Q1 [14pts]

Consider the feedback interconnection shown in Figure 1 where \( G \) and \( K \) are proper transfer functions. Here it is assumed that a transfer function is of the form \( \frac{n(s)}{d(s)} \) where \( n \) and \( d \) are polynomials in \( s \) with no common factors. For the following questions answer if the statement is true or false. If true provide an example and if false provide a proof.

1. [7pts] There exist single-input, single-output proper transfer functions \( G \) and \( K \) such that \( \frac{1}{1+GK} \) and \( \frac{G}{1+GK} \) are stable but \( \frac{K}{1+GK} \) is not.

**Solution:** Let \( G = \frac{s-1}{s^2+1} \) and \( K = \frac{s+1}{s^2+2} \). Here \( \frac{1}{1+GK} = \frac{s-1}{(s+1)(s^2+2)} \), \( \frac{G}{1+GK} = \frac{s+1}{(s+1)(s^2+2)} \), and \( \frac{K}{1+GK} = \left( \frac{s+1}{s^2+2} \right)^2 \) are stable but \( \frac{K}{1+GK} = \frac{(s+1)^2}{(s^2+1)(s^2+2)} \) is not.

2. [7pts] There exist single-input, single-output proper transfer functions \( G \) and \( K \) such that \( \frac{1}{1+GK} \) and \( \frac{G}{1+GK} \) are stable but \( \frac{K}{1+GK} \) is not.

**Solution:** Suppose \( G = \frac{n_g(s)}{d_g(s)} \) and \( K = \frac{n_k(s)}{d_k(s)} \) where \( n_g, d_g, n_k \) and \( d_k \) are polynomials in \( s \). Then it follows that

\[
\begin{align*}
\frac{1}{1+GK} &= \frac{d_g d_k}{n_g n_k + d_g d_k}, \\
\frac{G}{1+GK} &= \frac{n_g d_k}{n_g n_k + d_g d_k}, \\
\frac{K}{1+GK} &= \frac{n_k d_g}{n_g n_k + d_g d_k},
\end{align*}
\]

\[
\frac{1}{1+GK} = \frac{d_g d_k}{n_g n_k + d_g d_k},
\]

unstable implies that there is a \( s_0 \) in the right half plane (rhp) that \( (n_g n_k + d_g d_k)(s_0) = 0 \). Given that \( \frac{G}{1+GK} = \frac{n_g d_k}{n_g n_k + d_g d_k} \) and \( \frac{K}{1+GK} = \frac{n_k d_g}{n_g n_k + d_g d_k} \), are stable it follows that

\[
(n_g d_k)(s_0) = (n_k d_g)(s_0) = 0.
\]

As \( (n_g d_k)(s_0) = 0 \) there are two cases

**Case 1:** Suppose \( n_g(s_0) = 0 \)

Then \( d_g(s_0) \neq 0 \) and thus from \( (n_k d_g)(s_0) = 0, n_k(s_0) = 0 \). Thus \( d_k(s_0) \neq 0 \). Thus

\[
n_g(s_0)n_k(s_0) + d_g(s_0)d_k(s_0) = d_g(s_0)d_k(s_0) \neq 0
\]

which is a contradiction.

**Case 2:** Suppose \( d_k(s_0) = 0 \).

Then \( n_k(s_0) \neq 0 \). Thus from \( (n_k d_g)(s_0) = 0, d_g(s_0) = 0 \). Thus \( n_g(s_0) \neq 0 \) and thus

\[
n_g(s_0)n_k(s_0) + d_g(s_0)d_k(s_0) = n_g(s_0)n_k(s_0) \neq 0
\]

which is a contradiction.
Consider the bode plot of a minimum-phase transfer function $G(s)$ (the bode plot shows in the magnitude plot $20 \log_{10} |G(j\omega)|$ in db on the Y axis).

1. [3pts] Draw the asymptotes on the bode plot. Use the asymptotes to determine the transfer function $G(s)$.

2. [3pts] (a)Determine the gain-crossover frequency ($\omega_{gc}$) and the phase-crossover frequency ($\omega_{iso}$). (b) Determine the phase and gain margin.

3. [3pts] Suppose the plant $G$ is in a unity negative feedback interconnection with a controller $K$ (see Figure 1). With the controller $K = k_p$ a positive real constant, find the smallest value of $k_p$ such that the interconnection shown is unstable. (Hint: Use the gain margin to obtain the result).

4. [3pts] With $K = 1$ determine the steady state error due to a step input for the interconnection shown. Also, determine the steady state error due to a ramp input.

5. [3pts] Design a Proportional Integral (PI ) controller, $K = k_p + \frac{k_i}{s}$, to increase the type with specifications (i) the gain crossover frequency has to be 100 rad/sec (ii) the phase margin has to be at least 40 degrees.
Solution 1.
Solution: The breakpoint frequencies are shown on the bode-plot (next page).
They are at 1 rad/sec, 100 rad/sec and 10000 rad/sec. All are poles.
Thus, the transfer function is of the form

\[ G(s) = \frac{A}{(s+1)(s+100)(s+10000)}. \]

\[ \Rightarrow G(0) = \frac{A}{10^2 \cdot 10^4} = \frac{A}{10^6} \]

but from the Bode plot \( 20 \log_{10} |G(0)| = 20 \)
\[ \Rightarrow |G(0)| = 10 \]

Note that \( |G(0)| = 0 \); hence, \( A \) is a positive constant and \( A = 10^7 \)

\[ \Rightarrow G(s) = \frac{10^7}{(s+1)(s+100)(s+10000)} = \frac{10}{(s+1)(s+1000)(s+10^4)} \]
is the transfer function
Consider the bode plot of a plant $G$ as given below.
Solution 2.

Solution: \( \omega_{gc} \) is the frequency at which

\[
\left| G(j \omega_{gc}) \right| = 1 \quad \omega_{gc} = 10 \text{ rad/Sec}
\]

From the Bode plot

Similarly the phase crosses \( \omega_{180} \) is the frequency where

\[
\angle G(j \omega_{180}) = -180^\circ
\]

and this occurs at \( \omega_{180} = 1000 \text{ rad/Sec} \).
Determine the phase margin and the gain margin

**Solution:** The gain margin is given by

\[ GM = -20 \log_{10} |G(f(\pi/2)| \]

\[ = 60 \text{ dB} \]

\[ PM = 180 + \left| \angle G(f(\pi/2)) \right| \]

\[ = 180 - 90 = 90 \text{ degrees} \]

[See Bode plot earlier.]
3. With the controller $K=k_p$ a positive real constant, find the smallest value of $k_p$ such that the interconnection shown is unstable. Use the gain margin to obtain the result.

Solution:

The value of $K_p$ is given by

$$20 \log_{10} k_p = GM$$

$$= 60 \text{dB}.$$  

Thus, the smallest value of $k_p$ that will destabilize the feedback interconnection is $1000$. 

$$\log_{10} k_p = \frac{60}{20} = 3$$

$$\Rightarrow k_p = 10^3 = 1000.$$
Solution to 4.

With $K=1$, determine the steady state error due to a step input for the interconnection shown. Also, determine the steady state error due to a ramp input. Determine the type of the system.

\[ \text{Solution: } \begin{array}{c}
\xrightarrow{r} \quad e \quad \xrightarrow{K} \quad G \quad \xrightarrow{y}
\end{array} \]

In this case transfer function from $r$ to $e$ is

\[ \frac{1}{1+4K} = \frac{1}{1+2} \]

and steady state error due to step is

\[ \lim_{s \to 0} \mathcal{L} \{ e(s) \} = \lim_{s \to 0} \mathcal{L} \{ \left( \frac{1}{1+2} \right) \} = \frac{1}{1+2} \]

\[ = \lim_{s \to 0} \frac{1}{1+2} \]

\[ = \frac{1}{1+k_p}; \quad k_p = L(0) \]

\[ L(0) = G(0) \ast w \]  Such that

\[ 20 \log |L(0)| = 20 \log 8 \Rightarrow G(0) = 10 \]

and therefore $e_s = \frac{1}{10+1} = \frac{1}{11} \times 0.1$.
When \( r(t) = \text{ramp} \), \( e(t) = \left( \frac{1}{1+L} \right) \frac{1}{s^2} \)

and \( e(t) \) due to ramp is:

\[
\lim_{s \to 0} e(s) = \lim_{s \to 0} \frac{1}{s + 8L} \cdot \frac{1}{s^2} = \lim_{s \to 0} \frac{1}{s + 8L} 
\]

\[
= \lim_{s \to 0} \frac{1}{k_v} \cdot \frac{1}{s + 8L} 
\]

\[
k_v = \lim_{s \to 0} s \cdot e(s). \quad \text{As} \quad k = 1; \quad L = 6 \]

and \( k_v = \lim_{s \to 0} s \cdot e(s) = 0 \)

\[
\therefore \quad e(t) \text{ due to ramp} = 0 
\]

The system is Type 0; it does not track steps with zero steady state error.
Design a Proportional Integral (PI) controller, K, to increase the type. Additional specification is that the gain crossover frequency has to 100 rad/sec and to have a PM of 40 degrees.

**Solution:** The PI controller is

\[ K(s) = kp + \frac{ki}{s} = \frac{ki}{s} \left[ \frac{kp}{ki} s + 1 \right] \]

\[ = \frac{ki}{s} \left[ \frac{kp}{ki} + 1 \right] \]

which has a break frequency at \( \frac{ki}{kp} \).

\( G(s) \) has a phase of \(-135^\circ\) at \( \omega_g = 100 \text{ rad/sec} \).

Thus, \( PM_{\text{have}} = 180 - 135 = 45^\circ \)

\( PM_{\text{desired}} = 40 + 5 \) also, thus, controller \( K(s) \) cannot decrease the phase any further.

Thus, we choose \( \frac{ki}{kp} \leq \frac{\omega_g cd}{10} = \frac{100}{10} = 10 \)

Let us fix \( \frac{ki}{kp} = 10 \).
We also need to shift the gain crossover to 180 rad/sec. = \omega_{\text{g}}\text{ rad/s}

\therefore \quad |L(\omega_{\text{g}})| = 1

\Rightarrow \quad \left| \left( \frac{k_p + k_i}{s} \right) G(s) \right| = 1 \quad \text{at} \quad s = \omega_{\text{g}} \text{ rad/s}

\Rightarrow \quad \left| k_p \left( 1 + \frac{k_i}{k_p} \cdot \frac{1}{\omega_{\text{g}}^2} \right) \right| |G(\omega_{\text{g}})| = 1

\Rightarrow \quad \left| k_p \left( 1 - \left( \frac{k_i}{k_p} \cdot \frac{1}{100} \right) \right) \right| |G(\omega_{\text{g}})| = 1

\Rightarrow \quad k_p \left( 1 - \frac{1}{10} \right) |G(\omega_{\text{g}})| = 1

\Rightarrow \quad k_p \sqrt{1 + \frac{1}{100}} |G(\omega_{\text{g}})| = 1

\Rightarrow \quad k_p = \frac{1}{\sqrt{1.01}} = \frac{1}{1.0049} = 13.5

\Rightarrow \quad k_i = k_i \cdot k_p = \frac{10 \cdot 13.5}{k_p} = 135

\therefore \quad K(s) = 13.5 + \frac{135}{s}
Consider the unity gain loop depicted above, with open loop transfer function given by $KG(s) = K \frac{s+1}{s(s-1)}$. Let $K = k$ be a constant gain. Find the range of $k$ that give phase margins of at least $30^\circ$.

**Solution:** The closed-loop poles are given by the roots of the polynomial $f(s) = s^2 + (k-1)s + k$. It follows that the system is stable if and only if $k > 1$.

To find the phase margin, note that gain cross-over frequency is given by $\omega_{gc} = k$, since

$$|kG(j\omega_{gc})| = k \frac{|j\omega_{gc} + 1|}{|j\omega_{gc} - 1|} = k \frac{\omega_{gc}}{\omega_{gc}} = 1.$$  

Furthermore, the phase of $G(j\omega)$ is given by

$$\angle G(j\omega) = \angle(j\omega + 1) - 90^\circ - \angle(j\omega - 1)$$
$$= \angle(j\omega + 1) - 90^\circ - (180^\circ - \angle(j\omega + 1))$$
$$= 2\tan^{-1}(\omega) - 270^\circ.$$  

It follows that the phase margin is given by

$$\varphi_{PM} = 2\tan^{-1}(k) - 90^\circ.$$  

Thus $\varphi_{PM} \geq 30^\circ$ if and only if $\tan^{-1}(k) \geq (90^\circ + 30^\circ)/2 = 60^\circ$, which holds if and only if $k \geq \sqrt{3}$.  