Closed book, closed notes, calculators OK. The Communications problem consists of three parts, each comprising multiple questions, points as marked (point sum = 40 = perfect score); nominal duration = 1 hour.

**Part A: All bits are not equal [sum of points = 17]** The decision statistic at the output of the matched filter of a digital communication receiver can be written as $r = 3b_1 + b_2 + n$, where $b_1 \in \{-1, +1\}$, $b_2 \in \{-1, +1\}$ are independent bits, each one is equally likely to be $-1$ or $+1$, and $n$ is a zero-mean unit-variance Gaussian random variable that is independent of $b_1$, $b_2$.

- **[2 points]** Draw the symbol constellation and mark the point coordinates.
- **[3 points]** Draw and label the decision regions of the detector that minimizes the symbol error rate (SER).
- **[4 points]** For the above detector, compute the bit error rate (BER) for $b_1$; express the result in terms of the $Q(\cdot)$ function.
- **[5 points]** Do the same for the BER for $b_2$.
- **[3 points]** Can you think of an application or context where a system like this makes sense?

**Part B: Quadrature Amplitude Modulation vs. Orthogonal Modulation [sum of points = 17]**

- [**2 points**] Draw a 4-QAM constellation with average energy per bit $E_b = 1$ and compute the minimum distance between constellation points, $d_{\min}$. Your drawing should include the coordinates of the constellation points.
- [**3 points**] Repeat for a 16-QAM constellation with average energy per bit $E_b = 1$. For convenience, you may start with the prototype 16-QAM constellation $\{-3, -1, 1, 3\} \times \{-3, -1, 1, 3\}$, compute its average energy per bit, then scale it appropriately to bring it to $E_b = 1$.
- [**3 points**] Next, consider any $4-\perp$ modulation (i.e., orthogonal modulation of order 4) with average energy per bit $E_b = 1$. Write out the $4 \times 1$ coordinate vector for each of the 4 constellation points, and compute the associated $d_{\min}$.
- [**3 points**] Repeat for $16-\perp$ modulation with average energy per bit $E_b = 1$.
- [**6 points**] Compare QAM and $\perp$ modulation on the basis of (and what you can extrapolate from) the above. What is their fundamental difference? Is there a 'hidden' price paid for the apparent advantage of one vs. the other?

**Part C: Free lunch [sum of points = 6]** *In-phase / Quadrature modulation is special because it gives us a free orthogonal signaling dimension without expanding the bandwidth relative to pulse amplitude modulation.*

- **[3 points]** Derive a condition relating the carrier frequency $f_c$ to the symbol period $T$ which ensures exact orthogonality of $\cos(2\pi f_c t)w(t)$ and $\sin(2\pi f_c t)w(t)$, where $w(t) = u(t) - u(t-T)$ and $u(t)$ is the unit step function.
- **[3 points]** Derive an alternative condition which ensures approximate orthogonality of $\cos(2\pi f_c t)w(t)$ and $\sin(2\pi f_c t)w(t)$.