1) [7 points total] A finite sequence $b[n]$ is such that its $z$-Transform $B(z) = \sum_{n=-\infty}^{\infty} b[n]z^{-n}$ satisfies

$$B(z) + B(-z) = 2c,$$

where $c \neq 0$ is a real constant.

a) [5 points] What constraints does this place on the signal $b[n]$? In particular, what can be said about $b[n]$ for the following cases?

- $n = 0$,
- $n$ odd,
- $n$ even ($n \neq 0$)

b) [2 points] Give one example of such a sequence $b[n]$. 
2) [20 points total] Consider a real valued sequence $x[n]$ whose Fourier transform $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$ is known to satisfy 

$$X(e^{j\omega}) = 0, \quad \text{for } \frac{\pi}{4} \leq |\omega| \leq \pi.$$ 

One value of the sequence $x[n]$ may be corrupted; we would like to devise a strategy to recover it.

Let $g[n]$ denote the corrupted signal, which is real-valued, and let $n_0$ denote the (unknown) location of the corrupted value. Thus, we can write

$$g[n] = x[n], \quad n \neq n_0,$$

and

$$g[n] = x[n] + a\delta[n - n_0],$$

for some real-valued $a$. Our recovery approach begins by filtering $g[n]$ with an ideal high pass filter $h[n]$ with cutoff frequency $\omega_c = \pi/2$.

a) [5 points] Find an expression for the entries of the filter $h[n]$. (You might choose to use the Fourier inversion formula: $h[n] = \frac{1}{2\pi} \int_{2\pi} H(e^{j\omega}) e^{j\omega n} d\omega$.)

b) [2 points] Let $y[n] = g[n] * h[n] = (g[n] = x[n] + a\delta[n - n_0]) * h[n]$. Use your expression from part (a) to find a simple expression for $y[n]$. 
c) [5 points] Explain how you could identify $n_0$ by examining the values of the filtered sequence $y[n]$.
(Hint: It may help to sketch $y[n]$.)

d) [3 points] Find a simple expression for the unknown amplitude $a$ in terms of the original sequence $g[n]$.
(Your answer will be a function of $n_0$, which you would be able to identify following the procedure you describe in part (c)).

e) [5 points] Find a (simple) expression for the original sequence $x[n]$ in terms of the quantities $g[n], y[n]$, and the (now) known $n_0$. 
3) [13 points total] Suppose we wish to compute the autocorrelation function of an upsampled real signal using the system below

\[ x[n] \xrightarrow{L} x_u[n] \xrightarrow{\text{Ideal lowpass filter with cutoff } \pi/L} x_i[n] \xrightarrow{\text{Autocorrelate}} \phi_1[m] = \sum_{n=-\infty}^{\infty} x_i[n+m]x_i[n] \]

It was suggested that this can be accomplished by a different system, for an appropriate \( H(e^{j\omega}) \)

\[ x[n] \xrightarrow{\text{Autocorrelate}} \phi_2[m] \xrightarrow{L} \phi_{2u}[m] \xrightarrow{H(e^{j\omega})} \phi_3[m] \]

a) [5 points] Find an expression relating the Fourier transform of \( \phi_i[m] \), denoted \( \Phi_i(e^{j\omega}) \), to the Fourier transform of \( x_i[n] \), denoted \( X_i(e^{j\omega}) \).
b) [3 points] How does the bandwidth of $\Phi_1(e^{j\omega})$ compare with the bandwidth of $X_i(e^{j\omega})$?

c) [5 points] What should $H(e^{j\omega})$ be to ensure that the outputs of the two systems are the same (i.e., to ensure that $\phi_3[m] = \phi_1[m]$ for all $m$)?