Optimal Steering of a Linear Stochastic System to a Final Probability Distribution, Part II

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***Abstract*—We address the problem of steering the state of a linear stochastic system to a prescribed distribution over a finite horizon with minimum energy, and the problem to maintain the state at a stationary distribution over an infinite horizon with minimum power. For both problems the control and Gaussian noise channels are allowed to be distinct, thereby, placing the results of this paper outside of the scope of previous work both in probability and in control. The special case where the disturbance and control enter through the same channels has been addressed in the first part of this work that was presented as Part I. Herein, we present sufficient conditions for optimality in terms of a system of dynamically coupled Riccati equations in the finite horizon case and in terms of algebraic conditions for the stationary case. We then address the question of *feasibility* for both problems. For the finite-horizon case, provided the system is controllable, we prove that without any restriction on the directionality of the stochastic disturbance it is always possible to steer the state to any arbitrary Gaussian distribution over any specified finite time-interval. For the stationary infinite horizon case, it is not always possible to maintain the state at an arbitrary Gaussian distribution through constant state-feedback. It is shown that covariances of admissible stationary Gaussian distributions are characterized by a certain Lyapunov-like equation and, in fact, they coincide with the class of stationary state covariances that can be attained by a suitable stationary colored noise as input. We finally address the question of how to compute suitable controls numerically. We present an alternative to solving the system of coupled Riccati equations, by expressing the optimal controls in the form of solutions to (convex) semi-definite programs for both cases. We conclude with an example to steer the state covariance of the distribution of inertial particles to an admissible stationary Gaussian distribution over a finite interval, to be maintained at that stationary distribution thereafter by constant-gain state-feedback control.**

***Index Terms*—Covariance control, linear stochastic systems, Schrödinger bridges, stationary distributions, stochastic optimal control.**

I. Introduction

CONSIDER a linear system

$$\dot{x}(t) = Ax(t) + Bu(t),\ t \in [0, \infty) \tag{1}$$

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with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x(t) \in \mathbb{R}^n$, and $u(t) \in \mathbb{R}^m$, and the problem to steer (1) from the origin to a given point $x(T) = \xi \in \mathbb{R}^n$. This of course is possible for any arbitrary $\xi \in \mathbb{R}^n$ iff the system is *controllable*, i.e., the rank of $[B, AB, \ldots, A^{n-1}B]$ is n, that is, when (A, B) is a *controllable pair*. In this case it is well known that the steering can be effected in a variety of ways, including "minimum-energy" control, over any prespecified interval $[0, T]$. On the other hand, the problem to achieve and maintain a fixed value ξ for the state vector in a stable manner is not always possible. For this to be the case for a given ξ, using feedback and feedforward control

$$0 = (A - BK)\xi + Bu \tag{2}$$

must have a solution (u, K) for a constant value for the input u and a suitable value of K so that $A - BK$ is Hurwitz (i.e., the feedback system be asymptotically stable). It is easy to see that this reduces simply to the requirement that ξ satisfies

$$0 = A\xi + Bv$$

for some v; if there is such a v, we can always choose a suitable K so that $A - BK$ is Hurwitz and then, from v and K, we can compute the constant value u. Conversely, from u and K we can obtain $v = u - K\xi$.

In the present paper, we discuss an analogous and quite similar dichotomy between our ability to assign the state-covariance of a linear stochastically driven system by steering the system over an interval $[0, T]$, and our ability to assign the state-covariance of the ensuing stationary state process through constant state-feedback. It will be shown that the state-covariance can be assigned at the end of an interval through suitable feedback control if and only if the system is controllable. On the other hand, a positive semidefinite matrix is an admissible stationary state-covariance attained through constant feedback if and only if it satisfies a certain Lyapunov-like algebraic equation. Interestingly, the algebraic equation that specifies which matrices are admissible stationary state-covariances through constant feedback is the same equation that characterizes stationary state-covariances attained through colored stationary input noise in open loop.

Both of these problems, to steer and possibly maintain the state statistics of a stochastically driven system, represent generalizations of the classical *regulator problem* which is at the heart of many control applications and entails efficient and accurate steering to a target location. Prime examples, that brought the subject of control and estimation to prominence since the sixties include soft moon-landing, docking, and the guidance of space vehicles, aircraft navigation, robotics, and the steering of quantum mechanical systems, to name a few [1]–[4]. The paradigm that is being considered in this paper represents a "relaxed" version of the classical linear quadratic regulator (LQG) in that, hard constraints and penalties on the endpoint state, are replaced by "soft conditioning" on the state to be distributed according to a prescribed probability density. The theory departs sharply from classical LQG and is quite distinct as well from approaches that bound the probability of violating state constraints [5], [6]. However, the framework may be seen to have conceptual similarities to resource allocation and transport in networks and the tracking of spectral power in radar and antenna arrays [7].

A major source of motivation is provided by modern technological advances that allow us to manipulate micro and thermodynamic systems, and to measure physical properties with unprecedented accuracy. Many such advances rely heavily on our ability to limit state-uncertainty using feedback, e.g., in oscillators coupled to a heat bath or in steering the collective behavior of a swarm of particles experiencing stochastic forcing. Cutting edge examples include thermally driven atomic force microscopy, the control of molecular motors, laser driven reactions, the manipulation of macromolecules, the "active cooling" of devices aimed at measuring gravitational waves, and the focusing of particle beams (see [8]–[14]).

Historically, the problem to steer the probability density of Brownian particles in their trajectory across two points in time, has its origin in a study published in 1931/1932 by Erwin Schrödinger [15], [16, Section VII]. In this, Schrödinger asked for the most likely evolution of a cloud of particles that are observed, at the two end points of their path, to be distributed according to given empirical distributions. The answer he gave, which provides an updated probability law on path space, in fact relates to a minimum energy stochastic control problem [17]. The subject, which advanced with leaps and bounds over the past 80 years by contributions from Fortet, Beurling, Jamison, Föllmer, and many others, has come to be known as Schrödinger bridges. Yet, all prior work, was related to the case where the diffusive particles are modeled by non-degenerate diffusions where the noise affects directly all entries of (vectorial) stochastic process, and the link to minimum-energy optimal control was drawn primarily via the Girsanov transformation [17] for that case. Recent attempts to address linear stochastic systems were also limited to non-degenerate diffusions where the control and noise channel are identical [18], [19].

In a "sister paper" that precedes the present as part I [20], we presented a theory of Schrödinger bridges for general linear stochastic systems. This includes possibly degenerate linear diffusions and the theory entails two coupled *homogeneous* differential Riccati equations (which naturally reduce to two coupled Lyapunov equations) in the style of classical LQG theory. In contrast however, these differential equations are nonlinearly coupled through their boundary conditions and, furthermore, the boundary conditions are in general *sign indefinite*. Thus the theory falls outside the standard LQG framework, and yet, it is shown in [20], that the equations can be solved in *closed form* for the minimum-energy control. The reason for that is a salient feature of classical Schrödinger bridges and of the theory in [20] that the control which provides the needed drift to reconcile the empirical marginals *enters along the very same "directions" that the noise impacts*, i.e., control and noise channels are identical. The present work departs from [20] in that *control and noise channels may now differ*. Hence, no assumption on the directionality of control authority as compared to that of the random driving noise is being made; the cost functional is anyway quadratic in the control. In this more general case, while certain aspects parallel [20] (e.g., variational analysis, cf. Section II), the techniques needed to determine our ability to steer the state statistics and determine the corresponding control input are quite different.

The structure of the paper is as follows: In Section II we formulate both the finite horizon problem and the infinite horizon stationary problem, and present sufficient conditions for optimality. In Section III-A we consider the *feasibility* of steering the statistics over a finite interval by a suitable control action and in Section III-B we consider the possibility to maintain stationary state-statistics by constant state-feedback. In Section IV-A and B, we formulate the least-energy optimal control problem in each of the two cases, finite horizon and stationary statistics, as semidefinite programs. Finally, Section V highlights the theory with a numerical example to steer the statistics of inertial particles, in the phase-plane, in each of these two modalities, transient and stationary.

II. Optimal Steering

In this section we formulate the control problem to optimally steer a stochastic linear system to a final target Gaussian distribution at the end of a finite interval. In parallel, we formulate the problem to maintain a stationary Gaussian state distribution by constant state feedback for time-invariant dynamics. We also present sufficient conditions of optimality which in the case of finite-horizon take the form of a Schrödinger-like system of equations.

The ability to specify the mean value of the state-vector reduces to the problem discussed at the start of the introduction. More specifically, since $\mathbb{E}\{x(t)\} =: \bar{x}(t)$ satisfies (1), controllability of (A, B) is necessary and sufficient to specify $\bar{x}(T)$ at the end of the interval and this is effected by a deterministic mean value for the input process. Likewise, the mean value for a stationary input must satisfy (2) to attain $\bar{x}(t) \equiv \xi$ for a stationary state process. Thus, throughout and without loss of generality we assume that all processes have zero-mean and we only focus on our ability to assign the state-covariance in those two instances.

A. Finite-Horizon Optimal Steering

Consider the controlled evolution

$$\begin{aligned} dx^u(t) &= A(t)x^u(t)dt + B(t)u(t)dt + B_1(t)dw(t) \\ x^u(0) &= x_0 \text{ a.s.} \end{aligned} \tag{3}$$

where x_0 is an n-dimensional Gaussian vector independent of the standard p-dimensional Wiener process $\{w(t) | 0 \leq t \leq T\}$ and with density

$$\rho_0(x) = (2\pi)^{-n/2} \det(\Sigma_0)^{-1/2} \exp\left(-\frac{1}{2} x' \Sigma_0^{-1} x\right). \tag{4}$$

Here, A, B and B_1 are continuous matrix functions of t taking values in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times m}$, and $\mathbb{R}^{n\times p}$, respectively, Σ_0 is a symmetric positive definite matrix, and $T < \infty$ represents the end point of a time interval of interest. Suppose we also have a "target" Gaussian end-point distribution

$$\rho_T(x) = (2\pi)^{-n/2} \det(\Sigma_T)^{-1/2} \exp\left(-\frac{1}{2}x'\Sigma_T^{-1}x\right) \quad (5)$$

where we also assume Σ_T symmetric and positive definite. The uncontrolled evolution $x^{u\equiv 0} = \{x(t)|0 \le t \le T\}$ may be thought to represent a "prior," or reference evolution, for which, in general, $x(T)$ is not distributed according to ρ_T. Thus, we seek the least-effort strategy to steer (3) to the desired final probability density. To this end, let $\mathcal{U}$ represent the family of *adapted*, *finite-energy* control functions such that (3) has a strong solution and $x^u(T)$ is distributed according to (5). Thus, $u \in \mathcal{U}$ is such that $u(t)$ only depends on t and on $\{x^u(s)|0 \le s \le t\}$ for each $t \in [0,T]$, satisfies

$$J(u) := \mathbb{E}\left\{\int_0^T u(t)'u(t)\,dt\right\} < \infty$$

and *forces* $x^u(T)$ *to be distributed according to* (5). Therefore, $\mathcal{U}$ represents the class of *admissible* control inputs. The existence of such control inputs will be established in the following section, i.e., that $\mathcal{U}$ is not empty. At present, assuming this to be the case, we formulate the following *Bridge Problem*:

Problem 1: Determine $u^* := \arg\min_{u\in\mathcal{U}} J(u)$.

We point out that when $BB' \neq B_1B_1'$, no interpretation of this problem as a classical Schrödinger bridge [21] via the Girsanov transformation is possible since, in this case, the reference and controlled measures on path spaces are mutually singular; this is due to the fact that the diffusion coefficients differ and as a consequence the martingale part of the two evolutions are different. In spite of this, precisely the same completion of the squares argument used in [20, Section II] yields the sufficient conditions in Proposition 1 and shows that a *control-theoretic view* of the Schrödinger bridge problem [17] carries through in this more general setting.

Proposition 1: Let $\{\Pi(t)|0 \le t \le T\}$ be a solution of the matrix Riccati equation

$$\dot{\Pi}(t) = -A(t)'\Pi(t) - \Pi(t)A(t) + \Pi(t)B(t)B(t)'\Pi(t). \quad (6)$$

Define the feedback control law

$$u(x,t) := -B(t)'\Pi(t)x \quad (7)$$

and let $x^u = x^*$ be the Gauss-Markov process

$$dx^*(t) = (A(t) - B(t)B(t)'\Pi(t))\,x^*(t)dt + B_1(t)dw(t), \quad \text{with } x^*(0) = x_0 \text{ a.s.} \quad (8)$$

If $x^*(T)$ has probability density ρ_T, then $u(x^*(t),t) = u^*(t)$, i.e., it is the solution to Problem 1.

Now, in contrast to the standard LQG problem where the terminal cost provides a boundary value for the differential Riccati equation, here the boundary value $\Pi(0)$ is unspecified and needs to be selected so as to ensure that (7) drives the state to the desired final distribution. In [20], where $B = B_1$, the mapping between $\Pi(T)$ and Σ_T is onto with (6) having no finite escape-time and, thereby, that steering is always possible. However, it was also noted in [20] that $\Pi(T)$ may be indefinite, placing the analysis outside of standard LQG theory. Thus, in the present more general case we also need to resort to an approach that departs from classical LQG in order to determine the appropriate solutions of (6). Below we recast Proposition 1 in the form of a Schrödinger system.[1]

Let $\Sigma(t) := \mathbb{E}\{x^*(t)x^*(t)'\}$ be the state covariance of (8) and assume that the conditions of the proposition hold. Then

$$\dot{\Sigma}(t) = (A(t) - B(t)B(t)'\Pi(t))\,\Sigma(t) + \Sigma(t)\,(A(t) - B(t)B(t)'\Pi(t))' + B_1(t)B_1(t)' \quad (9)$$

holds together with the two boundary conditions

$$\Sigma(0) = \Sigma_0, \quad \Sigma(T) = \Sigma_T. \quad (10)$$

Further, since $\Sigma_0 > 0$, $\Sigma(t)$ is positive definite on $[0,T]$. Define

$$\mathrm{H}(t) := \Sigma(t)^{-1} - \Pi(t).$$

A direct calculation using (9) and (6) leads to (11b) below. We have therefore derived a *nonlinear* Schrödinger system

$$\dot{\Pi} = -A'\Pi - \Pi A + \Pi BB'\Pi \quad (11a)$$
$$\dot{\mathrm{H}} = -A'\mathrm{H} - \mathrm{H}A - \mathrm{H}BB'\mathrm{H} + (\Pi + \mathrm{H})\,(BB' - B_1B_1')\,(\Pi + \mathrm{H}) \quad (11b)$$
$$\Sigma_0^{-1} = \Pi(0) + \mathrm{H}(0) \quad (11c)$$
$$\Sigma_T^{-1} = \Pi(T) + \mathrm{H}(T). \quad (11d)$$

Indeed, in contrast to the case when $B = B_1$ (see [20]), the two Riccati equations in (11) are coupled not only through their boundary values (11c), (11d) but also in a nonlinear manner through their dynamics in (11b). Clearly, the case $\Pi(t) \equiv 0$ corresponds to the situation where the uncontrolled evolution already satisfies the boundary marginals and, in that case, $\mathrm{H}(t)^{-1}$ is simply the prior state covariance. We summarize our conclusion in the following proposition.

Proposition 2: Assume that $\{(\Pi(t), \mathrm{H}(t))|0 \le t \le T\}$ satisfy (11a)–(11d). Then the feedback control law (7) is the solution to Problem 1 and the corresponding optimal evolution is given by (8).

The existence and uniqueness of solutions for the Schrödinger system is quite challenging already in the classical case where the two dynamical equations are uncoupled and where major contributions are due to Fortet [22], Beurling [23], Jamison [24], Föllmer [21], see also [20], [25]. It is therefore hardly surprising that at present we don't know how to prove existence of solutions for (11a)–(11d)[2] A direct proof of existence of solutions for (11a)–(11d) would in particular imply *feasibility* of Problem 1, i.e., that $\mathcal{U}$ is nonempty and that there exists a minimizer. At present we do not have a proof that a minimizer exists. However, in Section III-A we establish that the set of admissible controls $\mathcal{U}$ is not empty and in Section IV we provide an approach that allows constructing suboptimal controls incurring cost that is arbitrarily close to $\inf_{u\in\mathcal{U}} J(u)$.

[1] In general, a Schrödinger system consists of a forward and a backward Kolmogoroff (partial) differential equation that are coupled through their boundary conditions, cf. [21]. Here, since the distributions are Gaussian, the Schrödinger system entails matrix differential equations.

[2] A numerical scheme based on successive approximations appears to be unstable and does not produce a fixed point in general. In this case, such a scheme could consist of solving (11a) backwards in time starting from $\Pi(T)$, computing initial conditions for (11b) using (11c), solving (11b) forward in time to compute $H(T)$ so as to update $\Pi(T)$ using (11d) and repeating the cycle. A similar idea was carried out by Fortet [22] in the classical setting, whereas a more powerful technique based on the Hilbert metric was explored recently in [25] for a Schrödinger system on finite spaces.

B. Infinite-Horizon Optimal Steering

Suppose now that A, B and B_1 do not depend on time and that the pair (A, B) is controllable. We seek a constant state feedback law $u(t) = -Kx(t)$ to maintain a stationary state-covariance $\Sigma > 0$ for (3). In particular, we are interested in one that minimizes the expected input power (energy rate)

$$J_{\text{power}}(u) := \mathbb{E}\{u'u\} \tag{12}$$

and thus we are led to the following problem.[3]

Problem 2: Determine u^* that minimizes $J_{\text{power}}(u)$ over all $u(t) = -Kx(t)$ such that

$$dx(t) = (A - BK)x(t)dt + B_1 dw(t) \tag{13}$$

admits

$$\rho(x) = (2\pi)^{-n/2} \det(\Sigma)^{-1/2} \exp\left(-\frac{1}{2}x'\Sigma^{-1}x\right) \tag{14}$$

as invariant probability density.

Interestingly, the above problem may not have a solution in general since not all values for Σ can be maintained by state feedback. In fact, Theorem 4 in Section III-B, provides conditions that ensure Σ is admissible as a stationary state covariance for a suitable input. Moreover, as it will be apparent from what follows, even when the problem is feasible, i.e., there exist controls which maintain Σ, an optimal control may fail to exist. The relation between this problem and Jan Willems' classical work on the Algebraic Riccati Equation [26] is provided after Proposition 3 below.

Let us start by observing that the problem admits the following finite-dimensional reformulation. Let $\mathcal{K}$ be the set of all $m \times n$ matrices K such that the corresponding feedback matrix $A - BK$ is Hurwitz. Observe that

$$\mathbb{E}\{u'u\} = \mathbb{E}\{x'K'Kx\} = \text{trace}(K\Sigma K')$$

Then Problem 2 reduces to finding a $m \times n$ matrix $K^* \in \mathcal{K}$ which minimizes the criterion

$$J(K) = \text{trace}(K\Sigma K') \tag{15}$$

subject to the constraint

$$(A - BK)\Sigma + \Sigma(A' - K'B') + B_1B_1' = 0. \tag{16}$$

Now, consider the Lagrangian function

$$\begin{aligned}\mathcal{L}(K, \Pi) &= \text{trace}(K\Sigma K')\\ &+ \text{trace}\left(\Pi\left((A - BK)\Sigma + \Sigma(A' - K'B') + B_1B_1'\right)\right)\end{aligned} \tag{17}$$

which is a simple quadratic form in the unknown K. Observe that $\mathcal{K}$ is *open*, hence a minimum point may fail to exist. Nevertheless, at any point $K \in \mathcal{K}$ we can take a directional derivative in any direction $\delta K \in \mathbb{R}^{m\times n}$ to obtain

$$\delta\mathcal{L}(K, \Pi; \delta K) = \text{trace}\left((\Sigma K' + K\Sigma - \Sigma\Pi B - B'\Pi\Sigma)\,\delta K\right).$$

Setting $\delta\mathcal{L}(K, \Pi; \delta K) = 0$ for all variations, which is a sufficient condition for optimality, we get the form

$$K^* = B'\Pi. \tag{18}$$

[3] An equivalent problem is to minimize $\lim_{T\to\infty}(1/T)\mathbb{E}\{\int_0^T u(t)'u(t)dt\}$ for a given terminal state covariance as $T \to \infty$.

To compute K^*, we calculate the multiplier Π as a maximum point of the dual functional

$$\begin{aligned}G(\Pi) &= \mathcal{L}(K^*, \Pi)\\ &= \text{trace}\left((A'\Pi + \Pi A - \Pi BB'\Pi)\Sigma + \Pi B_1B_1'\right).\end{aligned} \tag{19}$$

The unconstrained maximization of the concave functional G over symmetric $n \times n$ matrices produces matrices Π^* which satisfy (16), namely

$$(A - BB'\Pi^*)\Sigma + \Sigma(A' - \Pi^*BB') + B_1B_1' = 0. \tag{20}$$

There is no guarantee, however, that $K^* = B'\Pi^*$ is in $\mathcal{K}$, namely that $A - BB'\Pi^*$ is Hurwitz. Nevertheless, since (20) is satisfied, the spectrum of $A - BB'\Pi^*$ lies in the *closed* left half-plane. Thus, our variational analysis leads to the following result.

Proposition 3: Assume that there exists a symmetric matrix Π such that $A - BB'\Pi$ is a Hurwitz matrix and

$$(A - BB'\Pi)\Sigma + \Sigma(A - BB'\Pi)' + B_1B_1' = 0 \tag{21}$$

holds. Then

$$u^*(t) = -B'\Pi x(t) \tag{22}$$

is the solution to Problem 2.

We now draw a connection to some classical results due to Jan Willems [26]. In our setting, minimizing (12) is equivalent to minimizing

$$J_{\text{power}}(u) + \mathbb{E}\{x'Qx\} \tag{23}$$

for an arbitrary symmetric matrix Q since the portion

$$\mathbb{E}\{x'Qx\} = \text{trace}\{Q\Sigma\}$$

is independent of the choice of K. On the other hand, minimization of (23) for specific Q, but without the constraint that $\mathbb{E}\{xx'\} = \Sigma$, was studied by Willems [26] and is intimately related to the *maximal* solution of the Algebraic Riccati Equation (ARE)

$$A'\Pi + \Pi A - \Pi BB'\Pi + Q = 0. \tag{24}$$

Under the assumption that the Hamiltonian matrix

$$H = \begin{bmatrix} A & -BB' \\ -Q & -A' \end{bmatrix}$$

has no pure imaginary eigenvalues, Willems' result states that $A - BB'\Pi$ is Hurwitz and that (22) is the optimal solution.

Thus, starting from a symmetric matrix Π as in Proposition 3, we can define Q using

$$Q = -A'\Pi - \Pi A + \Pi BB'\Pi.$$

Since, by Willems' results, (24) has at most one "stabilizing" solution Π, the matrix in the proposition coincides with the maximal solution to (24). Therefore, if our original problem has a solution, this same solution can be recovered by solving for the maximal solution of a corresponding ARE, for a particular choice of Q. Interestingly, neither Π nor Q, corresponding to an optimal control law for which (21) holds, are unique, whereas K is. The computation and the uniqueness of the optimal gain K will be discussed later on in Section IV-B.

III. Controllability of State Statistics

We now return to the "controllability" question of whether there exist admissible controls to steer the controlled evolution

$$dx(t) = Ax(t)dt + Bu(t)dt + B_1 dw(t)$$
$$\text{with } x(0) = x_0 \text{ a.s.} \tag{25}$$

to a target Gaussian distribution at the end of a finite interval $[0, T]$, or, for the stationary case, whether a stationary Gaussian distribution can be achieved by constant state feedback. From now on, we assume that $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $B_1 \in \mathbb{R}^{n\times p}$, are time-invariant and that (A, B) is controllable. In view of the earlier analysis, we search over controls that are linear functions of the state, i.e.,

$$u(t) = -K(t)x(t), \text{ for } t \in [0, T] \tag{26}$$

and where K is constant and $A - BK$ Hurwitz for the stationary case.

A. Finite-Interval Steering by State-Feedback

We assume that $\mathbb{E}\{x_0\} = 0$ while $\mathbb{E}\{x_0 x_0'\} = \Sigma_0$. The state covariance

$$\Sigma(t) := \mathbb{E}\left\{x(t)x(t)'\right\}$$

of (3) with input as in (26) satisfies the Lyapunov differential equation

$$\dot{\Sigma}(t) = (A - BK(t))\,\Sigma(t) + \Sigma(t)\,(A - BK(t))' + B_1B_1' \tag{27}$$

and $\Sigma(0) = \Sigma_0$. Regardless of the choice of $K(t)$, (27) specifies dynamics that leave invariant the cone of positive semi-definite symmetric matrices

$$\mathcal{S}_n^+ := \{\Sigma | \Sigma \in \mathbb{R}^{n\times n},\ \Sigma = \Sigma' \geq 0\}.$$

To see this, note that the solution to (27) is of the form

$$\Sigma(t) = \hat{\Phi}(t,0)\Sigma_0\hat{\Phi}(t,0)' + \int_0^t \hat{\Phi}(t,\tau)B_1B_1'\hat{\Phi}(t,\tau)'d\tau$$

where $\hat{\Phi}(t, 0)$ satisfies

$$\frac{\partial\hat{\Phi}(t,0)}{\partial t} = (A - BK(t))\,\hat{\Phi}(t,0)$$

and $\hat{\Phi}(0, 0) = I$, the identity matrix; i.e., $\hat{\Phi}(t, 0)$ is the state-transition matrix of the system $\dot{x}(t) = (A - BK(t))x(t)$.

Assuming $\Sigma_0 > 0$, it follows that $\Sigma(t) > 0$ for all t and finite $K(\cdot)$. Our interest is in our ability to specify $\Sigma(T)$ via a suitable choice of $K(t)$. To this end, we define

$$U(t) := -\Sigma(t)K(t)'$$

we observe that $U(t)$ and $K(t)$ are in bijective correspondence provided that $\Sigma(t) > 0$, and we now consider the differential Lyapunov system

$$\dot{\Sigma}(t) = A\Sigma(t) + \Sigma(t)A' + BU(t)' + U(t)B'. \tag{28}$$

Reachability/controllability of a differential system such as (1), or (28), is the property that with suitable bounded control input $u(t)$, or $U(t)$, respectively, the solution can be driven to any finite value. Interestingly, if either (1) or (28) is controllable, so is the other. But, more importantly, when (28) is controllable, the control authority allowed is such that steering from one value for the covariance to another can be done by remaining within the non-negative cone. This is stated as our first theorem below.

Theorem 3: The Lyapunov system (28) is controllable iff (A, B) is a controllable pair. Furthermore, if (28) is controllable, given any two positive definite matrices Σ_0 and Σ_T and an arbitrary $Q \geq 0$, there is a smooth input $U(\cdot)$ so that the solution of the (forced) differential equation

$$\dot{\Sigma}(t) = A\Sigma(t) + \Sigma(t)A' + BU(t)' + U(t)B' + Q \tag{29}$$

satisfies the boundary conditions $\Sigma(0) = \Sigma_0$ and $\Sigma(T) = \Sigma_T$ and $\Sigma(t) > 0$ for all $t \in [0, T]$.

Proof: We first establish equivalence of the controllability of (1) and (28). Define $S(t) := e^{-At}\Sigma(t)e^{-A't}$. In these new "coordinates" (28) becomes

$$\dot{S}(t) = e^{-At}BU(t)'e^{-A't} + e^{-At}U(t)B'e^{-A't}$$

and upon re-naming $V(t) = e^{-At}U(t)$ as the input

$$\dot{S}(t) = e^{-At}BV(t)' + V(t)B'e^{-A't}. \tag{30}$$

Assuming that (A, B) is a controllable pair, the system

$$\dot{X}(t) = e^{-At}BV(t)' \tag{31}$$

where each column of $V(t)'$ serves as input that drives the corresponding column of $X(t)$ is clearly controllable since the controllability grammian

$$G(T) := \int_0^T e^{-A\tau}BB'e^{-A'\tau}d\tau$$

is invertible. Thus, by a suitable choice of $V(t)$ we can drive (31) to any final state $X(T)$ and, thus, we can drive (30) to any final state $S(T) = X(T) + X(T)'$.

The converse is straightforward. If (A, B) is not controllable, then there is a matrix C such that $Ce^{-At}B = 0$. It follows that $C\dot{S}(t)C' = 0$ and therefore $S(t)$ remains invariant when restricted to a certain subspace.

We now want to establish that there is a control input $U(t)$ so that the solution to (29) remains within the positive cone and satisfies the boundary conditions. We show this, and in fact, a stronger argument for a special case where A is a shift matrix and B is vectorial, and then explain why the general case can be reduced to this one.

So, we now establish that there is a smooth (infinitely differentiable) control input $U(t)$ so that $\Sigma(t)$ remains within the positive cone and satisfies the boundary conditions. We further claim (and show below) that such a control can always be chosen to satisfy arbitrary starting and ending boundary conditions $U(0)$ and $U(T)$ of its own. We show this for the special case where A is a shift matrix of size k. For specificity in the steps of the proof, we subscribe the size of matrices in the notation

$$A_k := \begin{bmatrix} 0_{k-1} & I_{k-1} \\ 0 & 0_{k-1}' \end{bmatrix} \text{ and } B_k := \begin{bmatrix} 0_{k-1} \\ 1 \end{bmatrix}. \tag{32}$$

Here also, I_k denotes the identity matrix of size k, and 0_k the column vector of size k that has all entries zero. We will show by induction on k that, for any $k \times k$ matrix $Q_k \geq 0$, the system

$$\dot{\Sigma}(t) = A_k\Sigma(t) + \Sigma(t)A_k' + B_kU_k(t)' + U_k(t)B_k' + Q_k \quad (33)$$

can be steered between positive-definite boundary values while $\Sigma(t)$, which is now $k \times k$, remains positive-definite and the control satisfies arbitrary starting and ending values. The statement is true for $k = 1$. In this case, the system is in the form

$$\dot{\Sigma}(t) = 2U_1(t) + Q \quad (34)$$

with all entries scalar. Positivity of $\Sigma(t)$ dictates that

$$\Sigma_0 + 2\int_0^t U_1(\tau)d\tau + Qt > 0 \text{ for all } t$$

while the boundary conditions dictate that

$$\Sigma_0 + 2\int_0^T U_1(\tau)d\tau + QT = \Sigma_T.$$

Clearly, these can be met along with any boundary conditions on $U_1(t)$ along with the smoothness requirement. An example of such an interpolating function is $\Sigma(t) = e^{h(t)} > 0$ where

$$h(t) = a_0 + b_0 t + \frac{a_T - a_0 - Tb_0}{T^2}t^2 + \frac{Tb_0 + Tb_T - 2a_T + 2a_0}{T^3}t^2(t - T)$$

and

$$\begin{aligned} a_0 &= \log(\Sigma_0) \\ a_T &= \log(\Sigma_T) \\ b_0 &= \frac{(2U_1(0) + Q)}{\Sigma_0} \\ b_T &= \frac{(2U_1(T) + Q)}{\Sigma_T}. \end{aligned}$$

The polynomial $h(t)$ is in fact a Hermite polynomial satisfying

$$\begin{aligned} &h(0) = a_0,\ h(T) = a_T \\ &\dot{h}(0) = b_0,\ \dot{h}(T) = b_T. \end{aligned}$$

It is easy to see that $\Sigma(t) = e^{h(t)}$ satisfies

$$\begin{aligned} \Sigma(0) &= \Sigma_0,\ \Sigma(T) = \Sigma_T \\ \dot{\Sigma}(0) &= 2U_1(0) + Q \\ \dot{\Sigma}(T) &= 2U_1(T) + Q \end{aligned}$$

and $U_1(t)$ can be computed from (34).

We now assume that the claim is valid for $k = n - 1$ and argue that it is also true for $k = n$. Let $\Pi := \Pi_{\mathcal{R}(B)^\perp}$ be the projection onto the orthogonal complement of the range of B

$$\Pi_{\mathcal{R}(B)^\perp} := I - B(B'B)^{-1}B'$$

where I denotes identity matrix; when B is not full column-rank, the inverse needs to be replaced by a pseudoinverse. Since $\Pi B = 0$, for any size of matrices, (29) implies that

$$\Pi\dot{\Sigma}(t)\Pi = \Pi A\Sigma(t)\Pi + \Pi\Sigma(t)A'\Pi + \Pi Q\Pi. \quad (35)$$

Conversely, if (35) holds, there exists a $U(t)$ so that (29) holds. To see this, let $\mathcal{S}_n$ denote the linear vector space of symmetric matrices of dimension n and note that the map

$$\mathfrak{g}_B : \mathcal{S}_n \to \mathcal{S}_n : Y \mapsto \Pi_{\mathcal{R}(B)^\perp} Y \Pi_{\mathcal{R}(B)^\perp} \quad (36)$$

is self-adjoint. Hence, the orthogonal complement of its range is precisely its null space, which according to the lemma in Appendix VI, is also the range of

$$\mathfrak{f}_B : \mathbb{R}^{n\times m} \to \mathcal{S}_n : X \mapsto BX' + XB'. \quad (37)$$

But

$$\dot{\Sigma}(t) - (A\Sigma(t) + \Sigma(t)A' + Q)$$

when projected onto the range of $\mathfrak{g}_B$ is identically zero (since (35) holds). Hence, (29) also holds for a suitable $U(t)$. (In other words, the extra directions that (35) does not already restrict can be freely adjusted by a proper choice of $U(t)$ since they are in the range of $\mathfrak{f}_B$.) The fact that we can always select $U(t)$ to be smooth, provided of course that $\Sigma(t)$ is smooth, follows since $\mathfrak{g}_B$ is linear. Also, in a similar manner as in the case where $k = 1$, we can select $U(t)$ to satisfy arbitrary boundary conditions $U(0)$ and $U(T)$ of its own.

Let us now return to the induction argument. Equation (33) for $k = n$, is equivalent to

$$\Pi_n\dot{\Sigma}(t)\Pi_n = \Pi_n A_n\Sigma(t)\Pi_n + \Pi_n\Sigma(t)A_n'\Pi_n + \Pi_n Q_n\Pi_n \quad (38)$$

where

$$\Pi_n = \begin{bmatrix} I_{n-1} & 0_{n-1} \\ 0_{n-1}' & 0 \end{bmatrix}.$$

If we partition

$$\Sigma(t) = \begin{bmatrix} \Sigma_1(t) & \sigma_2(t) \\ \sigma_2(t)' & \sigma_3(t) \end{bmatrix}$$

where Σ_1 is $(n-1) \times (n-1)$, σ_2 is a column vector, and σ_3 a scalar, then (38) becomes

$$\begin{bmatrix} \dot{\Sigma}_1(t) & 0_{n-1} \\ 0_{n-1}' & 0 \end{bmatrix} = M\begin{bmatrix} \Sigma_1(t) & 0_{n-1} \\ \sigma_2(t)' & 0 \end{bmatrix} + \begin{bmatrix} \Sigma_1(t) & \sigma_2(t) \\ 0_{n-1}' & 0 \end{bmatrix}M' + \begin{bmatrix} Q_1 & 0_{n-1} \\ 0_{n-1}' & 0 \end{bmatrix} \quad (39)$$

where Q_1 is the $(n-1) \times (n-1)$ block of Q and

$$\begin{aligned} M &= \Pi A_n \\ &= \begin{bmatrix} 0_{n-1} & I_{n-1} \\ 0 & 0_{n-1}' \end{bmatrix} \\ &= \begin{bmatrix} A_{n-1} & B_{n-1} \\ 0_{n-1}' & 0 \end{bmatrix} \end{aligned}$$

after we group its entries consistent with the partition of Σ. But now, (39) is in the form

$$\dot{\Sigma}_1(t) = A_{n-1}\Sigma_1(t) + \Sigma_1(t)A_{n-1}' + B_1\sigma_2(t)' + \sigma_2(t)B_1' + Q_1.$$

Since the matrices in this one are of size $(n-1)\times(n-1)$, by our hypothesis, we can find a control $U(t)$ which will then identify with $\sigma_2(t)$. The boundary conditions for $U(t)$ are dictated by the boundary conditions for $\Sigma(t)$. The final entry of $\Sigma(t)$, $\sigma_3(t)$ is not restricted in any way other than being in agreement with the boundary conditions of Σ. The values are the two ends, $\sigma_3(0)$ and $\sigma_3(T)$ are admissible since $\Sigma_0>0$ as well as $\Sigma_T>0$. Thus, we can choose a smooth function for $\sigma_3(t)$ that takes values large enough in $(0,T)$ so that $\Sigma(t)>0$ throughout.

A final point is needed to complete the proof. For an arbitrary controllable pair (A,B) it is well known that there exists a constant K and a vector v such that $(A-BK,Bv)$ is controllable (Heymann's lemma, see [27]). Further, K can be chosen so that $A-BK$ has all eigenvalues at the origin, hence it is equivalent to a shift matrix. Thus, we can choose K and v such that, after a similarity transformation,[4] $(A-BK,Bv)$ becomes (A_n,B_n) (in the notation of (32)). The statement of the theorem is invariant to similarity transformation as well as to action of the feedback group $A\mapsto A-BK$. Further, replacing B with Bv corresponds to selecting a portion of the allowed control authority, and we have already shown the theorem for this case which is more stringent. This completes the proof. ∎

Finite-Interval Steering via External Input: It is interesting to observe that, in the case when $B=B_1$, steering the state-covariance via state-feedback is equivalent to modeling the evolution of state-covariances as due to an external input process. Specifically, given the Gauss-Markov model

$$dx(t)=Ax(t)dt+Bdy(t)$$

and a path of state-covariances $\{\Sigma(t)|t\in[0,T]\}$ that satisfies (29) for some $U(t)$, the claim is that there is a suitable process $y(t)$ that can account for this state-covariance time-evolution. Indeed, starting from the Gauss-Markov process

$$\begin{aligned} d\xi(t)&=(A-BK(t))\,\xi(t)dt+Bdw(t)\\ dy(t)&=-K(t)\xi(t)dt+dw(t)\end{aligned}\tag{40}$$

with $\mathbb{E}\{\xi(0)\xi(0)'\}=\Sigma_0$ and

$$K(t)=-U(t)'\Sigma(t)^{-1}$$

we observe that

$$d\xi(t)=A\xi(t)dt+Bdy(t).$$

Therefore $\xi(t)$ and $x(t)$ share the same statistics. In the converse direction, the state covariance of (40) satisfies (29).

B. Assignability of Stationary State Covariances via State-Feedback

In this section we consider the problem to maintain the state process of a dynamical system at an equilibrium distribution with a specified state-covariance Σ via static state-feedback

$$u(t)=-Kx(t).\tag{41}$$

Due to linearity, the distribution will then be Gaussian. However, depending on the value of Σ this may not always possible.

[4]For (A,b) a controllable pair with A having all eigenvalues at the origin, take $T=[A^{n-1}b,\dots Ab,b]$ and define $b_1:=T^{-1}b$ and $A_1=T^{-1}AT$. It is easy to see that $b_1=[0,\dots 0,1]'$ while A_1 is a shift matrix.

The precise characterization of admissible stationary state-covariances is provided in Theorem 4 given below.

Assuming that $A-BK$ is a Hurwitz matrix, which is necessary for the state process $\{x(t)|t\in[0,\infty)\}$ to be stationary, the (stationary) state-covariance $\Sigma=\mathbb{E}\{x(t)x(t)'\}$ satisfies the algebraic Lyapunov equation

$$(A-BK)\Sigma+\Sigma(A-BK)'=-B_1B_1'.\tag{42}$$

Thus, the equation

$$A\Sigma+\Sigma A'+B_1B_1'+BX'+XB'=0 \quad \text{can be solved for } X \tag{43a}$$

which in particular can be taken to be $X=-\Sigma K'$. The solvability of (43a) is obviously a necessary condition for Σ to qualify as a stationary state-covariance attained via feedback. Alternatively, (43a) is equivalent to the statement that

$$A\Sigma+\Sigma A'+B_1B_1'\in\mathcal{R}(\mathfrak{f}_B).\tag{43b}$$

The latter can be expressed as a rank condition [28, Proposition 1] in the form

$$\operatorname{rank}\begin{bmatrix}A\Sigma+\Sigma A'+B_1B_1' & B\\ B & 0\end{bmatrix}=\operatorname{rank}\begin{bmatrix}0 & B\\ B & 0\end{bmatrix}.\tag{43c}$$

In view of Lemma 6, (43b) is equivalent to

$$A\Sigma+\Sigma A'+B_1B_1'\in\mathcal{N}(\mathfrak{g}_B).\tag{43d}$$

Therefore, the conditions (43a)–(43d), which are all equivalent, are necessary for the existence of a state-feedback gain K that ensures $\Sigma>0$ to be the stationary state covariance of (3).

Conversely, given $\Sigma>0$ that satisfies (43) and X the solution to (43a), then (42) holds with $K=-X'\Sigma^{-1}$. Provided $A-BK$ is a Hurwitz matrix, Σ is an admissible stationary covariance. The property of $A-BK$ being Hurwitz can be guaranteed when $(A-BK,B_1)$ is a controllable pair. In turn, controllability of $(A-BK,B_1)$ is guaranteed when $\mathcal{R}(B)\subseteq\mathcal{R}(B_1)$. Thus, we have established the following.

Theorem 4: Consider the Gauss-Markov model (3) and assume that $\mathcal{R}(B)\subseteq\mathcal{R}(B_1)$. A positive-definite matrix Σ can be assigned as the stationary state covariance via a suitable choice of state-feedback if and only if Σ satisfies any of the equivalent statements (43a)–(43d).

Interest in (43d) was raised in [29] where it was shown to characterize state-covariances that can be maintained by state-feedback. On the other hand, conditions (43a)–(43c) were obtained in [28] and [30], for the special case when $B=B_1$, as being necessary and sufficient for a positive-definite matrix to materialize as the state covariance of the system driven by a stationary stochastic process (not-necessarily white). It should be noted that in [28], the state matrix A was assumed to be already Hurwitz so as to ensure stationarity of the state process. However, if the input is generated via feedback as above, A does not need to be Hurwitz whereas, only $A-BK$ needs to be.

Assignability via External Input: We now turn to the question of which positive definite matrices materialize as state covariances of the Gauss-Markov model

$$dx(t)=Ax(t)dt+Bdy(t)\tag{44}$$

with (A, B) controllable and A Hurwitz, when driven by some stationary stochastic process $y(t)$. The characterization of admissible state covariances was obtained in [28] and amounts to the condition that

$$A\Sigma + \Sigma A' \in \mathcal{R}(\mathfrak{f}_B)$$

which coincides with the condition that Σ can be assigned as in Theorem 4 by state-feedback. As in Section III-A, a feedback system can be implemented, separate from (44), to generate a suitable input process to give rise to Σ as the state covariance of (44). Specifically, let X be a solution of

$$A\Sigma + \Sigma A' + BX' + XB' = 0 \tag{45}$$

and

$$d\xi(t) = (A - BK)\xi(t)dt + Bdw(t)$$
$$dy(t) = -K\xi(t)dt + dw(t)$$

with

$$K = \frac{1}{2}B'\Sigma^{-1} - X'\Sigma^{-1}. \tag{46}$$

Trivially

$$d\xi(t) = A\xi(t)dt + Bdy(t)$$

and therefore, $\xi(t)$ shares the same stationary statistics with $x(t)$. But if $S = \mathbb{E}\{\xi(t)\xi(t)'\}$

$$(A - BK)S + S(A - BK)' + BB' = 0$$

which, in view of (45), (46), is satisfied by $S = \Sigma$.

IV. Numerical Computation of Optimal Control

Having established feasibility for the problem to steer the state-covariance to a given value at the end of an interval, it is of interest to design efficient methods to compute the optimal controls of Section II. As an alternative to solving the generalized Schrödinger system (11a)–(11d), we formulate the optimization as a semidefinite program in Section IV-A, and likewise for the infinite-horizon problem in Section IV-B. See [31] for other applications of semidefinite programming in control theory.

A. Finite Interval Minimum Energy Steering of State Statistics

We are interested in computing an optimal choice for feedback gain $K(t)$ so that the control signal $u(t) = -K(t)x(t)$ steers (3) from an initial state-covariance Σ_0 at $t = 0$ to the final Σ_T at $t = T$. The expected control-energy functional

$$J(u) := \mathbb{E}\left\{\int_0^T u(t)'u(t)dt\right\}$$
$$= \int_0^T \operatorname{trace}\left(K(t)\Sigma(t)K(t)'\right)dt \tag{47}$$

needs to be optimized over $K(t)$ so that (27) holds as well as the boundary conditions

$$\Sigma(0) = \Sigma_0, \text{ and } \Sigma(T) = \Sigma_T. \tag{48a}$$

If instead we sought to optimize over $U(t) := -\Sigma(t)K(t)'$ and $\Sigma(t)$, the functional (47) becomes

$$J = \int_0^T \operatorname{trace}\left(U(t)'\Sigma(t)^{-1}U(t)\right)dt$$

which is jointly convex in $U(t)$ and $\Sigma(t)$, while (27) is replaced by

$$\dot{\Sigma}(t) = A\Sigma(t) + \Sigma(t)A' + BU(t)' + U(t)B' + B_1B_1' \tag{48b}$$

which is now linear in both. Thus, finally, the optimization can be written as a semi-definite program to minimize

$$\int_0^T \operatorname{trace}\left(Y(t)\right)dt \tag{48c}$$

subject to (48a), (48b) and

$$\begin{bmatrix} Y(t) & U(t)' \\ U(t) & \Sigma(t) \end{bmatrix} \geq 0. \tag{48d}$$

This can be solved numerically after discretization in time via `cvx` ([32]) and a corresponding (suboptimal) gain recovered as $K(t) = -U(t)'\Sigma(t)^{-1}$. The constraints imposed by discretization in time of (48b), e.g., using an Euler scheme, have a block-sparse structure and therefore it will be advantageous to develop customized optimization algorithms for large problems.

B. Minimum Energy Control to Maintain Stationary State Statistics

As noted earlier, a positive definite matrix Σ is admissible as a stationary state-covariance provided (43a) holds for some X and $A + BX'\Sigma^{-1}$ is a Hurwitz matrix. The condition $\mathcal{R}(B) \subseteq \mathcal{R}(B_1)$ is a sufficient condition for the latter to be true always, but it may be true even if $\mathcal{R}(B) \subseteq \mathcal{R}(B_1)$ fails (see the example in Section V). Either way, the expected input power (energy rate) is

$$\mathbb{E}\{u'u\} = \operatorname{trace}(K\Sigma K')$$
$$= \operatorname{trace}(X'\Sigma^{-1}X) \tag{49}$$

expressed in either in K, or X. Thus, assuming that $\mathcal{R}(B) \subseteq \mathcal{R}(B_1)$ holds, and in case (43a) has multiple solutions, the optimal constant feedback gain K can be obtained by solving the convex optimization problem

$$\min\left\{\operatorname{trace}(K\Sigma K') \mid (43a) \text{ holds}\right\}. \tag{50}$$

Remark 5: In case $\mathcal{R}(B) \not\subseteq \mathcal{R}(B_1)$, the condition that $A - BK$ be Hurwitz needs to be verified separately. If this fails, we cannot guarantee that Σ is an admissible stationary state-covariance that can be maintained with constant state-feedback. However, it is always possible to maintain a state-covariance that is arbitrarily close. To see this, consider the control

$$K_\epsilon = K + \frac{1}{2}\epsilon B'\Sigma^{-1}$$

for $\epsilon > 0$. Then, from (42)

$$(A - BK_\epsilon)\Sigma + \Sigma(A - BK_\epsilon)' = -\epsilon BB' - B_1B_1' \\ \leq -\epsilon BB'.$$

The fact that $A - BK_\epsilon$ is Hurwitz is obvious. If now Σ_ϵ is the solution to

$$(A - BK_\epsilon)\Sigma_\epsilon + \Sigma_\epsilon(A - BK_\epsilon)' = -B_1B_1'$$

the difference $\Delta = \Sigma - \Sigma_\epsilon \geq 0$ and satisfies

$$(A - BK_\epsilon)\Delta + \Delta(A - BK_\epsilon)' = -\epsilon BB'$$

and hence is of $o(\epsilon)$.

V. Examples

Example 1: Consider particles that are modeled by

$$dx(t) = v(t)dt + dw(t) \\ dv(t) = u(t)dt.$$

Here, $u(t)$ is the control input (force) at our disposal, $x(t)$ represents position and $v(t)$ velocity (integral of acceleration due to input forcing), while $w(t)$ represents random displacement due to impulsive accelerations. The purpose of the example is to highlight a case where the control is handicapped compared to the effect of noise. Indeed, the displacement $w(t)$ is directly affecting the position while the control effort needs to be integrated before it impacts the position of the particles.

Another interesting aspect of this example is that $\mathcal{R}(B) \not\subseteq \mathcal{R}(B_1)$ since $B = [0, 1]'$ while $B_1 = [1, 0]'$. If we choose

$$\Sigma_1 = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \quad (51)$$

as a candidate stationary state-covariance, it can be seen that (43a) has a unique solution X giving rise to $K = [1, 1]$ and a stable feedback since $A - BK$ is Hurwitz.

We wish to steer the spread of the particles from an initial Gaussian distribution with $\Sigma_0 = 2I$ at $t = 0$ to the terminal marginal Σ_1 at $t = 1$, and from there on, since Σ_1 is an admissible stationary state-covariance, to maintain with constant state-feedback control.

Fig. 1 displays typical sample paths in phase space as functions of time. These are the result of using the optimal feedback strategy derived following (48c) over the time interval [0, 1]. The optimal feedback gains $K(t) = [k_1(t), k_2(t)]$ are shown in Fig. 2 as functions of time over the interval [0, 1], where the state-covariance transitions to the chosen admissible steady-state value Σ_1. The corresponding cost is $J(u) = 9.38$. Past the point in time $t = 1$, the state-covariance of the closed-loop system is maintained at this stationary value in (51). Fig. 3 shows representative sample paths in phase space under the now constant state feedback gain $K = [1, 1]$ over time window [1, 5]. Finally, Fig. 4 displays the corresponding control action for each trajectory over the complete time interval [0, 5], which consists of the "transient" interval [0, 1] to the target (stationary) distribution and the "stationary" interval [1, 5].

Example 2: Consider a second-order process with random acceleration (e.g., modeling the dynamics of inertial particles)

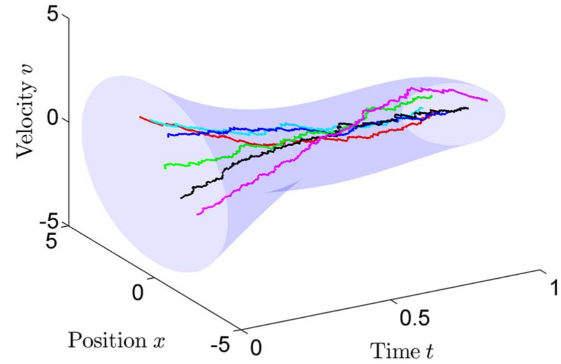

Fig. 1. Finite-interval steering in phase space (Example 1).

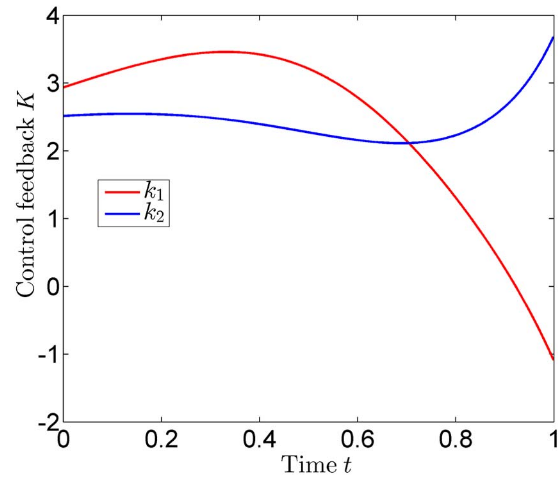

Fig. 2. Optimal feedback gains in finite-interval steering.

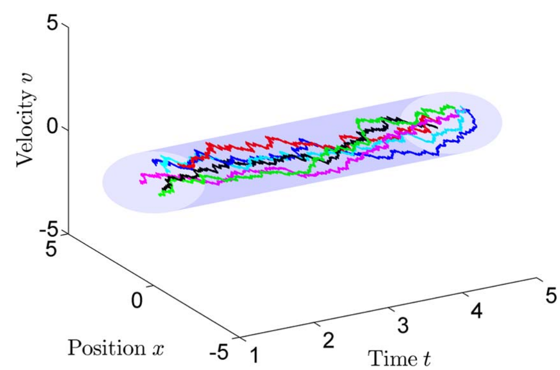

Fig. 3. Steady state trajectories in phase space (Example 1).

where the control is again "handicapped" by the lag in actuation dynamics. More specifically, consider

$$dx(t) = v(t)dt \\ dv(t) = x_c(t)dt + dw(t) \\ dx_c(t) = -x_c(t)dt + u(t)dt. \quad (52)$$

Here, x_c is the 1-dimensional state/output of the actuator and represents force, while $u(t)$ represents the control signal to the actuator. We (arbitrarily) select

$$\Sigma_{x,v} = \begin{bmatrix} \frac{7}{4} & 0 \\ 0 & \frac{3}{4} \end{bmatrix}$$

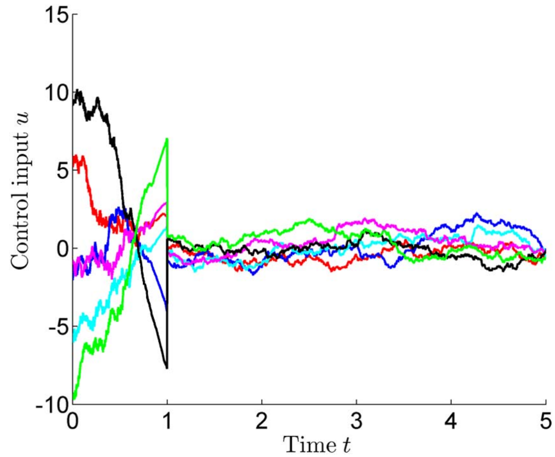

Fig. 4. Control inputs for steering over [0,1] and for steady state operation for times ≥1 (Example 1).

as a desirable steady-state covariance for the projection of the process onto the phase-plane of the particle dynamics (x, v). (The additional component x_c corresponding to the actuator is not shown.) First, we need to determine whether $\Sigma_{x,v}$ is indeed an admissible steady-state covariance and, if so, to determine an optimal choice for the constant state-feedback gain that ensures the state is distributed accordingly. To this end, we seek a choice of a variance Σ_{x_c} for x_c and of a cross-covariance Y between x_c and (x, v) so that

$$\Sigma = \begin{bmatrix} \Sigma_{x,v} & Y \\ Y' & \Sigma_{x_c} \end{bmatrix} > 0$$

is a admissible stationary state-covariance for (52). For this to be true, we need to verify that (43c) holds with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

This is indeed the case for $Y = [-3/4\ \ -1/2]'$, $\Sigma_{x_c} = 3/4$.

For the above choice of Σ, the optimal gain and power are found to be $K = [1\,3\,2]$ and $J = 5/2$ by solving (50) using e.g., [32]. For this solution, stationary state trajectories, projected onto the (x, v)-coordinates are now displayed in Fig. 6.

As before the steering between specified Gaussian probability densities over an interval $[0, 1]$ follows Section IV-A. For completeness, we display in Fig. 5 sample paths corresponding to the transition between marginals with covariance matrices $\Sigma_0 = 3I$ to $\Sigma_1 = \Sigma$, respectively. The figure shows the projection onto the (x, v)-component of the process that corresponds to position and velocity.

APPENDIX

Lemma 6: Consider the maps $\mathfrak{f}_B$ and $\mathfrak{g}_B$ defined in (36), (37). The range of $\mathfrak{f}_B$ coincides with the null space of $\mathfrak{g}_B$, that is

$$\mathcal{R}(\mathfrak{f}_B) = \mathcal{N}(\mathfrak{g}_B).$$

Proof: It is immediate that

$$\mathcal{R}(\mathfrak{f}_B) \subseteq \mathcal{N}(\mathfrak{g}_B).$$

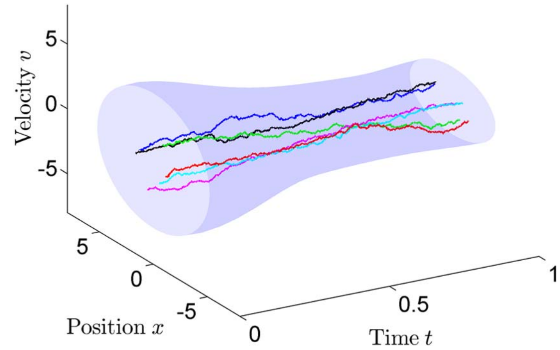

Fig. 5. Finite-interval steering in phase space (Example 2).

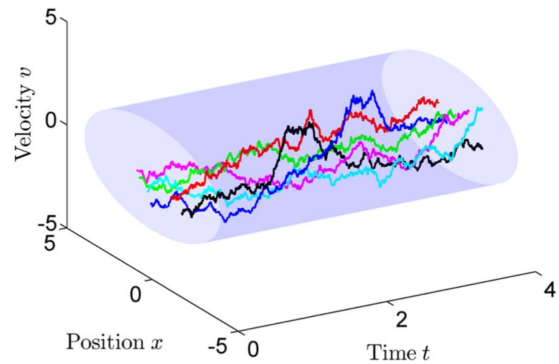

Fig. 6. Steady state trajectories in phase space (Example 2).

To show equality it suffices to show that $(\mathcal{R}(\mathfrak{f}_B))^\perp \subseteq \mathcal{N}(\mathfrak{g}_B)^\perp$. To this end, consider

$$M \in \mathcal{S}_n \cap (\mathcal{R}(\mathfrak{f}_B))^\perp.$$

Then

$$\mathrm{trace}\,(M(BX + X'B')) = 0$$

for all $X \in \mathbb{R}^{m\times n}$. Equivalently, for $Z = MB \in \mathbb{R}^{n\times m}$, $\mathrm{trace}(ZX) + \mathrm{trace}(X'Z') = 0$ for all X. Thus, $\mathrm{trace}(ZX) = 0$ for all X and hence $Z = 0$. Since $MB = Z = 0$, then $M\Pi_{\mathcal{R}(B)} = 0$ or, equivalently, $M\Pi_{\mathcal{R}(B)^\perp} = M$. Therefore $\Pi_{\mathcal{R}(B)^\perp} M \Pi_{\mathcal{R}(B)^\perp} = M$, i.e., $M \in (\mathcal{R}(\mathfrak{g}_B))$. Therefore

$$(\mathcal{R}(\mathfrak{f}_B))^\perp \subseteq (\mathcal{R}(\mathfrak{g}_B)) = \mathcal{N}(\mathfrak{g}_B)^\perp$$

since $\mathfrak{g}_B$ is self-adjoint, which completes the proof. ■

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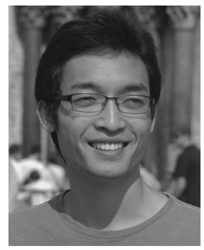

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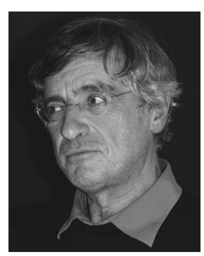

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