Completion of partially known turbulent flow statistics

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Turbulent flows

- prevent/suppress turbulence
- turbulent drag reduction

\{ Economic impact \}
Control-oriented modeling of turbulent flows

- **Stochastically forced NS equations**

\[ \psi_t = A \psi + f \]
\[ \mathbf{v} = C \psi \]

Farrell & Ioannou, Phys. Fluids ’93
Bamieh & Dahleh, Phys. Fluids ’01
Jovanović & Bamieh, J. Fluid Mech. ’05
Moarref & Jovanović, J. Fluid Mech. ’12
Control-oriented modeling of turbulent flows

- Stochastically forced NS equations

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Moarref & Jovanović, J. Fluid Mech. '12

Observation

- White-in-time stochastic excitation too restrictive!

Jovanović & Georgiou, APS '10
Proposed approach

- utilize second-order statistics $\rightarrow$ formulate inverse problem
- shape forcing statistics to account for available data
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- utilize second-order statistics  →  formulate inverse problem
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new class of structured matrix-completion problems
Stochastically forced linearized dynamics

Lyapunov equation

- **white-in-time** $u$

\[
AX + XA^* = -BWB^*
\]

- **colored-in-time** $u$

\[
AX + XA^* = -(BH^* + HB^*)
\]

Georgiou, IEEE TAC ’02
Chen, Jovanović & Georgiou, CDC ’13
**Problem data**

**Knowns**
- partially available correlations
- dynamical generator $A$

**Unknowns**
- unavailable statistics
- disturbance dynamics
  - input matrix $B$
  - input power spectrum
Covariance matrix completion problem

\[
\begin{align*}
\text{minimize} \quad & \|M\|_* \\
\text{subject to} \quad & AX + XA^* + M = 0 \\
& \text{trace}(T_k X) = g_k, \quad k = 1, \ldots, N \\
& X \succeq 0
\end{align*}
\]

- \(\|M\|_* := \sum \sigma_i(M) \) → proxy for rank minimization

Fazel, Recht, Parrilo, Candès, ...

- Small problems: use general-purpose SDP solvers
Customized algorithm

Reformulation of constraints

Interior point method - ADMM
Reformulation of constraints

- Linear constraint
  \[ \text{trace}(T_k X) = g_k \quad \rightarrow \quad \mathcal{T}(X) = g \]
  \[ k = 1, \ldots, N \]

- Parameterization
  \[ X = X_0 + \sum_{j=1}^{q} x_j X_j \]
  \[ \mathcal{T}(X_0) = g \quad \mathcal{T}(X_j) = 0 \]

\[
M = -(AX + XA^*) = M_0 + \sum_{j=1}^{q} x_j M_j
\]

Lin, Jovanović & Georgiou, CDC ’13
Interior point method

\[
\begin{align*}
\text{minimize} \quad & \|M\|_* - \frac{1}{\tau} \log \det (X_0 + \sum_{j=1}^{q} x_j X_j) \\
\text{subject to} \quad & \sum_{j=1}^{q} x_j M_j - M + M_0 = 0
\end{align*}
\]
for fixed $\tau$:

\[
\begin{align*}
\text{minimize} & \quad \|M\|_* - \frac{1}{\tau} \log \det (X_0 + \sum_{j=1}^{q} x_j X_j) \\
\text{subject to} & \quad \sum_{j=1}^{q} x_j M_j - M + M_0 = 0
\end{align*}
\]
ADMM

▶ Augmented Lagrangian

\[ \mathcal{L}_\rho(x, M, \Lambda) = \|M\|_* - \frac{1}{\tau} \log \det (X_0 + \sum_{j=1}^{q} x_j X_j) + \text{trace} \left( \Lambda^* \left( \sum_{j=1}^{q} x_j M_j - M + M_0 \right) \right) \]

\[ + \frac{\rho}{2} \| \sum_{j=1}^{q} x_j M_j - M + M_0 \|_F^2 \]

▶ iterative ADMM steps

\[ x^{k+1} := \arg \min_x \mathcal{L}_\rho(x, M^k, \Lambda^k) \]

\[ M^{k+1} := \arg \min_M \mathcal{L}_\rho(x^{k+1}, M, \Lambda^k) \]

\[ \Lambda^{k+1} := \Lambda^k + \rho \left( \sum_{j=1}^{q} x_j^{k+1} M_j - M^{k+1} + M_0 \right) \]
Turbulent channel flow

\[ \Phi(\kappa) := \lim_{t \to \infty} \mathcal{E}(v v^*) \]

\[ v = [v_1 \ v_2 \ v_3]^T \]

► Matrix completion

stochastic forcing \hspace{2cm} \text{linearized dynamics} \hspace{2cm} \text{velocity fluctuations} +

\[
\begin{bmatrix}
\Phi_{11} & \Phi_{12} & \Phi_{13} \\
\Phi_{21} & \Phi_{22} & \Phi_{23} \\
\Phi_{31} & \Phi_{32} & \Phi_{33}
\end{bmatrix}
\]
One-point velocity correlations

- Numerical simulations
- Matrix completion problem (MCP)
Low rank solution \( M = -(AX +XA^*) \)

\[ \sigma_i (M) \text{ feasibility problem} \quad \longrightarrow \quad \text{no clear cut} \]

\[ \sigma_i (M) \text{ covariance matrix completion problem} \quad \longrightarrow \quad \text{approximately low-rank} \]
Matrix completion

Simulations

\[ \Phi_{11} \]

MCP

\[ \Phi_{33} \]
Concluding remarks

- **Motivation**: completion of partially available flow statistics
- account for forcing models of low complexity
- customized interior-point algorithm
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- **Motivation**: completion of partially available flow statistics
- account for forcing models of low complexity
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**Ongoing work**

- model-based control
- filter design
Key ADMM steps

▶ **x-minimization step**  \(\rightarrow\) smooth

\[
\begin{align*}
\text{minimize} & \quad - \frac{1}{\tau} \log \det (X_0 + \sum_{j=1}^{q} x_j X_j) + \frac{\rho}{2} \left\| \sum_{j=1}^{q} x_j M_j - U^k \right\|^2_F
\end{align*}
\]

▶ **M-minimization step**  \(\rightarrow\) analytical solution

\[
\begin{align*}
\text{minimize} & \quad \| M \|_* + \frac{\rho}{2} \left\| M - V^k \right\|^2_F
\end{align*}
\]

\[
M^{k+1} = U \text{ diag} \left( \sigma_i - 1/\rho \right)_+ U^*
\]

Relation to matrix completion problems

\[ \begin{align*} 
\text{minimize} \quad & \operatorname{rank} (M) \\
\text{subject to} \quad & AX + XA^* + M = 0 \\
& \operatorname{trace} (T_i X) = g_i, \quad i = 1, \ldots, N \\
& X = X^* \succeq 0 
\end{align*} \]

Setting \( A = -\frac{1}{2} I \) \( T_i = e_i e_j^T \)

\[ \begin{align*} 
\text{minimize} \quad & \operatorname{rank} (X) \\
\text{subject to} \quad & X_{jl} = Q_{jl}, \quad (j, l) \in \Omega \\
& X = X^* \succeq 0 
\end{align*} \]