(1) (12 points) Suppose Z_i, $i = 1, \ldots, n$ are i.i.d. random variables with the following distribution:

$$Z_i = \begin{cases} -1 & \text{with prob. } p \\ 0 & \text{with prob. } 1-2p \\ 1 & \text{with prob. } p. \end{cases}$$

Let,

$$Z = \sum_{i=1}^{n} Z_i.$$

a. Find $\mathbb{E}Z$ and $\mathbb{E}Z^2$. 4

b. Find the characteristic function of Z. 5

Now consider a discrete time signal X_i is transmitted over an additive-noise channel. At the output of the channel we obtain $i = 1, \ldots, n$:

$$Y_i = X_i + Z_i.$$

If $[X_1, X_2, \ldots, X_n]$ is a Gaussian vector with covariance matrix $= 10I$, where I is an $n \times n$ identity matrix, then,

c. Find the Signal to Noise ratio for this channel when $p = 0.2$. 3

Solution.

a. $\mathbb{E}Z_i = -p + 0 + p = 0$.

$$\mathbb{E}Z = \sum_i \mathbb{E}Z_i = 0.$$

$\mathbb{E}Z_i^2 = p + 0 + p = 2p$.

$$\mathbb{E}Z^2 = \sum_i \mathbb{E}Z_i^2 + \sum_i \sum_{j \neq i} \mathbb{E}Z_i \mathbb{E}Z_j = \sum_i \mathbb{E}Z_i^2 = 2np.$$

b. Let $i^2 = -1$.

$$\mathbb{E}e^{itZ} = \mathbb{E}e^{it\sum_j Z_j} = \mathbb{E}\prod_j e^{itZ_j} = \left(\mathbb{E}e^{itZ_j}\right)^n = \left(pe^{-it} + (1-2p) + pe^{it}\right)^n.$$

c. Signal power = 10. Noise power = $2 \times 0.2 = 0.4$. SNR $= 10/0.4 = 25$. ($10 \log 10/0.4 = 13.98$ dB.)

(2) (8 points) Suppose, you need to sample and transmit the following signal using an 8-bit PCM (pulse code modulation):

$$X(t) = 32\cos(8\pi t).$$

a. What is the bits/sec. transfer rate (assume Nyquist rate sampling)? 3

b. What is the mean square quantization error in the PCM? You can assume that the quantization noise is uniformly distributed. 5

Solution.

a. Here, the frequency of the signal is 4 Hz. Hence, Nyquist rate is 8 Hz. Each sample uses 8 bits. Therefore the transfer rate is $8 \cdot 8 = 64$ bits per second.

b. The range for the amplitude of the signal is $[-32, +32]$. There are 2^8 levels of quantizations in the PCM. Hence, the quantization step size $\Delta \equiv 64/2^8 = \frac{1}{4}$. Assume the quantization noise is uniform. I.e., the pdf of quantization noise is $= \frac{1}{\Delta}$ in the interval $[-\Delta/2, +\Delta/2]$.

So, mean square quantization error is,

$$\frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} x^2 dx = \frac{1}{3\Delta}\left(\Delta^3/8 + \Delta^3/8\right) = \frac{\Delta^2}{12} = \frac{1}{192}.$$

(3) (20 points) Consider the following Octal Phase-Shift-Keying (PSK) scheme. Each of the 8 signal waveforms are represented as (below $f_c = 1$ Hz):

$$s_n(t) = \left|t - \frac{1}{2}\right| \cos\left(2\pi f_c t + \frac{\pi}{4}(n-1)\right), \quad n = 1, 2, \ldots, 8, \quad 0 \le t \le 1.$$

a. Compute the energy of the signal waveform $s_n(t)$. 5

b. Write an basis for this set of signals. There should be only two signals in this basis. 5

c. Represent $s_1(t)$ and $s_2(t)$ as a linear combination of above two basis signals. 5

d. How many bits of information can be sent in the interval $0 \le t \le 1$? Suppose, signals with adjacent phases can be confuseed at the receiver. How many bits can still be sent so that information retrieval with certainty is possible? 5

Solution.

a. The PSK signals are time-shifts of one-another. Hence the energy of each of them is same. So without loss of generality we can take $n = 1$ and by straightforward integration ,

$$\int_0^1 (s_n(t))^2 dt = \frac{1}{2}\int_0^1 (t - 1/2)^2 dt = \frac{1}{6}(1/8 + 1/8) = \frac{1}{24}.$$

b.

$$\begin{aligned} s_n(t) &= \left|t - \frac{1}{2}\right| \cos\left(2\pi f_c t + \frac{\pi}{4}(n-1)\right) \\ &= \left|t - \frac{1}{2}\right| \cos 2\pi f_c t \cos\frac{\pi}{4}(n-1) - \left|t - \frac{1}{2}\right| \sin 2\pi f_c t \sin\frac{\pi}{4}(n-1) \\ &= \left[\cos\frac{\pi}{4}(n-1) \quad -\sin\frac{\pi}{4}(n-1)\right] \times \left[\left|t - \frac{1}{2}\right| \cos 2\pi f_c t \quad \left|t - \frac{1}{2}\right| \sin 2\pi f_c t\right]^T \end{aligned}$$

Hence the the basis is $\left\{\left|t - \frac{1}{2}\right| \cos 2\pi f_c t, \left|t - \frac{1}{2}\right| \sin 2\pi f_c t\right\}$.

c.

$$s_1(t) = \left|t - \frac{1}{2}\right| \cos 2\pi f_c t.$$

$$s_2(t) = \frac{1}{\sqrt{2}}\left|t - \frac{1}{2}\right| \cos 2\pi f_c t - \frac{1}{\sqrt{2}}\left|t - \frac{1}{2}\right| \sin 2\pi f_c t.$$

d. 3 bits.

2 bits, as one can send only 4 non-adjacent phases.