poles cannot be outside unit circle due to Schur polynomial property

f) This is an all-pass filter.

$$|H(e^{jw})| = \frac{|a e^{j3w} + ab e^{j2w} + b e^{jw} + 1|}{|e^{j3w} + b e^{j2w} + ab e^{jw} + a|}$$

$$= \frac{|e^{j3w}| \, |a + ab e^{-jw} + b e^{-j2w} + e^{-j3w}|}{|e^{j3w} + b e^{j2w} + ab e^{jw} + a|}$$

$|e^{j3w}| = 1$, $|H(e^{jw})| = 1$

since, remaining numerator is conjugate of denominator.

Poles and zeros are in reciprocal conjugate pairs.

g) $\sum h^2(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{jw})|^2 dw = \frac{1}{2\pi} \int_{-\pi}^{\pi} dw = 1$

h) $H(-1) = \frac{1 - b + ab - a}{-1 + b - ab + a} = -1$, $V(n) = (-1)^{n+1}$