1) [15 points] Consider discrete-time LTI system depicted below.

**Figure 1:** Discrete-time LTI system.

Suppose that $H(z) = (1 - 2z^{-1})(1 - \frac{1}{2}z^{-1})$ with ROC $|z| > 0$. Find the impulse response $g[n]$ of a stable system $G$ such that $w[n] = x[n-2]$.

*Hint: Recall the following z-Transform relations, which you may find useful:*

$x[n] = \alpha^n u[n] \leftrightarrow X(z) = \frac{1}{1 - \alpha z^{-1}}, \text{ ROC: } |z| > |\alpha|$

$x[n] = -\alpha^n u[-n-1] \leftrightarrow X(z) = \frac{1}{1 - \alpha z^{-1}}, \text{ ROC: } |z| < |\alpha|$.

**SOLUTION:**

The problem asks for $G(z)$ such that

$H(z)G(z) = z^{-2},$

or

$G(z) = \frac{z^{-2}}{(1 - 2z^{-1})(1 - \frac{1}{2}z^{-1})},$

where the constraint that $G(z)$ be stable implies that the unit circle $|z| = 1$ must be in the ROC of $G(z)$.

To find the impulse response $g[n]$, we expand $G(z)$ using partial fraction expansion. Note that an equivalent expression for $G(z)$ is

$G(z) = \frac{1}{(z-2)(z-\frac{1}{2})},$

and to factor $G(z)$, we seek $A, B$ such that

$G(z) = \frac{1}{(z-2)(z-\frac{1}{2})} = \frac{A}{z-2} + \frac{B}{z-\frac{1}{2}}.$

The solutions are obtained by

$A = (z-2)G(z) \bigg|_{z=2}, \quad B = \left(z-\frac{1}{2}\right)G(z) \bigg|_{z=\frac{1}{2}},$

giving $A = 2/3$, and $B = -2/3.$
Thus,

\[
G(z) = \frac{1}{(z - 2)(z - \frac{1}{2})} = \frac{(2/3)}{z - 2} \frac{(2/3)}{z - \frac{1}{2}} = \frac{(2/3)z^{-1}}{1 - 2z^{-1}} - \frac{(2/3)z^{-1}}{1 - \frac{1}{2}z^{-1}}.
\]

Now, since the ROC of \(G(z)\) must contain the unit circle (\(|z| = 1\)), we must choose the ROC of the first term to include the region \(|z| < 2\), and the ROC for the second term to include \(|z| > 1/2\). Using the expressions given in the hint, and the fact that the \(z^{-1}\) term simply induces a delay of one unit, we obtain

\[
g[n] = -(2/3)2^{n-1}u[-n] - (2/3) \left(\frac{1}{2}\right)^{n-1} u[n - 1]
\]
2) [10 points] Recall that the discrete-time convolution operation satisfies several standard arithmetic properties. Namely, for all functions \( f[n], g[n], \) and \( h[n] \), we have:

- **Commutativity**: \( f[n] * g[n] = g[n] * f[n] \)
- **Associativity**: \( f[n] * (g[n] * h[n]) = (f[n] * g[n]) * h[n] \)
- **Distributivity**: \( f[n] * (g[n] + h[n]) = (f[n] + g[n]) * (f[n] + h[n]) \)

Note, however, that for arbitrary functions \( g[n], h[n] \), the relationship

\[
f[n] (g[n] * h[n]) = (f[n]g[n]) * (f[n]h[n])
\]

(1)

does not hold for all \( f[n] \), though it does hold for the trivial cases \( f[n] = 1 \) and \( f[n] = 0 \).

Besides the trivial examples given above, find another function \( f[n] \) for which (1) holds for all \( g[n], h[n] \).

**SOLUTION:**

Recall the definition of the discrete-time convolution sum: for discrete time functions \( g[n], h[n] \),

\[
g[n] * h[n] = \sum_{k=-\infty}^{\infty} g[k]h[n-k].
\]

Using this, we can express the relationship (1) as

\[
f[n] \sum_{k=-\infty}^{\infty} g[k]h[n-k] = \sum_{k=-\infty}^{\infty} f[k]g[k]f[n-k]h[n-k],
\]

or

\[
\sum_{k=-\infty}^{\infty} f[n]g[k]h[n-k] = \sum_{k=-\infty}^{\infty} f[k]f[n-k]g[k]h[n-k],
\]

from which we see that the relationship (1) is satisfied provided \( f[n] \) satisfies

\[
f[n] = f[k]f[n-k]
\]

for all \( k \). This condition is satisfied if \( f[n] \) is any exponential function of the form

\[
f[n] = \alpha^n,
\]

where \( \alpha \in \mathbb{C} \). Choosing \( \alpha \neq \{0, 1\} \) gives a nontrivial function \( f[n] \) for which (1) holds.
3) Consider the system shown below.

The anti-aliasing filter $H_c(j\Omega)$ is an ideal low-pass continuous-time filter with unit gain and cutoff frequency $\pi/T$. The frequency response of the LTI discrete time system between the converters is given by:

$$H_d(e^{j\omega}) = j\omega e^{-j\omega/3}, \quad |\omega| < \pi.$$

Assume that the C/D and D/C converters are ideal.

a) **[5 points]** Find the impulse response $h_d[n]$ of the discrete-time LTI system.

**SOLUTION:**

Recall the inversion formula for discrete time Fourier transforms:

$$h_d[n] = \frac{1}{2\pi} \int_{2\pi} H_d(e^{j\omega})e^{jn\omega} d\omega,$$

where the integral is over any interval of length $2\pi$. Choosing the interval of integration to be $[-\pi, \pi]$, we have

$$h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} j\omega e^{-j\omega/3} e^{jn\omega} d\omega$$

$$= \frac{j}{2\pi} \int_{-\pi}^{\pi} \omega e^{j\omega(n-1/3)} d\omega.$$

Perform integration by parts: let $u = \omega$ and $dv = e^{j\omega(n-1/3)} d\omega$. Then $du = d\omega$ and $v = \frac{e^{j\omega(n-1/3)}}{j(n-1/3)}$.

Now,

$$h_d[n] = \frac{j}{2\pi} \int_{-\pi}^{\pi} \omega e^{j\omega(n-1/3)} d\omega$$

$$= \frac{j}{2\pi} \left[ \omega \frac{e^{j\omega(n-1/3)}}{j(n-1/3)} \right]_{-\pi}^{\pi} - \frac{j}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-1/3)} \frac{1}{j(n-1/3)} d\omega.$$

For the first term, note that

$$\frac{j}{2\pi} \left[ \omega \frac{e^{j\pi(n-1/3)}}{j(n-1/3)} \right]_{-\pi}^{\pi} = \frac{j}{2\pi} \left[ \frac{e^{j\pi(n-1/3)}}{j(n-1/3)} + \frac{e^{j\pi(n-1/3)}}{j(n-1/3)} \right]$$

$$= \frac{\cos[\pi(n-1/3)]}{(n-1/3)}.$$
For the second, we have

\[-\frac{j}{2\pi} \int_{-\pi}^{\pi} \frac{e^{j\omega(n-1/3)}}{j(n-1/3)} d\omega = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{j\omega(n-1/3)}}{(n-1/3)} d\omega = -\frac{1}{2\pi} \left[ \frac{e^{j\pi(n-1/3)} - e^{-j\pi(n-1/3)}}{2j} \right] \]

Combining the results, we have

\[h_d[n] = \frac{\cos[\pi(n-1/3)]}{(n-1/3)} - \frac{\sin[\pi(n-1/3)]}{\pi(n-1/3)^2}.\]
b) **[5 points]** What is the effective continuous-time frequency response of the overall system, \( H(j\Omega) \)?

**SOLUTION:**

Note that because of the filter \( H_c(j\Omega) \), \( x(t) \) is bandlimited \( (X(j\Omega) = 0 \text{ for } |\Omega| > \pi/T) \). Thus, the continuous-time input and continuous-time output of the subsystem shown in Figure 3 are related by

\[
Y(j\Omega) = X(j\Omega) \cdot H_d(e^{j\omega}) \bigg|_{\omega=\Omega T}.
\]

This implies \( H_{C\rightarrow D\rightarrow C}(j\Omega) = H_d(e^{j\Omega T}) = j\Omega T e^{-j\Omega T/3} \). Overall, the frequency response of the entire system is given by

\[
H(j\Omega) = \begin{cases} 
    j\Omega T e^{-j\Omega T/3}, & |\Omega| < \pi/T \\
    0, & \text{otherwise}
\end{cases}
\]

Given this overall frequency response, we see that for **bandlimited inputs** the net effect of this system would be amplification by the factor \( T \), differentiation, and time delay by \( T/3 \).

c) **[5 points]** Choose the most accurate statement:

(i) \( y(t) = x(t - 1/3) \)

(ii) \( y(t) = \frac{d}{dt} x(3t) \)

(iii) \( y(t) = T \frac{d}{dt} x(t - 1/3) \)

(iv) \( y(t) = T \frac{d}{dt} x(t - T/3) \) \( \Leftarrow \text{by the above argument} \)

(v) \( y(t) = \frac{d}{dt} x(t - 1/3) \)

(vi) \( y(t) = \frac{d}{dt} x(t - T/3) \)