Problem 1 [40 pts] Assume that an uncompensated operational amplifier (Op-Amp) has the following transfer function from the differential voltage $V_p - V_n$ to the output voltage $V_o$.

$$a(s) = \frac{10^5}{(1 + 10^{-4}s)(1 + 10^{-6}s)}$$

![Figure 1: The Op-Amp with the static compensator](image)

1. (5pts) Carefully sketch the bode plot corresponding to $a(s)$. Make sure that your axis are correctly labeled and the asymptotes are evident.

2. (10pts) If the Op-Amp is compensated by the resistive network shown in Figure 1, Provide values for the two compensator resistors so that the closed loop transfer function from $V_p$ to $V_o$ has a DC gain equal to 10.

To answer this question, assume that $V_o$ is not affected by the load provided by the compensator network, and that the current flowing into the (-) terminal is zero, thus that $V_n$ is determined by the compensator transfer function. The resistive network realizes a proportional controller; assume this gain to be $K$.

3. (10 pts) Describe what is the unit step response you expect from the compensated amplifier, and why. Hint: Find the gain cross-over frequency and phase margin.

4. (5pts) Provide the relationship between the pole and zero of a first order lead compensator. To improve the response, the control engineers decide to use a lead compensator. Such a compensator can be built by putting a capacitor in parallel to one of the resistors. Modify the compensator circuit using a capacitor that can realize a lead compensator behavior.

5. (10 pts) Keeping the values of the resistors you have found in Part 2, find a value of the capacitor, for the lead compensator in Part 4, that improves the closed loop step response without reducing the closed loop bandwidth. Explain. Hint: Increase the phase margin.

Solution

1. The Bode plot is shown in Figure 2.

2. The compensator is static gain of value

$$K = \frac{R_2}{R_1 + R_2}.$$ 

Using the assumption that that the output impedance is zero and the input impedance is infinity, we have that

$$V_o(s) = a(s)(V_p(s) - V_n(s)) = a(s)(V_p(s) - KV_o(s))$$
Thus, the closed loop transfer function is given by the following expression:

\[
\frac{V_o(s)}{V_p(s)} = \frac{a(s)}{1 + a(s)K}.
\]

**DC gain** = \(\frac{a(0)}{1 + a(0)K} \approx \frac{1}{K}\) for \(a(0) \gg 1\).

It follows that \(K = 0.1\) satisfies the DC gain specification. Such \(K\) can be achieved by selecting

\[R_1 = 90k\Omega, \quad R_2 = 10k\Omega\]

3. Since the closed loop system is stable for all \(K > 0\), the DC gain corresponds to the steady state step response.

However, the open loop transfer function \(L(s) = a(s)K\), which, with \(K = 0.1\), has a cross-over frequency at \(\omega_0 \approx 10^7\text{rad/sec}\). At \(\omega_0\) the phase margin is very small (less than 5°). This result in a closed loop system with a lightly damped resonant mode around \(\omega_0\).

Thus, although the unit step response converges to 10 in steady state, the transient has a large overshoot and high frequency lightly damped oscillations.

4. By adding a capacitor in parallel to \(R_1\), the compensator becomes dynamic with the following transfer function:

\[
K(s) = \frac{R_2}{R_1 + \frac{1}{sC} + R_2} = \frac{R_2}{R_1 + \frac{1}{sC} + R_2} = \frac{R_2(1 + sCR_1)}{R_1 + R_2 + sCR_1R_2} = \frac{R_2}{R_1 + R_2} \frac{1 + sCR_1}{R_1 + R_2 + sC\frac{R_1R_2}{R_1 + R_2}}
\]

The compensator zero is at \(z = -\frac{1}{sCR_1}\) and the pole is at \(p = -\frac{1}{sCR_1} - \frac{1}{sCR_2}\). With \(C = 2pF = 2\cdot10^{-12}F\), we obtain

\[\omega \approx 5.55\cdot10^6\text{rad/sec}, \quad p \approx 5.55\cdot10^7\text{rad/sec}\]

These correspond to

\[
K(s) = \frac{1.8\cdot10^{-8}s + 0.1}{1.8\cdot10^{-8}s + 1}
\]

The compensated open loop Bode plot can be sketched easily using asymptotes since \(z\) and \(p\) are a decade apart for each other. The new cross-over frequency is about \(1.1\cdot10^7\text{rad/sec}\) which correspond to a phase margin of about 40° degree.
Figure 3: Lead compensated open loop Bode plot.