1)

a) Since $X(j\Omega)$ is band limited (and the sampling rate T exceeds the Nyquist rate, by assumption), we have that the transfer function of the overall continuous-time system is given by

$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T, \\ 0, & |\Omega| \geq \pi/T. \end{cases} .$$

Thus, here, our aim is to have

$$H_{\text{eff}}(j\Omega) = \begin{cases} e^{-j\Omega t_0}, & |\Omega| < \pi/T, \\ 0, & |\Omega| \geq \pi/T. \end{cases}$$

so that the system performs the time-shift (by the fixed parameter t_0) for the band limited inputs. It follows that we should choose the function $H(e^{j\omega})$ so that $H(e^{j\omega}) = e^{-j\omega t_0/T}$ for $|\omega| < \pi$, and so that it is 2π-periodic.

b) Using the inversion formula,

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{2\pi} H(e^{j\omega})e^{j\omega n} d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-t_0/T)} d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-t_0/T)}}{j(n-t_0/T)}\right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi(n-t_0/T)}\left[\frac{e^{j\pi(n-t_0/T)} - e^{-j\pi(n-t_0/T)}}{2j}\right] \\ &= \frac{1}{\pi(n-t_0/T)}\sin(\pi(n-t_0/T)), \end{aligned}$$

which is just a sinc function.

c) The sinc function is zero when $\pi(n-t_0/T) = k\pi$ for $k = \pm 1, \pm 2, \ldots$, and nonzero otherwise. Thus, the impulse response will in general be infinite (in both directions) unless t_0/T is an integer. In this special case, we have that $t_0 = sT$ for some integer s, and $h[n] = \delta[n-s]$. Thus, $h[n]$ is FIR whenever $t_0 = sT$ for an integer s.

d) For $h[n]$ to be causal we require that $h[n] = 0$ for $n < 0$. By the argument above, we see that this is only the case here when $t_0 = sT$ for an integer $s \geq 0$.

e) A discrete-time system is BIBO stable if and only if its impulse response is absolutely summable, so that

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty.$$

Clearly, when t_0/T is an integer (say s) this is true, since in that case $h[n] = \delta[n-s]$. For other values of t_0 the impulse response takes the form of samples of a sinc function, and such sequences are not absolutely summable. Overall, the system is only BIBO stable when t_0/T is integer, so NOT BIBO stable for all t_0.

2)

a) First, note that we may write $Y(j\Omega) = e^{-j\Omega t_0}X(j\Omega)$ and $\widetilde{Y}(j\Omega) = e^{-j\Omega \widetilde{t}_0}X(j\Omega)$. Now, by Parseval's equation,

$$\begin{aligned}
\int_{-\infty}^{\infty} |y(t) - \widetilde{y}(t)|^2 dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(j\Omega) - \widetilde{Y}(j\Omega)|^2 d\Omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\Omega)|^2 \cdot |e^{-j\Omega t_0} - e^{-j\Omega \widetilde{t}_0}|^2 d\Omega \\
&= \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} 1 \cdot |e^{j\Omega t_0} - e^{-j\Omega \widetilde{t}_0}|^2 d\Omega \\
&= \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} 2 - \left(e^{j\Omega(t_0 - \widetilde{t}_0)} + e^{-j\Omega(t_0 - \widetilde{t}_0)}\right) d\Omega \\
&= \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} 2 - 2\cos(\Omega(t_0 - \widetilde{t}_0))\ d\Omega \\
&= \frac{1}{\pi}\left[\Omega - \frac{\sin(\Omega(t_0 - \widetilde{t}_0))}{t_0 - \widetilde{t}_0}\right]_{-\Omega_c}^{\Omega_c} \\
&= \frac{2}{\pi}\left[\Omega_c - \frac{\sin(\Omega_c(t_0 - \widetilde{t}_0))}{t_0 - \widetilde{t}_0}\right] \\
&= \frac{2\Omega_c}{\pi}\left[1 - \frac{\sin(\Omega_c(t_0 - \widetilde{t}_0))}{\Omega_c(t_0 - \widetilde{t}_0)}\right].
\end{aligned}$$

Finally, since the sinc function is symmetric, we have that

$$\int_{-\infty}^{\infty} |y(t) - \widetilde{y}(t)|^2 dt = \frac{2\Omega_c}{\pi}\left[1 - \frac{\sin(\Omega_c\delta)}{\Omega_c\delta}\right],$$

where $\delta = \widetilde{t}_0 - t_0$.

b) Let

$$\mathcal{E}(\delta, \Omega_c) \triangleq \frac{2\Omega_c}{\pi}\left[1 - \frac{\sin(\Omega_c\delta)}{\Omega_c\delta}\right].$$

Since

$$\lim_{\delta\to\infty}\left(\frac{2\Omega_c}{\pi}\right)\frac{\sin(\Omega_c\delta)}{\Omega_c\delta} = \lim_{\delta\to\infty}\frac{2\sin(\Omega_c\delta)}{\pi\delta} = 0,$$

for any $\Omega_c > 0$, we have that

$$\lim_{\delta\to\infty}\mathcal{E}(\delta, \Omega_c) = \frac{2\Omega_c}{\pi}.$$

2)

c) The function looks like an "upside-down" sinc function, shifted up by 2 units on the y-axis. Further, the sinc function becomes zero for $\delta = k$ for integers $k = \pm 1, \pm 2, \pm 3, \ldots$, so at those values of δ the function takes the value 2. Overall,