Q. 1

Consider the feedback system shown in the figure with transfer function $G(s) = \frac{e^{-1}}{(s + 10)^2}$ and a controller of the form $C(s) = \frac{k}{s^2 + as + b}$, where $k > 0$.

1. Determine values of $a$, $b$ and a range of values for $k$, so that the feedback system can track a ramp input $r(t) = t$, $t > 0$ with zero steady state error.

2. A block diagram of a servo system for motion control is shown in the Figure below:

The plant transfer function is given by $G(s) = \frac{1}{s + 10}$. For the unity feedback setup, design a Proportional-Derivative controller such that the resulting closed loop system has a damping $\zeta = \frac{1}{\sqrt{2}} = 0.707$ and a natural frequency $\omega_n = 8 \text{ rad/s}$. (Note that a second order system of the form $\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ has a damping $\zeta$ and natural frequency $\omega_n$.)

Q 1(a): The open loop gain is $L = GK = \frac{k(s+1)}{(s^2 + as + b)(s + 10)^2}$. The transfer function with the error $E$ as output and the input as $R$ is

$$\frac{E(s)}{R(s)} = \frac{1}{1 + L}.$$ 

If $R = \frac{1}{s^2}$ if $r$ is a ramp input. If the ramp has to be tracked with zero steady state error then from the internal model principle, the controller $K$ has to include two poles at zero. Setting $a = b = 0$ we get the controller to be $k/s^2$ which has two poles at zero. Now we have to determine $k$ for stability. The characteristic polynomial in this case is

$$s^2(s + 10)^2 + k(s + 1) = s^4(s^2 + 20s + 100) + ks + 1 = s^4 + 20s^3 + 100s^2 + ks + k$$

We use the Routh Hurwitz criterion

$$s^4: \quad 1 \quad 100 \quad k$$
$$s^3: \quad 20 \quad k$$
$$s^2: \quad \frac{2000-k}{20} \quad \frac{20k}{20}$$
$$s^1: \quad \frac{(100-k/20)(10k-20k)}{100-k/20} \quad k$$
$$s^0: \quad$$
Thus we need \(100 - k/20 > 0\), \(k > 0\) and \(\frac{(100-k/20)k-20k}{100-k/20} > 0\). Thus \(0 < k < 2000\) and \(100 - k/20 - 20 > 0\). Thus \(0 < k < 1600\). Thus the controller can be of the form \(k/s^2\) where \(0 < k < 1600\).

**Q 1 (b):** Note that \(L = \frac{k_p + k_d s}{s(s+10)}\) and the closed-loop transfer function is \(\frac{L}{1+L} = k_p + k_d s^2 + (k_d + 10)s + k_p\). Thus we need \(k_p = \omega_n^2 = 8^2 = 64\) and \(k_d + 10 = 2\zeta\omega_n = 16/\sqrt{2} = 8\sqrt{2}\). Thus \(k_p = 64\) and \(k_d = 8\sqrt{2} - 10 = 1.313\).
Q. 2
Consider the figure given below. Design a lead controller \( G_c(s) = \frac{T s + 1}{\alpha T s + 1} \) for the plant \( G(s) = \frac{1}{s^2} \) such that the phase margin is 45°. The bode plot of \( G(s) \) is given below. *Hint:* The frequency \( \omega_m \) at which the phase of \( G_c(j\omega) \) is a maximum is \( \omega_m = \frac{1}{T \sqrt{\alpha}} \).

![Bode Plot](image)

**Solutions to Q. 2:** The phase of the plant is constant at -180°. Thus to have a phase margin of 45°, the controller needs to have a phase of 45° at gain crossover. We will choose the gain crossover frequency \( \omega_c = \omega_m \) where \( \omega_m = \frac{1}{T \sqrt{\alpha}} \), where the lead controller has the maximum phase. Thus we need \( \phi_m := \angle \{G_c(j\omega_m)\} = \pi/4 \).

This implies that

\[
\sin \phi_m = \frac{1 - \alpha}{1 + \alpha} \Rightarrow \frac{1}{\sqrt{2}} = \frac{1 - \alpha}{1 + \alpha} \Rightarrow \alpha = 0.1716
\]

We also need \( \omega_m \) to be the gain crossover and therefore

\[
\frac{|G_c(j\omega_m)||G(j\omega_m)|}{\sqrt{1 + \omega_m^2 T^2}} = 1
\]

\[
\Rightarrow \sqrt{1 + \omega_m^2 T^2} |G(j\omega_m)| = 1
\]

\[
\Rightarrow \frac{1 + \omega_m^2 T^2}{\sqrt{1 + \omega_m^2 T^2}} |G(j\omega_m)| = 1
\]

\[
\Rightarrow \frac{1 + \omega_m^2 T^2}{\sqrt{1 + \omega_m^2 T^2}} = \frac{1}{\omega_m T}
\]

Thus \( \omega_m = 1.5537 \) and \( T = \frac{1}{\omega_m \sqrt{\alpha}} = 1.5537 \) and \( G_c = \frac{1.5537 s + 1}{0.2666 s + 1} \).
Q. 3
Consider the systems with the following transfer functions:

\[ G_A(s) = \frac{1}{s^2 + 0.5s + 1} \]
\[ G_B(s) = \frac{4}{s^2 + 1s + 4} \]
\[ G_C(s) = \frac{3}{s^2 + 2s + 3} \]
\[ G_D(s) = \frac{6}{s^2 + 2s + 4} \]
\[ G_E(s) = \frac{4}{s^2 + 2s + 4} \]
\[ G_F(s) = \frac{12}{s^2 + 2s + 8} \]

You are required to match these with the unit step responses shown below (Hint: calculate the damping \( \zeta \), the natural frequency \( \omega_n \), for each system and the corresponding steady state output values).

Solutions to Q. 30)

\[ G_A(s) = \frac{1}{s^2 + 0.5s + 1}, \quad \omega_n = 1, \quad \zeta = 0.25, \quad x_\infty = 1 \]
\[ G_B(s) = \frac{4}{s^2 + 1s + 4}, \quad \omega_n = 2, \quad \zeta = 0.25, \quad x_\infty = 1 \]
\[ G_C(s) = \frac{3}{s^2 + 2s + 3}, \quad \omega_n = 1, \quad \zeta = 0.5, \quad x_\infty = 1.5 \]
\[ G_D(s) = \frac{6}{s^2 + 2s + 4}, \quad \omega_n = 2, \quad \zeta = 0.5, \quad x_\infty = 1.5 \]
\[ G_E(s) = \frac{4}{s^2 + 2s + 4}, \quad \omega_n = 2, \quad \zeta = 0.5, \quad x_\infty = 1 \]
\[ G_f(s) = \frac{12}{s^2 + 2s + 8}, \ \omega_n = 2, \ \zeta = 0.25, \ \infty = 1.5 \]

(I), (II), (IV) have \( \infty = 1 \). Thus these have to be matched with \( G_A \), \( G_B \) and \( G_E \). I has the least peak (indicates greatest damping and thus \( G_E \) corresponds to I. Also (IV) settles slower than II and thus \( G_A \) corresponds to IV and \( G_B \) to II.

Of the cases with \( \infty = 1.5 \), V is the least damped and thus \( G_F \) corresponds to V. III settles faster than VI and thus \( G_D \) corresponds to III and \( G_C \) to VI.