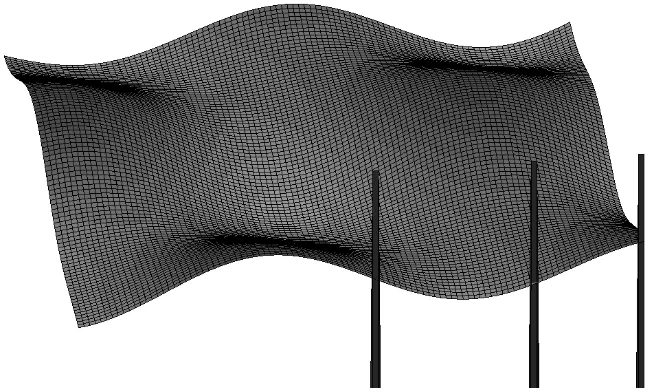
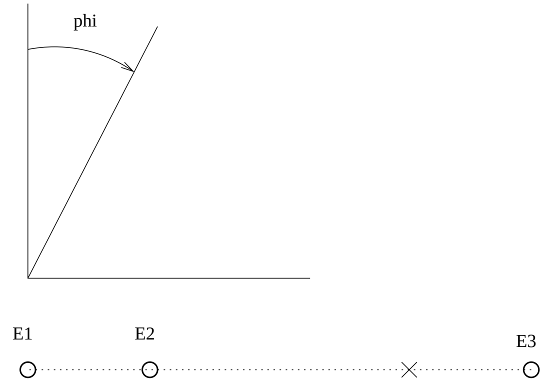
Moment Problems and Relative Entropy

Sensor Arrays, Source localization and Multidimensional Spectral Estimation. . .

Tryphon Georgiou
University of Minnesota





Sensor readings: $u_\ell(t) = \int A(\theta) e^{j(\omega t - p x_\ell \cos(\theta) + \phi(\theta))} d\theta$

Correlations: $\boxed{R_k = E\{u_{\ell_1}\bar{u}_{\ell_2}\} := \int \overbrace{e^{-jk\cos(\theta)}}^{g_k(\theta)} \overbrace{f(\theta)d\theta}^{d\mu}}$ with $f = A(\theta)^2$,

$$k \in \{0, 1, \sqrt{2}, \sqrt{2}+1\}.$$

Given R_0, R_1, $R_{\sqrt{2}}$, $R_{\sqrt{2}+1}$

(i) how can we tell they come from an $f > 0$?

(ii) how can we recover f?

(iii) how can we describe all admissible f's?

$$\int \left(\begin{bmatrix} 1 \\ e^{-j\tau} \\ e^{-j\sqrt{2}\tau} \end{bmatrix} \overbrace{f(\theta)d\theta}^{d\mu} \begin{bmatrix} 1 & e^{j\tau} & e^{j\sqrt{2}\tau} \end{bmatrix} \right) = \begin{bmatrix} R_0 & R_1 & R_{\sqrt{2}+1} \\ \bar{R}_1 & R_0 & R_{\sqrt{2}} \\ \bar{R}_{\sqrt{2}+1} & \bar{R}_{\sqrt{2}} & R_0 \end{bmatrix} \geq 0$$

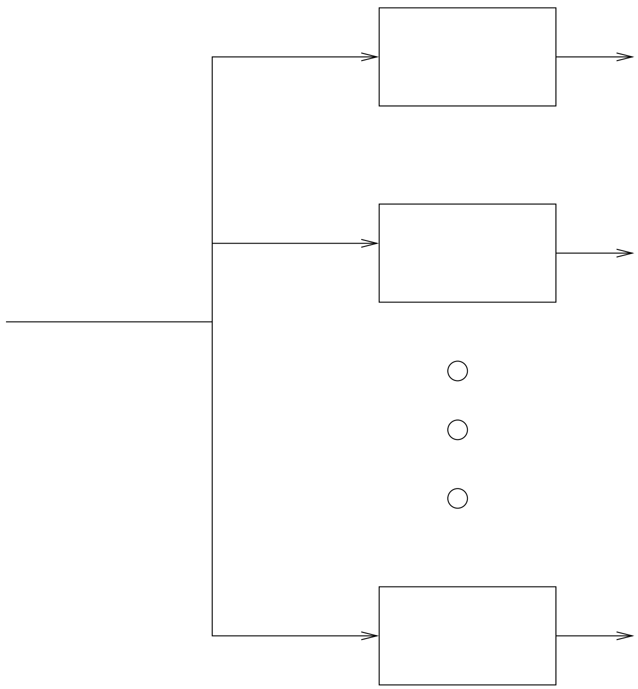
necessary but not sufficient

Given R_0, R_1, R_{100}, with $R_2, \ldots R_{99}$ missing,
$\exists$? values for the missing R's so that $T_{100} > 0$?

$$T_{100} := \begin{bmatrix} R_0 & R_1 & x_2? & x_3? & x_4? & \ldots & x_{98}? & R_{99} \\ R_1 & R_0 & R_1 & x_2? & x_3? & \ldots & x_{97}? & x_{98}? \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \end{bmatrix}$$

Solvable in principle via LMI's.

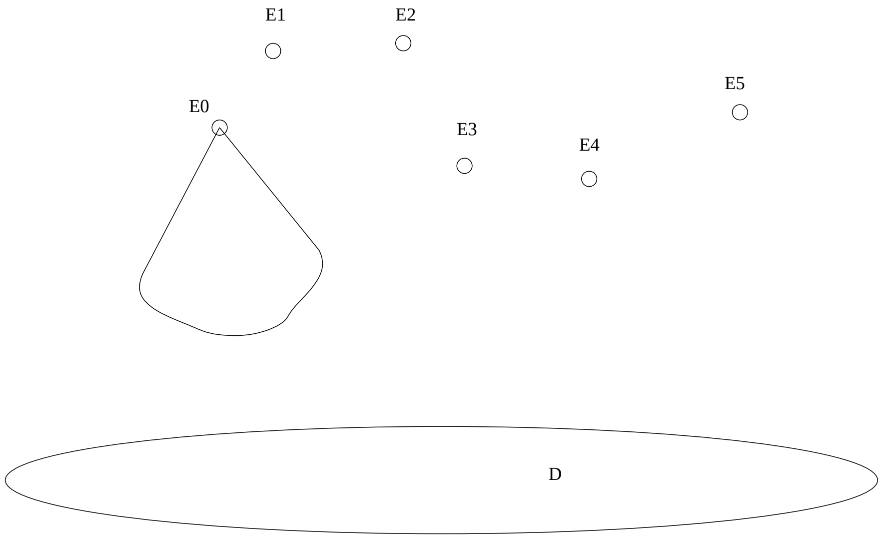
- Low-pass "sensors" $H_k(i\omega) = 1/(1 + i\omega/\tau_k)$
- stochastic input with spectral measure $d\mu(\omega)$
- knowledge of output covariances

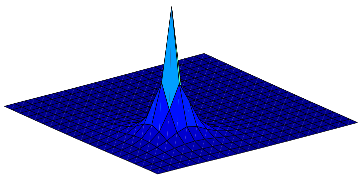


$r_k = \int_{-\infty}^{\infty} g_k(\omega) d\mu(\omega)$, with $g_0 = 1$, $g_k(\omega) = \frac{1}{\omega^2/\tau_k^2+1}$, $k = 0, 1, 2, 3$.

- Scattered "sensors" $E_0, E_1, \ldots$
with Green's functions/transfer functions/etc. $g_k(\omega, \theta)$
- stochastic excitation with spectral measure $d\mu(\omega, \theta)$
- knowledge of correlations of sensor readings

$$r_{k,\ell} = \int g_k(\omega, \theta) d\mu(\omega, \theta) g_\ell(\omega, \theta)$$





- Determine $d\mu$
- Effect of g_k's

Interpolation problem:

$F(z) = \frac{1}{2\pi}\int_0^{\pi} \frac{1+e^{-j\theta}}{1-e^{-j\theta}} f(\theta) d\theta.$

Find $F(z)$ analytic with positive real part so that:

$F(0) = R_0$, $\frac{d}{dz}F(z)|_{z=0} = R_1$, $\frac{d^{1/2}}{dz^{1/2}}F(z)|_{z=0} = \hat{R}_{1/2}$,

or, e.g., more important,

$R_{\sqrt{2}}$, R_{π}, $R_{1.534}$,etc.

$$R = E\{xy\} = \int_{\mathcal{S}} g_\ell d\mu g_r$$

- Characterize R
- Given R "find" $d\mu$
- Parametrize all $d\mu$'s
- What is the effect of the g's

Moment problem – late 1800's early 1900's

Chebysev, Markov, Stieljes, Shohat, Tamarkin, Achiezer, Krein, Nudelman, ...

Analytic interpolation – early 1900's ...

Caratheodory, Toeplitz, Schur, Nevanlinna, Pick... Krein, Arov, Sarason, Sz-Nagy, Foias, Ball, Helton, Gohberg, Dym...

Circuit theory, Stochastic processes, Control: 1950's ...

Levinson, Youla, Helton, Tannenbaum, Zames, Kalman, Kimura...

Entropy and relative entropy functionals ...

von Neumann, Shanon... Kullback, Leibler,...

- Motivating non-classical problem
- Classical moment problems

  Necessity - positivity of a quadratic form

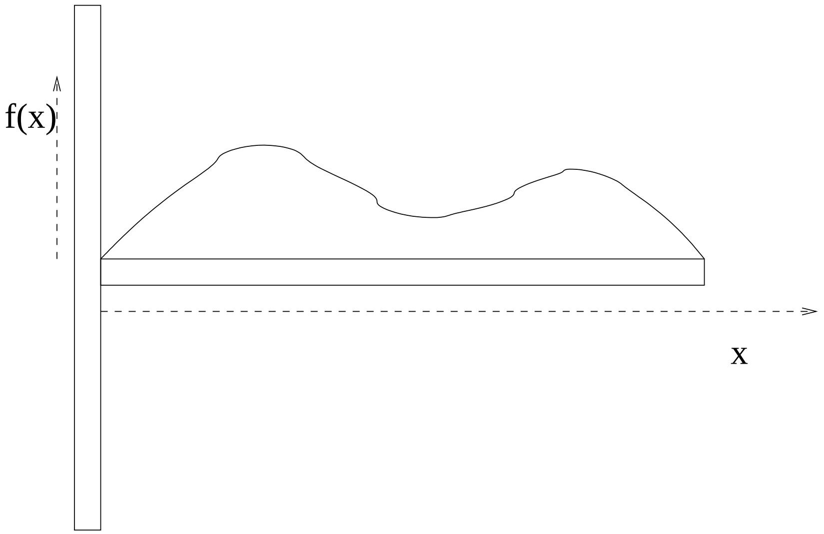
  Sufficiency - constructive-canonical equations, entropy principle
- Classical + non-classical problems

  existence & parametrization of solutions

  —via minimizers of relative entropy

  —homotopy methods

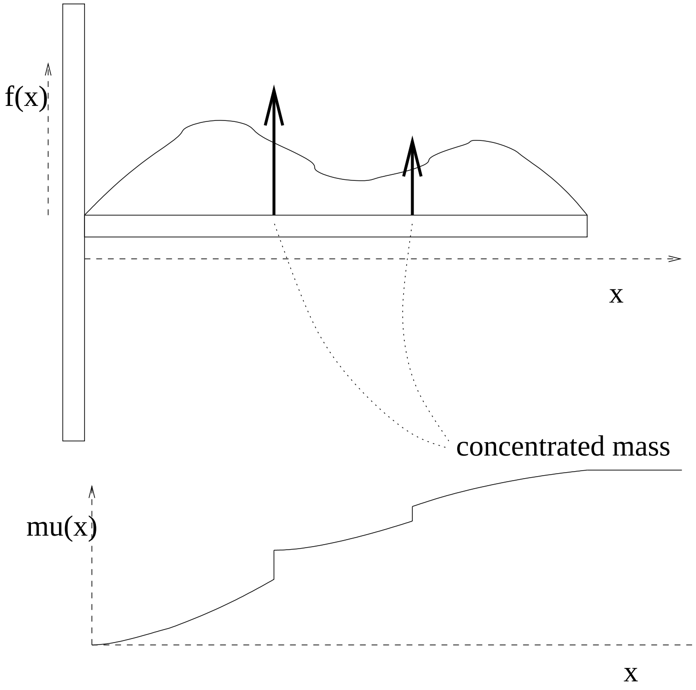
  matricial/bi-tangential generalizations
- a theorem in analytic interpolation
- analogs in LMI's (if time permits)



What can we infer about an unknown mass density $f(x)$ from a set of moments:

$$R_k := \int_0^\infty x^k f(x) dx, \quad k = 0, 1, 2, \ldots?$$

R_0 : total mass
R_1 : torque to hold the beam
etc.



$$R_k := \int_0^\infty x^k d\mu(x), \quad k = 0, 1, 2, \ldots,$$

$\mu(x)$	non-decreasing, distribution function
$f(x) = \dot{\mu}(x)$	non-negative density function

In general $d\mu(x) = f(x)dx + d\mu_j(x) + d\mu_s(x)$
into “absolutely continuous,” “jump,” and “singular” parts.

Basic questions:

- Given $R_0, R_1, \ldots$, determine whether $\exists\ \mu(x)$
- If yes, determine one such $\mu(x)$
- If possible, describe all such $\mu(x)$'s
- Determine bounds on integrals $\int g(x) d\mu(x)$, etc.

Existence question:

Fact: Polynomials in x, which are positive on $[0, \infty)$ include $(a_0 + a_1 x + \ldots)^2$ and $x(b_0 + b_1 x + \ldots)^2$.

Necessity condition:

$$\boxed{\int (a_0 + a_1 x + \ldots)^2 d\mu \geq 0 \text{ and } \int x(b_0 + b_1 x + \ldots)^2 d\mu(x) \geq 0}$$

$$\Leftrightarrow$$

$$\begin{pmatrix} R_0 & R_1 & \ldots & R_n \\ R_1 & R_2 & \ldots & R_{n+1} \\ \ldots & & & \\ R_n & R_{n+1} & \ldots & R_{2n} \end{pmatrix} \geq 0, \ \underline{\text{and}} \ \begin{pmatrix} R_1 & R_2 & \ldots & R_{n+1} \\ R_2 & R_3 & \ldots & R_{n+2} \\ \ldots & & & \\ R_{n+1} & R_{n+2} & \ldots & R_{2n+1} \end{pmatrix} \geq 0$$

Sufficient condition: Same!

Variants of the problem

- Support of μ:
  $[0, \infty)$ (Stieljes),
  $(-\infty, \infty)$ (Hamburger),
  $[0, 1]$ (1-D Hausdorff), etc.
- Moment kernels:
  $g_k(x) = x^k$, $x \in \mathbb{R}$ or $[0, 1]$
  g_k being trigonometric, or
  $g_k(\theta) = e^{jk\theta}$, $\theta \in [-\pi, \pi]$
- Index set: $0, 1, \ldots, n$, or $\mathbb{N}$

Solvability

Non-negativity of quadratic forms

e.g., non-negativity of a Pick or Toeplitz matrix, Sarason operator, etc.

Trigonometric moments (finite indexing set):

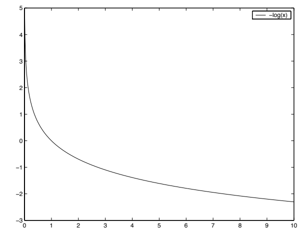
$$R_k = \int_{-\pi}^{\pi} e^{-jkx} d\mu(x),\ \ k = 0, \pm 1, \pm 2, \ldots, \pm n.$$

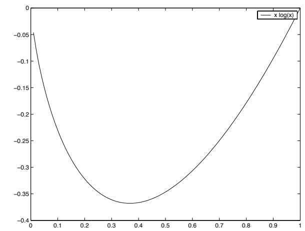
L. Fejer and F. Riesz: Any non-negative trigonometric polynomial is of the form $p(e^{jx}) = |a_0 + a_1 e^{jx} + \ldots a_n e^{njx}|^2$

Solvability condition:

$$\int_{-\pi}^{\pi} p(e^{jx}) d\mu(x) = \begin{bmatrix} \bar{a}_0 & \bar{a}_1 & \ldots & \bar{a}_n \end{bmatrix} \begin{bmatrix} R_0 & R_1 & \ldots & R_n \\ R_{-1} & R_0 & \ldots & R_{n-1} \\ \ldots & & & \\ R_{-n} & R_{-n+1} & \ldots & R_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \ldots \\ a_n \end{bmatrix} \geq 0, \forall a's$$

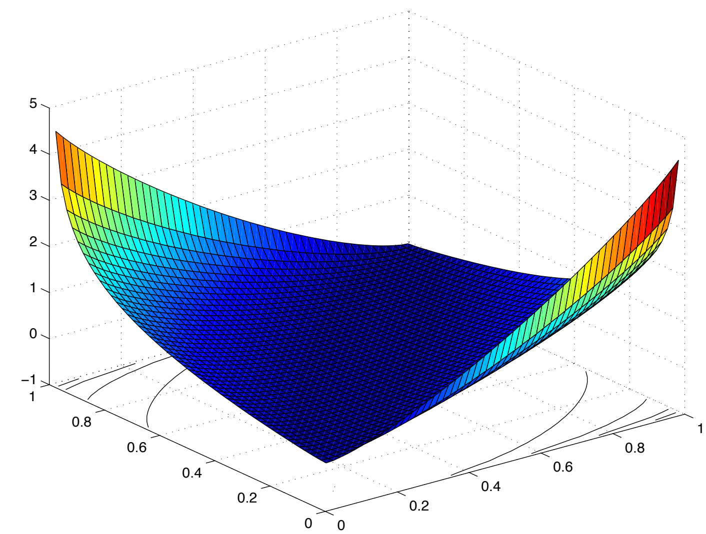
Distance between:





$f = \big(p,\ 1-p\big)$ and $\hat{f} = \big(\hat{p},\ 1-\hat{p}\big)$

$$\mathbb{S}(f||\hat{f}) = p\log\frac{p}{\hat{p}} + (1-p)\log\frac{1-p}{1-\hat{p}}$$



In general: $\mathbb{S}(f||\hat{f}) = \int \Big(f\log(f) - f\log(\hat{f})\Big)$, and

$\mathbb{S}(A||B) = \mathrm{trace}(A\log A - A\log B)$,

are jointly convex in their arguments

Kullback-Leibler, von Neumann, Lieb, ...

Example: Find $f = \big(f_1,\ f_2\big)$, $f_i > 0$ such that

$$f = \arg\min\{\log(f_1) + \log(f_2)\ :\ f_1 a + f_2 b = R\}$$

Answer:
$f_1 = \frac{R}{2a}$ and $f_2 = \frac{R}{2b}$

Parametrization of solutions via a choice of $\hat{f}$ in:
$f = \arg\min\{\mathbb{S}(\hat{f}||f) = \hat{f}_1 \log(f_1) + \hat{f}_2 \log(f_2)\ :\ f_1 a + f_2 b = R\}$

“Entropy rate” (concave)

$$\mathbb{I}_f := \int \log f(x) dx$$

“distance” to 1 (convex)

$$\mathbb{S}(1||f) := \int (1 \cdot \log(1) - 1 \cdot \log(f(x)) dx = -\mathbb{I}_f$$

Find $\text{argmax}(\mathbb{I}_f)$

subject to $\int G(e^{jx}) f(x) dx = R$

where $G := \begin{bmatrix} e^{-jnx} \\ \vdots \\ 1 \\ \vdots \\ e^{jnx} \end{bmatrix}$ and $R := \begin{bmatrix} R_n \\ \vdots \\ R_0 \\ \vdots \\ R_{-n} \end{bmatrix}$.

Analysis: $L(f, \lambda) := \int \log f dx - \lambda(\int G f dx - R)$

$$\delta L(f, \lambda; \delta f) \equiv 0 \Rightarrow \int (\frac{1}{f} - \lambda G) \delta f dx \equiv 0$$
$$\Rightarrow f = \frac{1}{\lambda G}$$

Set

$$\lambda G = \sum_{-n}^{n} \lambda_k e^{jkx} =: |\sum_{0}^{n} a_k e^{jkx}|^2 = |a(e^{jx})|^2$$

Then

$$\int \frac{\bar{a}(e^{-jx})}{|a(e^{jx})|^2} = \int \frac{1}{a(e^{jx})} = \frac{1}{a_0}, \text{ and } \int \frac{e^{jkx}\bar{a}(e^{-jx})}{|a(e^{jx})|^2} = \int \frac{e^{jkx}}{a(e^{jx})} = 0, k \geq 1,$$

together with

$$\boxed{R_k = \int e^{-jkx} \frac{1}{|a(e^{jx})|^2}, \text{ for } k = 0, \pm 1, \ldots,}$$

gives

$$\begin{bmatrix} R_0 & R_1 & \ldots & R_n \\ R_{-1} & R_0 & \ldots & R_{n-1} \\ \ldots & & & \\ R_{-n} & R_{-n+1} & \ldots & R_0 \end{bmatrix} \begin{bmatrix} \bar{a}_0 \\ \bar{a}_1 \\ \vdots \\ \bar{a}_n \end{bmatrix} = \begin{bmatrix} \frac{1}{a_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \boxed{a\text{'s !}}$$

Relative entropy, Kullback-Leibler-von Neumann distance

Fact: Let f, g non-negative functions, then

$$\mathbb{S}(f||g) := \int f\log(f) - f\log(g)$$

is jointly convex. Given one of f, g and specifying moments for the other, leads to a minimizer of a particular form.

Idea:

> Choose "parameter" g,
> then determine the minimizer f which agrees with the moments.

Similarly, repeat with the roles of f and g reversed.

$$\mathbb{S}(f_1||f_2) = \int f_1 \log f_1 - f_1 \log f_2$$

Given ψ find	$\text{argmin}(\mathbb{S}(f\|\|\psi))$
subject to	$\int G(e^{jx})f(x)dx = R$

Given ψ find	$\text{argmin}(\mathbb{S}(\psi\|\|f))$
subject to	$\int G(e^{jx})f(x)dx = R$

If $\exists$ a solution f, then it belongs to:

$\mathfrak{F}_{\text{exp}} := \{\psi(\theta)e^{-<\lambda, G(\theta)>}\}$ or, respectively, $\mathfrak{F}_{\text{rat}} := \{\frac{\psi(\theta)}{<\lambda, G(\theta)>} \;:\;$ with $\lambda G > 0\}$

for any $\psi(\theta) > 0$.

We need to solve $\int G(e^{jx})f(x)dx = R$

For general G, there exists

no representation of positive elements

$$\lambda G = \sum_{\text{index set}} \lambda_k g_k$$

hence, no canonical equations, ...

Key observation:

If

$$h \ : \ \lambda \mapsto R = \int_{\mathcal{S}} G(\theta) \frac{\psi(\theta)}{<\lambda, G(\theta)>} d\theta$$

then

$$\nabla h := \frac{\partial R}{\partial \lambda} = - \int_{\mathcal{S}} G(\theta) \left(\frac{\psi(\theta)}{<\lambda, G(\theta)>} \right)^2 G(\theta)' d\theta$$

is non-singular $\forall \lambda G > 0$.

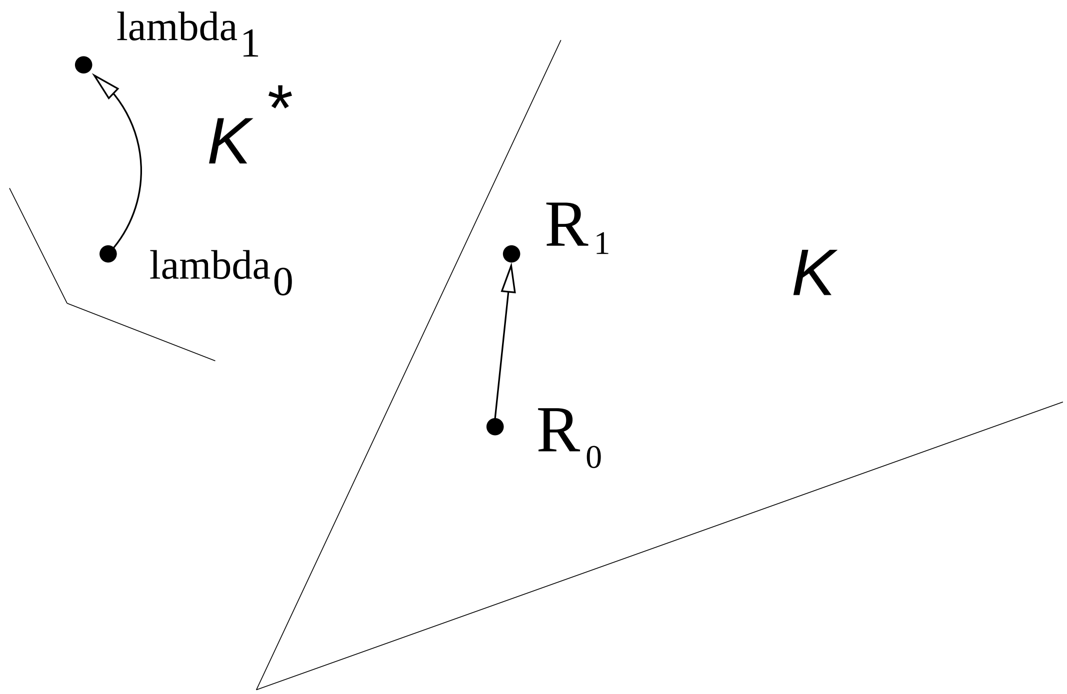
If

$$\kappa \ : \ \lambda \mapsto R = \int_{\mathcal{S}} G(\theta) \psi(\theta) e^{-<\lambda, G(\theta)>} d\theta$$

then

$$\nabla \kappa := \frac{\partial R}{\partial \lambda} = - \int_{\mathcal{S}} G(\theta) \psi(\theta) e^{-<\lambda, G(\theta)>} G(\theta)' d\theta$$

is non-singular $\forall \lambda$.



Homotopy on the R's

Construction of homotopy:

$R_\rho := R_0 + \rho(R_1 - R_0)$ for $\rho \in [0, 1]$

$$\frac{dR_\rho}{d\rho} = R_1 - R_0, \text{ with } R_{\rho=0} = R_0 = \int_{\mathcal{S}} G(\theta) f(\lambda_0, \theta) d\theta$$

$$\frac{d\lambda_\rho}{d\rho} = \left(\frac{\partial R}{\partial \lambda}\bigg|_{\lambda_\rho} \right)^{-1} (R_1 - R_0).$$

$$\frac{dR_\rho}{d\rho} = (1-\rho)(R_1 - R_\rho)$$

$$\frac{d\lambda_\rho}{d\rho} = (1-\rho)\left(\left.\frac{\partial R}{\partial \lambda}\right|_{\lambda_\rho}\right)^{-1}(R_1 - R_\rho).$$

and, for $\rho = 1 - e^{-t}$,

$$\boxed{\frac{d\lambda_t}{dt} = \left(\left.\frac{\partial R}{\partial \lambda}\right|_{\lambda_t}\right)^{-1}\left(R_1 - \int_{\mathcal{S}} G(\theta) f(\lambda_t, \theta) d\theta\right)}$$

THM: Let λ_0 such that $\lambda_0 G > 0$, and

$$\frac{d\lambda_t}{dt} = \left(\left. \frac{\partial R}{\partial \lambda} \right|_{\lambda_t} \right)^{-1} \left(R_1 - \int_{\mathcal{S}} G(\theta) f(\lambda_t, \theta) d\theta \right)$$

for $t \geq 0$ and

$$f(\lambda_t, \theta) = \frac{\psi(\theta)}{\langle \lambda_t, G(\theta) \rangle}.$$

If $R_1 \in \mathrm{int}(\mathcal{K})$, then $\lambda_t \in \mathcal{K}_+^*$ for all $t \in [0, \infty)$, $\lambda_t \to \hat{\lambda} \in \mathcal{K}_+^*$, and

$$R_1 = \int_{\mathcal{S}} G(\theta) f(\hat{\lambda}, \theta) d\theta.$$

If $R_1 \notin \mathrm{int}(\mathcal{K})$, then $\|\lambda_t\| \to \infty$.

THM: For any λ_0 and

$$\boxed{\frac{d\lambda_t}{dt} = \left(\left. \frac{\partial R}{\partial \lambda} \right|_{\lambda_t} \right)^{-1} \left(R_1 - \int_{\mathcal{S}} G(\theta) f(\lambda_t, \theta) d\theta \right)}$$

for $t \geq 0$ and

$$f(\lambda_t, \theta) = \psi(\theta) e^{-\langle \lambda_t, G(\theta) \rangle}.$$

If $R_1 \in \operatorname{int}(\mathcal{K})$, then $\lambda_t \to \hat{\lambda}$, remains bounded, and

$$R_1 = \int_{\mathcal{S}} G(\theta) f(\hat{\lambda}, \theta) d\theta.$$

If $R_1 \notin \operatorname{int}(\mathcal{K})$, then $\|\lambda_t\| \to \infty$.

In both cases,

$$V(\lambda) = \|R_1 - \int_{\mathcal{S}} G(\theta) f(\lambda_t, \theta) d\theta\|^2$$

is a Lyapunov function , satisfying

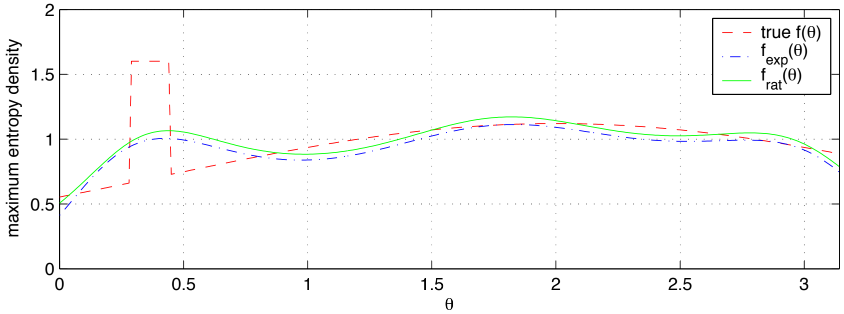
$$\frac{dV(\lambda_t)}{dt} = -2V(\lambda_t),$$

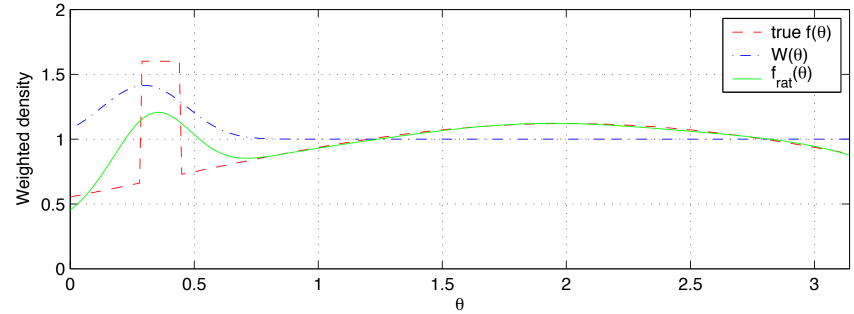
- All positive densities solving $R = \int Gf$ can be obtained, with a suitable choice of ψ, via one of the two constructions.

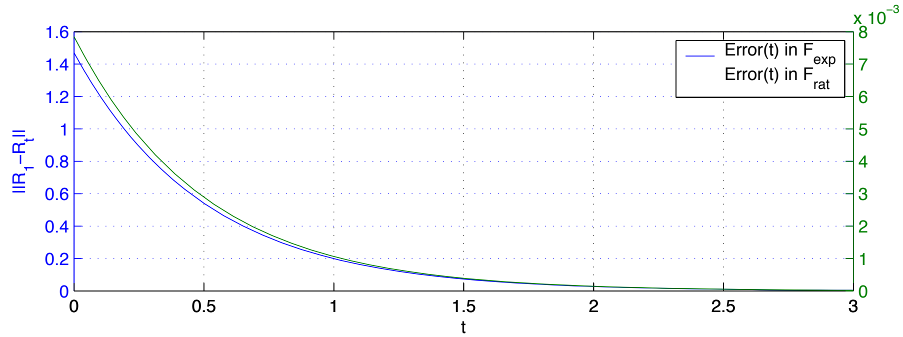
- Given R, G, ψ either solution, $\psi / \langle \lambda, G \rangle$ or, $\psi e^{-\langle \lambda, G \rangle}$, is unique.

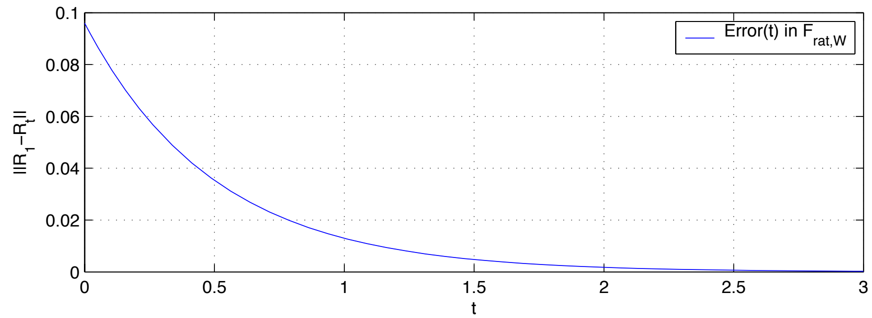
- Convergence is “fast.”

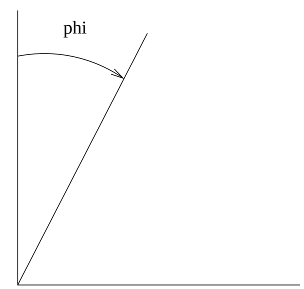
- Failure to converge $\Rightarrow$ no solution exists and $\lambda \to \infty$.











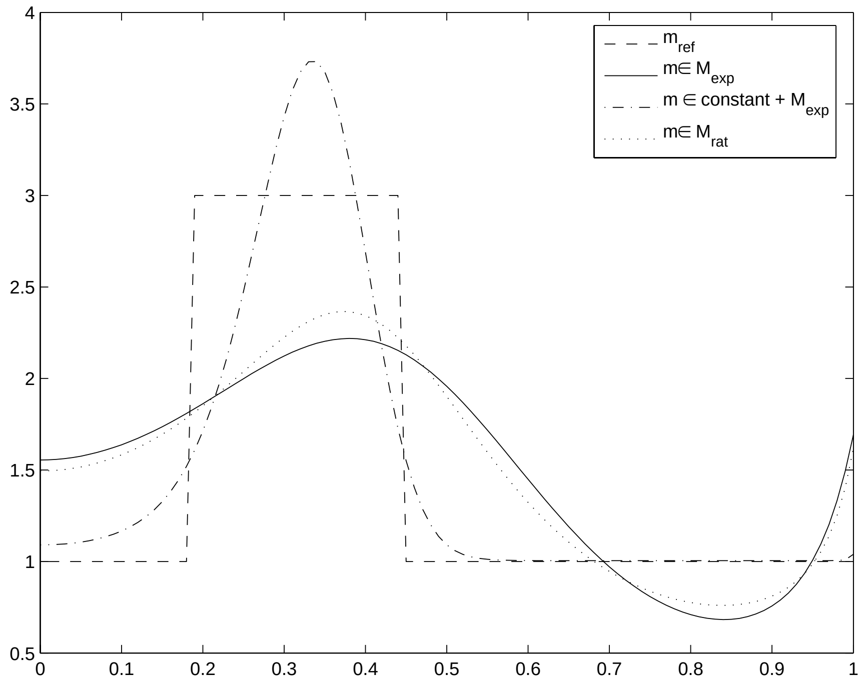
$$G(\theta) := \left[1 \;\; e^{-j\tau} \;\; e^{-j\sqrt{2}\tau} \;\; e^{-j(\sqrt{2}+1)\tau}\right]'$$

- Compute $R_{1,\text{white}} = \int G(\theta)d\theta$
- Problem:

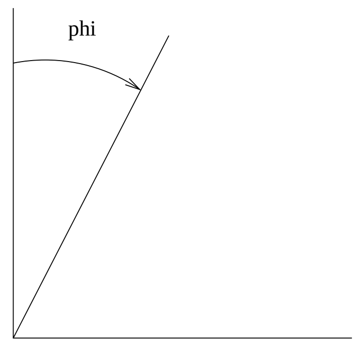
$$p_0 = \text{argmax}\{p \; : \; R_1 - pR_{1,\text{white}} \in \mathcal{K}\}.$$

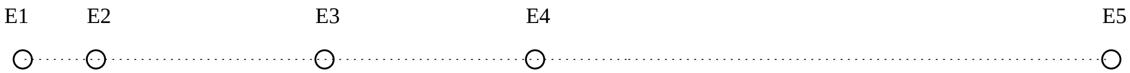
- $d\mu$? such that

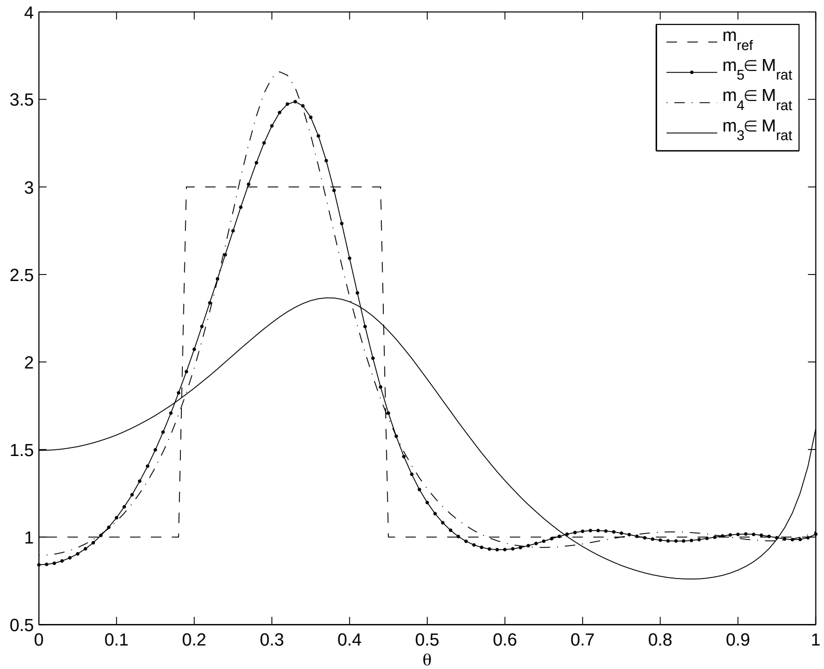
$$R_1 = \int_0^1 G(\theta)(p_0 d\theta + d\mu(\theta)).$$

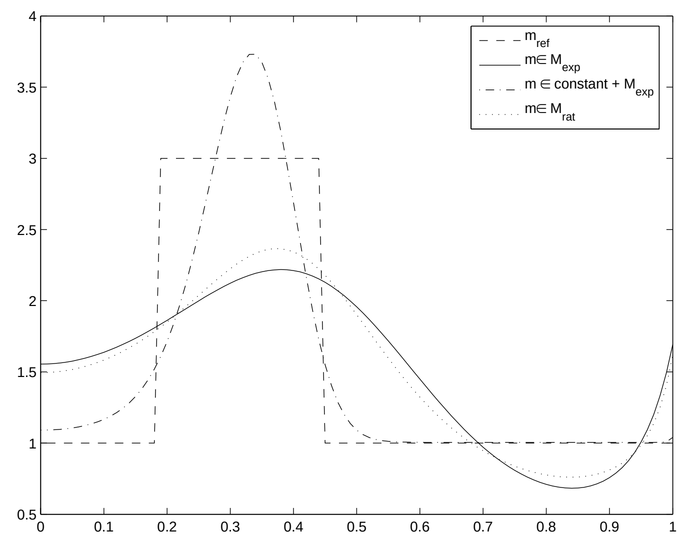


EXAMPLE: MORE "SAMPLES"



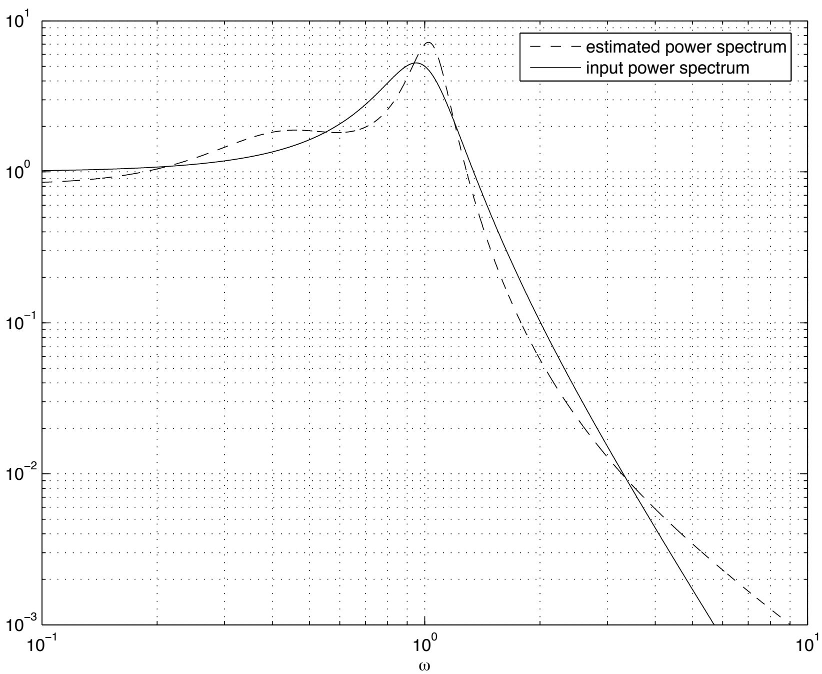


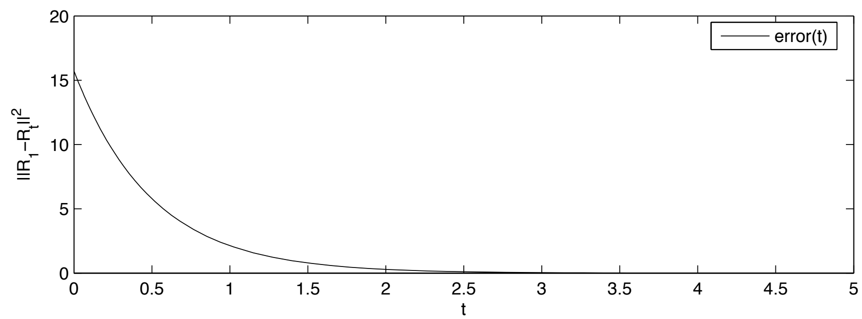


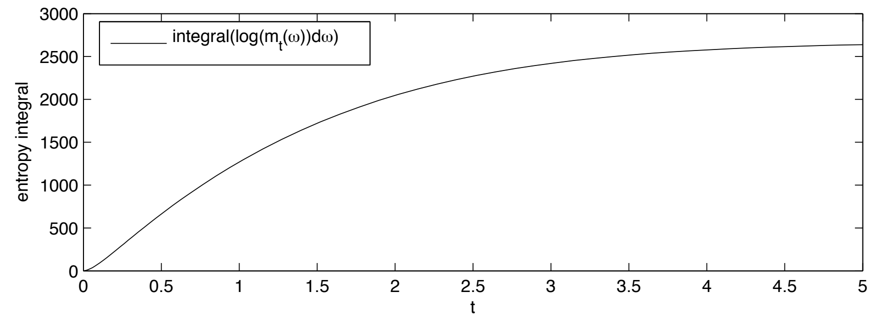


- Low-pass "sensors" $H_k(i\omega) = 1/(1 + i\omega/\tau_k)$
- stochastic input with spectral measure $d\mu(\omega)$
- knowledge of output covariances

$r_k = \int_{-\infty}^{\infty} g_k(\omega) d\mu(\omega)$, with $g_0 = 1$, $g_k(\omega) = \frac{1}{\omega^2/\tau_k^2 + 1}$, $k = 0, 1, 2, 3.$



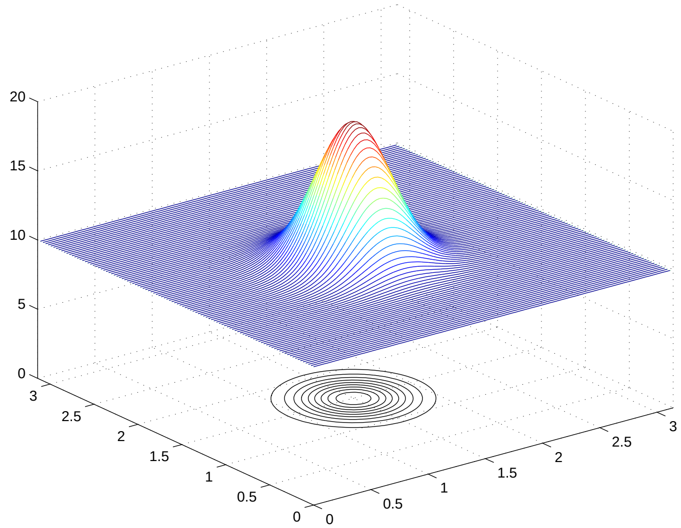


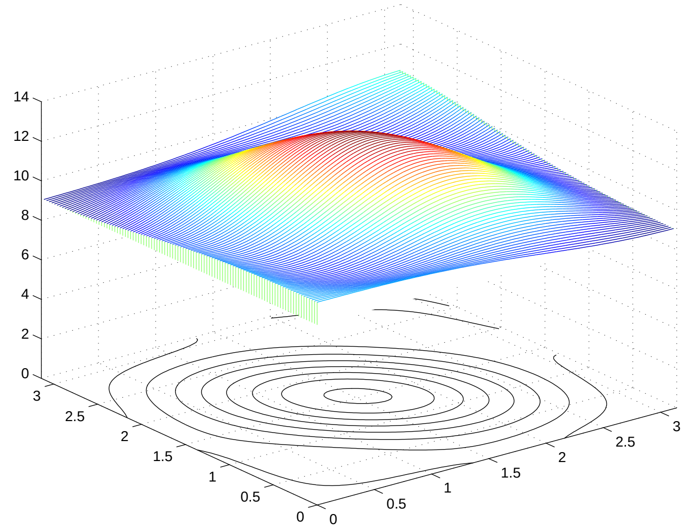


- Begin with $m_{\text{ref}}(\theta,\phi)$, and $g_{k,\ell}(\theta,\phi) = \cos(k\theta + \ell\phi), \quad k,\ell \in \{0,1,2\}$

$$R_1 = \left[\int_{\mathcal{S}} g_{k,\ell}(\theta,\phi) m_{\text{ref}}(\theta,\phi) d\theta d\phi \right]_{k,\ell=0}^{k,\ell=2} = \begin{bmatrix} 33.0129 & 0.3140 & -1.1417 \\ 0.3140 & -14.0469 & -0.2502 \\ -1.1417 & -0.2502 & 1.0310 \end{bmatrix}$$

- m_{ref} and $e^{-\langle\lambda,G\rangle}$ (for comparison).





$$R = \int_{\mathcal{S}} (G_{left} d\mu G_{right})$$

G_{left}: $\mathbb{C}^{p\times m}$-valued (C^2 and $\mathcal{S} \subset \mathbb{R}^1$)
G_{right}: $\mathbb{C}^{m\times q}$-valued (same)
$d\mu$: $m \times m$ Hermitian non-negative measure
$R \in \mathbb{C}^{p\times q}$.

(i) Given G_{left}, G_{right} and R, $\exists ?\ d\mu > 0$?
(ii) It $\exists$, then find a particular one.
(iii) Parametrize all $d\mu$'s.

Cf. tangential interpolation

The homotopy construction generalizes to
$\mathfrak{F}_{\mathrm{rat}}$: hermitian positive matrix-valued functions of the form

$$f = \psi^{1/2} \left((G_{right} \lambda G_{left})_{Hermitian} \right)^{-1} \psi^{1/2}$$

leading to:

$$\boxed{\frac{d\lambda}{dt} = (\nabla h(\lambda))^{-1} (R_1 - \int (G_{left} f(\lambda) G_{right}))}$$

$$h \ : \ \lambda \mapsto R = \int (G_{left} \psi^{1/2} \left((G_{right} \lambda G_{left})_{Hermitian} \right)^{-1} \psi^{1/2} G_{right}) dx$$

$\nabla h \ : \ \delta\lambda \mapsto \delta R$ ...... is invertible, ...

Given G_{left}, G_{right}, R
choose $\psi > 0$ on $\mathcal{S}$, and λ_0 s.t. $(G_{right}\lambda_0 G_{left})_{Hermitian} > 0$.
Then integrate the diff. equation $\Rightarrow \lambda(t)$:

- If $\exists\, d\mu > 0 : R = \int G_{left} d\mu G_{right}$ then:
  (1) $\lambda(t) \to \hat{\lambda}$.
  (2) $f = \psi^{1/2} \left((G_{right}\hat{\lambda}G_{left})_{Hermitian} \right)^{-1} \psi^{1/2} > 0$
  and $R = \int G_{left} d\mu G_{right}$, with $d\mu = f d\theta$.
  (3) $\hat{\lambda}$ does not depend on λ_0.
  (4) $V(\lambda) = \|R_1 - \int (G_{left} f(\lambda) G_{right})\|_F^2$ a Lyapunov function.
  (5) convergence of $\dot{\lambda} = (\nabla h)^{-1}(R_1 - \int G_{left} f G_{right})$ exponentially fast.
  (6) <u>all</u> admissible $d\mu$'s can be obtained with suitable ψ.

- If $\nexists\, d\mu > 0$, then $\|\lambda\| \to \infty$

The homotopy construction generalizes to

$\mathfrak{F}_{\exp}$: hermitian positive matrix-valued functions of the form

$$f = \psi^{1/2} e^{-\left((G_{right}\lambda G_{left})_{Hermitian}\right)} \psi^{1/2}$$

leading to:

$$\boxed{\frac{d\lambda}{dt} = (\nabla h(\lambda))^{-1}(R_1 - \int_{\mathcal{S}} (G_{left} f(\lambda) G_{right}))}$$

with $\mathcal{S} \subset \mathbb{R}^k$, $k \geq 1$.

Moments/Interpolation problems, in absence of a “shift”

Main points: Existence and parametrization of solutions via minimizers of relative entropy

Construction via homotopy methods

Generalization matricial/bi-tangential

Applications: nonuniform sampling, irregular bases (e.g., wavelets), nonuniform arrays, and spacial distribution of sensors

Discussion?

$\mathbb{D}$ open unit disc
$H^2 \subset L^2(\partial\mathbb{D})$
$U : L^2 \to L^2 \;:\; f(z) \mapsto zf(z)$

Beurling-Lax:
All U-invariant subspaces of H_2 are of the form ϕH_2, with ϕ inner in H^∞

Sarason: Let $\mathcal{K} := H^2 \ominus \phi H^2$, $S := \Pi_{\mathcal{K}} U|_{\mathcal{K}}$, and $T : \mathcal{K} \to \mathcal{K}$ such that $TS = ST$:

- $\exists\, f \in H^\infty$ such that $T = f(S)$ and $\|T\| = \|f\|_\infty$.
- If T has a maximal vector then f is unique and $f = \frac{b}{a}$, $a, b \in \mathcal{K}$.

Thm (BGLM):
If $\|T\| < 1$ and ψ arbitrary outer in $\mathcal{K} = H^2 \ominus \phi H^2$, then

$$\exists!\ a, b \in \mathcal{K}:$$

- $f = \frac{b}{a} \in H^\infty$
- $\|f\|_\infty \leq 1$
- $f(S) = T$ and
- $|a|^2 - |b|^2 = |\psi|^2$.

Proof based on maximizing $\int_{\partial\mathbb{D}} |\psi|^2 \log(1 - |\underbrace{w + \phi v}_{f}|^2) dm$ over v

w is any H_∞-function : $w(S) = T$

PROBLEM: Given $L_i = L_i' \in \mathbb{R}^{n \times n}$, determine $x_i \in \mathbb{R}$:

$$L(x) := L_0 + L_1 x_1 + \ldots + L_k x_k > 0.$$

Below:

a particular interior point method (special path of analytic centers)

Caveat:

- strict dual feasibility
- not clear if any advantage…

$$\mathcal{L} := \{M \ : \ M = \sum_{i=1}^{k} L_i x_i, \text{ with } x \in \mathbb{R}^k\},$$
$$\mathcal{G} := \{M \ : \ \langle M, L\rangle = 0, \forall L \in \mathcal{L}\}$$

$$\langle M_1, M_2\rangle := \mathrm{trace}(M_1 M_2)$$

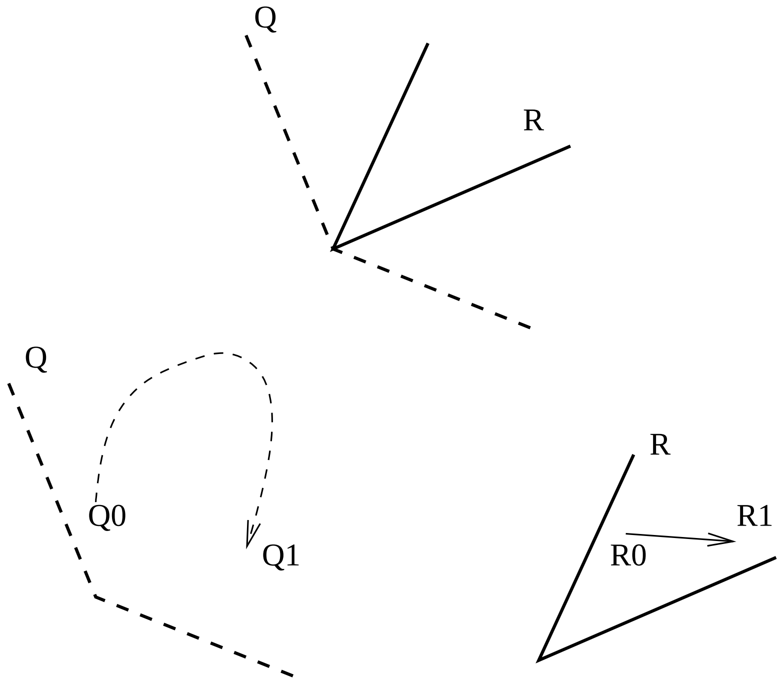
$$\begin{aligned} \mathcal{H} &:= \{M \ : \ M = L_0 + L \text{ with } L \in \mathcal{L}\} \\ &= \{M \in \mathbb{M} \ : \ \Pi_{\mathcal{G}} M = \Pi_{\mathcal{G}} L_0 =: R_1\} \end{aligned}$$

$$\mathfrak{R} := \{R \in \mathcal{G} \;:\; R = \Pi_{\mathcal{G}} M \text{ with } M > 0\}.$$

LMI PROBLEM:
Is $R_1 \in \mathfrak{R}$?
If yes, then determine all $M > 0$ such that $R_1 = \Pi_{\mathcal{G}} M$.

$\mathfrak{R} = \{R \in \mathcal{G} \ : \ R = \Pi_{\mathcal{G}} M$ with $M > 0\}$ is a convex cone

$\mathcal{Q} := \{Q \in \mathcal{G} \ : \ \langle Q, R \rangle \geq 0, \forall R \in \mathfrak{R}\}$ is its dual.



we consider $Q \mapsto R = \Pi_{\mathcal{G}} Q^{-1}$
and again "pull back" the homotopy $R_\rho = (1-\rho)R_0 + \rho R_1$.

- map between $\mathcal{Q}$ and $\mathfrak{R}$:

$$\begin{aligned} h &: \mathcal{Q}_+ \rightarrow \mathfrak{R} \\ &: Q \mapsto R = \Pi_{\mathcal{G}} Q^{-1}, \end{aligned}$$

- Jacobian (tangent map) at a $Q \in \mathcal{Q}_+$:

$$\begin{aligned} \nabla h &: \mathcal{G} \rightarrow \mathcal{G} \\ &: \delta Q \mapsto \delta R = -\Pi_{\mathcal{G}}(Q^{-1} \delta Q Q^{-1}). \end{aligned}$$

$\nabla h|_Q$ is finite and invertible for all $\mathcal{Q}_{\ni} Q > 0$

Also for $Q \mapsto R = \Pi_{\mathcal{G}}(\Psi Q^{-1} \Psi), \ldots$

Set $Q(0) \in \mathcal{Q}_+$ and integrate:

$$\frac{dQ(t)}{dt} = \left(\nabla h|_{Q(t)}\right)^{-1} \left(R_1 - \Pi_{\mathcal{G}} Q(t)^{-1}\right).$$

• If $R_1 \in \mathfrak{R}$, then $Q(t) \in \mathcal{Q}_+$ for all $t \in [0, \infty)$, $\lim_{t\to\infty} Q(t) =: Q_1$ exists, and

$$R_1 = \Pi_{\mathcal{G}} Q_1^{-1}.$$

Moreover, $V(Q) := \langle R_1 - \Pi_{\mathcal{G}} Q^{-1}, R_1 - \Pi_{\mathcal{G}} Q^{-1} \rangle$ satisfies $\frac{dV(Q(t))}{dt} = -2V(Q(t))$.

• If $R_1 \notin \mathfrak{R}$ and if $t_c \in (0, \infty)$ denotes the max value s.t. $[R(0), R(t_c)) \subset \mathfrak{R}$, then as $t \to t_c$, either $\|Q(t)\| \to \infty$ or $\|Q(t)^{-1}\| \to \infty$.

Moment (and LMI-type) problems:
find $M > 0$ s.t. $R_1 = \mathbf{\Pi} M$, parametrize all such

- as minimizers of relative entropy: existence & parametrization
- homotopy on the moments

Discussion?