PhD Preliminary Written Exam
Controls
Fall 2013

There are two Parts, assigned 2 points each, for a total of 4 points.

Part I (2 points):

A typical electromechanical position system, such as those encountered in applications of

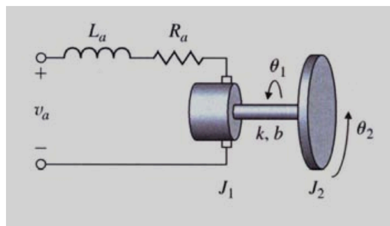

Figure 1: DC motor plus inertial mass

computer-disk-head control, tape drives, etc., consists of an electrical motor driving a load that has one dominant vibration mode. A schematic for such a system is provided in Figure 1. The motor has an electrical constant K_e, a torque constant K_t, an armature inductance L_a, and a resistance R_a. The rotor has an inertia J_1 and a viscous friction B. The load has an inertia J_2. The two inertias are connected by a "mass-less" shaft with a spring constant k and an equivalent viscous damping b. This means that, since the mass of the shaft is considered negligible, the torque applied to the shaft at each end has magnitude $|k(\theta_1 - \theta_2) + b(\dot{\theta}_1 - \dot{\theta}_2)|$ and the two torques have opposite signs. Write the equations of motion. In the scematic, θ_1 and θ_2 represent the angular position of the two ends of the shaft, since this is assumed flexible. Note that a back emf voltage at the terminals of the motor is generated by the rotation of the shaft and is equal to $K_e\dot{\theta}_1$. Also, the torque applied by the motor to the shaft is $T_m = K_t i_a$ where i_a is the current through the motor.

Do the following:

1. Write the equations of motion for the complete system.
2. Indicate what the order of the system is and what is a possible choice for a state vector (without necessarily giving the state equations).
3. For suitable values of parameters and in suitable units, the transfer function from v_a to the load position θ_2 becomes

$$G(s) = \frac{1}{s(s^2 + s + 1)}.$$

Assume that the system is initially at rest at $t = 0$ and that an input voltage

$$v_a(t) = \sin(t)$$

is applied to it. Determine the transient and steady-state responses.

You are given:

Laplace transform table	
$F(s)$	$f(t)$ (= 0 for $t < 0$)
$\frac{1}{s+a}$	e^{-at}
$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos(bt)$
$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin(bt)$

Solution:
1) The equations of motion are as follows.
— Circuit:

$$v_a = K_e\dot{\theta}_1 + L_a\frac{d}{dt}i_a + R_a i_a.$$

— Torque (linking the electrical to the mechanical part):

$$T_m = K_t i_a.$$

— Rotor:

$$J_1\ddot{\theta}_1 = T_m - B\dot{\theta}_1 - b(\dot{\theta}_1 - \dot{\theta}_2) - k(\theta_1 - \theta_2).$$

— Load:

$$J_2\ddot{\theta}_2 = -b(\dot{\theta}_2 - \dot{\theta}_1) - k(\theta_2 - \theta_1).$$

2) The order of the system is 3 and a possible choice for a state vector is

$$x(t) := \begin{pmatrix} i \\ \theta \\ \dot{\theta} \end{pmatrix}$$

3) We compute

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1} \times \frac{1}{s(s^2+s+1)}\right)$$

using partial fractions and the above tables and we obtain:

$$1 - \sin(t) - e^{-t/2}\left(\cos(\frac{\sqrt{3}t}{2}) - \frac{\sqrt{3}}{3}\sin(\frac{\sqrt{3}t}{2})\right)$$

The part

$$-e^{-t/2}\left(\cos(\frac{\sqrt{3}t}{2}) - \frac{\sqrt{3}}{3}\sin(\frac{\sqrt{3}t}{2})\right)$$

is transient and goes to 0 as $t \to \infty$. The steady state response is

$$1 - \sin(t).$$

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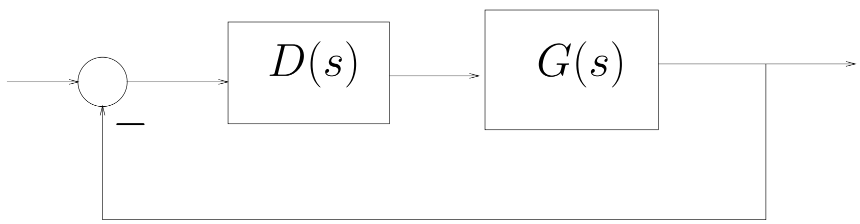


Figure 2: Closed loop control

Part II (2 points):
You are given a system with transfer function

$$G(s) = \frac{1}{s(s^2 + s + 1)}.$$

A negative-unity feedback control system is shown in Figure 2 where $D(s)$ represents the transfer function of a compensator.

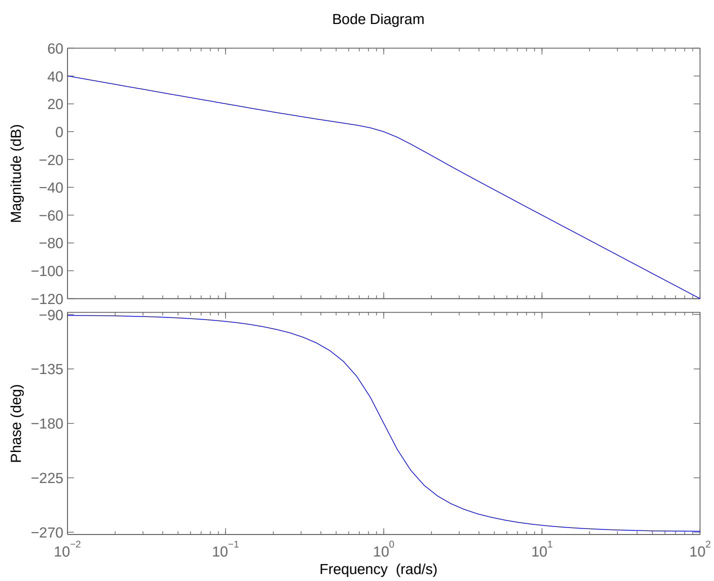
Do the following:

1. For a choice $D(s) = K$, determine the range of the constant gain K for which the feedback system is stable.
2. Draw the root locus when $D(s) = K$. You need to draw separately the loci corresponding to $K > 0$ and $K < 0$.
3. Draw approximately the Nyquist plot.
4. You are given values of the Bode plot and the frequency response of the open loop system $G(j\omega)$ in the next page. Assume that you choose a control law $D(s) = 0.5$. Determine the gain margin, the phase margin, and the amount of minimal time-delay that if introduced in the feedback loop the closed loop system would become unstable. (Time-delay may be due to processing delays by a digital controller or transmission delays, in case the controller and the system are physically separated.)
5. Suppose you decide to implement proportional plus derivative control, i.e.,

$$D(s) = K_p + K_d s$$

for a suitable choice of constants K_p, K_d. Is it possible to achieve zero overshoot for the closed loop system (i.e., is it possible to ensure that it is overdamped)? If this is possible, determine suitable values for the parameters, and if not, explain why.

You are given the Bode plot of $G(s)$ as well as numerical values of gain and phase of $G(j\omega)$ for a range of frequencies. You may interpolate, or approximately interpolate between those if needed.



ω [rad/sec]	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$ [degrees]
0.3162	3.3150	-109.3596
0.3728	2.8592	-113.4084
0.4394	2.4770	-118.5696
0.5179	2.1536	-125.2924
0.6105	1.8712	-134.2270
0.7197	1.6041	-146.1854
0.8483	1.3193	-161.7151
1.0000	1.0000	-180.0000
1.1788	0.6833	-198.2849
1.3895	0.4303	-213.8146
1.6379	0.2600	-225.7730
1.9307	0.1550	-234.7076
2.2758	0.0923	-241.4304
2.6827	0.0552	-246.5916
3.1623	0.0331	-250.6404

Provide your answer below and in the next page.

Solution:
1) Using the Routh test we obtain that $\boxed{0 < K < 1}$ for stability.
2) Root locus for $K > 0$ and for $K < 0$ are given in Figure 3.

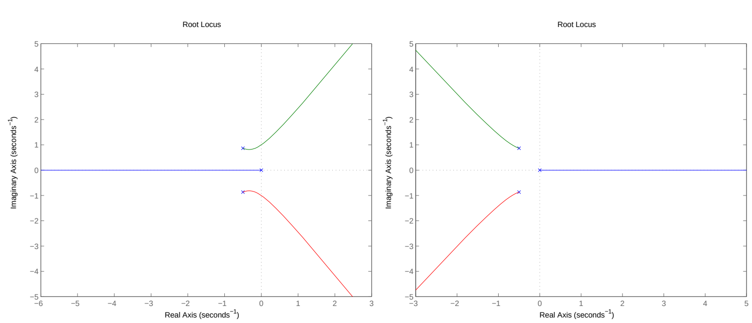


Figure 3: Root locus for $K > 0$ and for $K < 0$, respectively

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Solution:
3) The Nyquist plot is given in Figure 4.

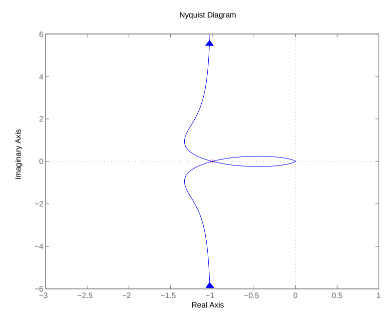

Figure 4: Nyquist plot

4) The gain margin is $\boxed{\text{GM} = 2}$ which we can of course determine using the Routh test as well as by noting that $G(j) = -1$, and hence $\frac{1}{2}G(j) = \frac{1}{2}$. The phase margin is about $\boxed{50^o}$ and the gain-crossover frequency, with $D(s) = 0.5$ in place, is between 0.6 and 0.5 [rad/sec]. At that point, where $0.5G(j\omega) = 1$, i.e., $G(j\omega) = 2$ the phase of $G(j\omega)$ is about 130^o and hence the phase margin is about 50^o. The delay margin is obtained by solving

$$\omega\tau = P.M.$$

for $\omega = 0.55$ and $P.M. = \frac{50\times\pi}{180}$. Therefore the delay margin is approximately $\boxed{\tau = 1.5 \text{ [sec]}}$

5) No. The addition of an open-loop zero will not alter the asymptotes for the complex poles (see Root locus rules). For any choice of K_p, K_d the closed loop will have complex poles.