Closed book, closed notes, calculators OK. The Signal Processing problem consists of two parts, each comprising multiple questions, points as marked (point sum = 40 = perfect score); nominal duration = 1 hour.

Part A: Filtering, (down-)sampling, and the roots of unity [sum of points = 25]

- **[3 points]** For \( a \in \mathbb{C} \), define \( s(N) := 1 + a + a^2 + \cdots + a^{N-1} \). Express \( s(N+1) \) as a function of \( s(N) \) in two different ways, and use the resulting expressions to derive a closed-form expression for \( s(N) \).

  \[
  s(N+1) = s(N) + a^N = 1 + as(N) \Rightarrow (1 - a)s(N) = 1 - a^N \Rightarrow s(N) = \frac{1 - a^N}{1 - a}.
  \]

- **[5 points]** Consider a discrete-time filter with impulse response

  \[ h(k) = \begin{cases} 
  1, & 0 \leq k \leq N - 1 \\
  0, & \text{otherwise}
  \end{cases} \]

  where \( k \) is the discrete time variable, and \( N \) is a non-negative integer constant (the length of the impulse response). Compute the discrete-time Fourier transform \( H(e^{j\omega}) := \sum_{k=-\infty}^{+\infty} h(k)e^{-j\omega k} \).

  \[
  H(e^{j\omega}) = \sum_{k=0}^{N-1} e^{-j\omega k} = \sum_{k=0}^{N-1} (e^{-j\omega})^k = \frac{1 - e^{-jN\omega}}{1 - e^{-j\omega}}.
  \]

- **[5 points]** Suppose that a discrete-time signal \( x(k) \) is input to the above filter, and the output \( y(k) = 0 \), \( \forall k \). What can you say about (i.e., can you characterize) the input signal \( x(k) \) in this case? Be as specific as you can.

  Notice that \( H(e^{j\omega}) \) is zero at all \( \omega \) for which \( e^{-j\omega N} = 1 \) (and \( e^{-j\omega} \neq 1 \)), i.e., at \( \omega N = \ell 2\pi \Leftrightarrow \omega = \frac{2\pi \ell}{N}, \ell \in \{1, \ldots, N - 1\} \) (\( \pm \) integer multiples of \( 2\pi \)). At \( \omega = 0 \), on the other hand, \( H(e^{j0}) = \sum_{k=0}^{N-1} e^{-j0} = N \). Thus the filter suppresses all frequency content at \( \omega = \frac{2\pi \ell}{N}, \ell \in \{1, \ldots, N - 1\} \) and passes all other frequencies. When \( y(k) = 0 \), \( \forall k \), what you can say is that \( x(k) \) is confined to have content only at \( \omega = \frac{2\pi \ell}{N}, \ell \in \{1, \ldots, N - 1\} \). In particular, \( x(k) \) need not be the all-zero signal.

- **[5 points]** Now assume that the discrete-time Fourier transform \( X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x(k)e^{-j\omega k} \) of \( x(k) \) satisfies \( X(e^{j\omega}) = 0 \), \( \forall |\omega| > \omega_c \). Find a condition on \( \omega_c \) that allows recovery of the input signal \( \{x(k)\}_{k=-\infty}^{+\infty} \) from the output signal \( \{y(k)\}_{k=-\infty}^{+\infty} \).

  The first zero of \( H(e^{j\omega}) \) occurs at frequency \( \omega = \frac{2\pi \ell}{N} \), so if \( \omega_c < \frac{2\pi \ell}{N} \), then recovery of the input signal is theoretically possible via ‘inverse filtering’ by \( \frac{1}{H(e^{j\omega})} \) over the sub-band of interest.
[2 points] Suppose you down-sample the filter’s output \( \{y(k)\}_{k=-\infty}^{\infty} \) by a factor of \( N \), and let \( \{v(k)\}_{k=-\infty}^{\infty} \) with \( v(k) := y(Nk), \forall k \in \mathbb{Z} \), be the down-sampled signal. Under what condition can you recover the signal \( \{y(k)\}_{k=-\infty}^{\infty} \) from the signal \( \{v(k)\}_{k=-\infty}^{\infty} \)? (no need to prove the (down-)sampling theorem here, simply state the condition).

The condition is that \( \{y(k)\}_{k=-\infty}^{\infty} \) should be bandlimited to within \( \omega_c < \frac{\pi}{N} \). This will be true if and only if \( \{x(k)\}_{k=-\infty}^{\infty} \) is bandlimited to within \( \omega_c < \frac{\pi}{N} \).

[5 points] Write out an expression for \( v(k) \) as a function of the filter’s input signal. Combining the above results, state a condition under which you can exactly recover the filter’s input signal \( \{x(k)\}_{k=-\infty}^{\infty} \), from the down-sampled version \( \{v(k)\}_{k=-\infty}^{\infty} \) of the filter’s output signal. Comment on the result.

\[ v(k) = \sum_{m=1}^{N} x((k-1)N + m). \]

The condition is simply that \( \{x(k)\}_{k=-\infty}^{\infty} \) should be bandlimited to within \( \omega_c < \frac{\pi}{N} \). Under this condition, \( \{x(k)\}_{k=-\infty}^{\infty} \) can be perfectly recovered from its partial non-overlapping sums. This speaks for the ‘severity’ of the bandlimited assumption, which is often taken ‘lightly’.
Part B: Phase retrieval? [sum of points = 15]

- [5 points] Show by simple counter-example that it is generally impossible to recover a discrete-time signal \( \{ x(k) \}_{k=-\infty}^{\infty} \) from only the magnitude \( |X(e^{j\omega})| \) of its discrete-time Fourier transform \( X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k} \). I.e., show that we need the phase of \( X(e^{j\omega}) \) too, \( |X(e^{j\omega})| \) is not enough.

A simple counter-example is that \( x(k) \) and \( x(k-\ell) \) have the same Fourier transform magnitude, so you cannot recover the ‘correct’ shift. E.g., \( \delta(k) \leftrightarrow 1 \) and \( \delta(k-\ell) \leftrightarrow e^{-j\omega \ell} \).

- [10 points] Describe how you can construct a non-trivial class of signals \( \{ x(k) \}_{k=-\infty}^{\infty} \) for which perfect recovery of \( \{ x(k) \}_{k=-\infty}^{\infty} \) is possible from \( |X(e^{j\omega})| \) only. Hint: How can you generate \( X(e^{j\omega}) \geq 0 \) (non-negative real) \( \forall \omega \)?

Let \( x(k) \) be the convolution of \( h(k) \) and \( g(k) := h^*(-k) \), where \( * \) stands for complex conjugate. Then

\[
G(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h^*(-k)e^{-j\omega k} = \sum_{k=-\infty}^{\infty} h^*(k)e^{j\omega k} = \left( \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \right)^* = H^*(e^{j\omega}),
\]

and since convolution in time corresponds to multiplication in frequency,

\[
X(e^{j\omega}) = H(e^{j\omega})G(e^{j\omega}) = H(e^{j\omega})H^*(e^{j\omega}) = |H(e^{j\omega})|^2 \geq 0, \ \forall \omega.
\]

So we only need to take the inverse Fourier transform of \( |X(e^{j\omega})| \), which is equal to the inverse Fourier transform of \( X(e^{j\omega}) \).

Remark: This is in fact the only way to generate \( X(e^{j\omega}) \geq 0 \) (non-negative real) \( \forall \omega \). The only if part is the Fejér-Riesz spectral factorization theorem, but I don’t ask you to prove this here.