Confinement Condition for a Gaussian Beam

Consider a Gaussian beam whose width (beam radius) \( W(z) \) and wavefront radius of curvature \( R(z) \) are given by

\[
W(z) = W_0 \sqrt{1 + \left( \frac{z}{z_0} \right)^2} \quad \quad R(z) = z \left[ 1 + \left( \frac{z_0}{z} \right)^2 \right]
\]

A Gaussian beam reflected from a spherical mirror will retrace the incident beam if the radius of curvature of its wavefront is the same as that of the mirror radius. Let’s fit a Gaussian beam to two mirrors separated by a distance \( d \), as shown below. Their radii of curvature are \( R_1 \) and \( R_2 \). Both mirrors are taken to be concave (i.e. \( R_1 < 0 \) and \( R_2 < 0 \)).

The center of the beam is assumed to be at the origin \( z = 0 \); mirrors \( R_1 \) and \( R_2 \) are located at positions \( z_1 \), and \( z_2 = z_1 + d \).

Prove the following relations:

*Pay careful attention to the sign of \( R(z) \): A concave mirror has a negative radius (i.e. \( R_1 < 0 \) and \( R_2 < 0 \)). But the beam radius of curvature is defined to be positive for \( z > 0 \) (right mirror) and negative for \( z < 0 \), meaning \( R_1 = R(z_1) \), \( -R_2 = R(z_2) \)

\[
\begin{align*}
    z_1 &= \frac{-d(R_2 + d)}{R_2 + R_1 + 2d} \\
    z_0^2 &= \frac{-d(R_1 + d)(R_2 + d)(R_1 + R_2 + d)}{(R_2 + R_1 + 2d)^2}
\end{align*}
\]

(a) (1 point) \quad (b) (1 point)

In order that the above solution represents a Gaussian beam, \( z_0 \) must be real. Using that condition, derive the following condition for the confinement of a Gaussian beam:

\[
0 \leq \left( 1 + \frac{d}{R_1} \right) \left( 1 + \frac{d}{R_2} \right) \leq 1
\]

(c) (2 points)