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SPECTRAL FACTORIZATION AND NEVANLINNA-PICK INTERPOLATION*

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Abstract. We develop a spectral factorization algorithm based on linear fractional transformations and on the Nevanlinna-Pick interpolation theory. The algorithm is recursive and depends on a choice of points (z_k, $k=1, 2, \cdots$) inside the unit disk. Under a mild condition on the distribution of the z_k's, the convergence of the algorithm is established. The algorithm is flexible and convergence can be influenced by the selection of z_k's.

Key words. spectral factorization, interpolation theory, positive-real functions, Nevanlinna-Pick interpolation

AMS(MOS) subject classifications. 93C05, 30E05, 30D50

1. Introduction. Interpolation theory for complex analytic functions has a long history in mathematics and engineering. The origin of the subject can be traced back at least to the work of Caratheodory, Schur, Nevanlinna and Pick (see [7]) and continues with the recent works of Adamjan, Arov and Krein [1], Sarason [24], Sz. Nagy and Foias [20], and Ball and Helton [3] which have extended the theory to a general operator theoretic setting. In engineering, interpolation theory has been used in a variety of problem areas. Passive circuit synthesis, optimal control, stability theory, representation and prediction theory for stochastic processes, and control theory are some of the engineering disciplines where interpolation theory of complex analytic functions has played a significant role. For these, see for example [10], [11], [18], [19], [25], [26] and the references therein.

In the present work we use interpolation theoretic ideas and, in particular, linear fractional transformations to develop a general scheme for spectral factorization. Spectral factorization is a key problem in a variety of engineering fields, and has been investigated extensively. In particular, see [2], [5], [6], [11], [12], [15], [21], [23]. The approach we have taken leads to a connection with ideas from interpolation theory and in particular to the use of linear fractional transformations. *Our main contribution in this paper is a new and versatile theoretical algorithm for spectral factorization.* However, numerical properties of this algorithm are not addressed here and will be pursued elsewhere.

We denote by $\mathbb{U}$ the unit ball in H^∞; i.e., $\mathbb{U} := \{f(z)$ analytic in $\mathbb{D}$ such that $|f(z)| \leqq 1$ for all $z \in \mathbb{D}\}$, where $\mathbb{D}$ denotes the *open unit disc* in the complex plane. The classical Nevanlinna-Pick interpolation problem requires finding a function f in $\mathbb{U}$ that satisfies the following interpolation conditions:

$$f^{(m)}(z_k) = w_{m,k}, \qquad m = 0, 1, \cdots, N_k, \quad \text{for } k = 1, 2, \cdots, N.$$

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(The case $N = 1$ is known as Caratheodory-Schur interpolation.) The *Nevanlinna-Pick recursion* allows the solution, i.e., existence and characterization of all solutions, of this problem by iteratively reducing it to an equivalent one with fewer interpolation constraints. The general form of the solutions is then presented in terms of a linear fractional transformation

$$f(z) = [A(z) + B(z)h(z)] \times [C(z) + D(z)h(z)]^{-1}, \tag{1.1}$$

where A, B, C and D are functions depending on the data of the problem and $h(z)$ is an arbitrary function in $\mathbb{U}$ that parametrizes the set of solutions.

With every function f in $\mathbb{U}$, such that

$$\ln [1 - |f(e^{i\theta})|^2] \text{ is in } L^1 := L^1[-\pi, \pi],$$

there is associated a unique outer function (see [22])

$$g_f(z) := \exp\left\{ (4\pi)^{-1} \int_{-\pi}^{\pi} [e^{i\theta} + z][e^{i\theta} - z]^{-1} \ln [1 - |f(e^{i\theta})|^2] \, d\theta \right\}$$

such that $|g_f(e^{i\theta})|^2 = 1 - |f(e^{i\theta})|^2$ a.e. on $[-\pi, \pi]$. The function g_f is known as the **canonical spectral factor** of f and plays an important role in several problem areas such as representation of stochastic processes, optimal control, network synthesis, etc. Under fairly general conditions g_f can in fact be defined as a meromorphic function on the whole complex plane (see [8]). This is certainly true for the important case where f is also a rational function, and this is precisely the case we consider in the present paper.

If g_f and g_h denote the spectral factors of f and h respectively that are related as in (1.1), then it can be shown that g_f and g_h under certain conditions have the same zeros. (The general problem of describing invariants of the action of the semigroup of linear fractional transformations has been considered by Helton [17].) Utilizing the invariance of the so-called spectral zeros (or transmission zeros) under linear fractional transformations, we developed a spectral factorization algorithm along the lines of Caratheodory-Schur interpolation [14], [15]. The present work extends our earlier results to the Nevanlinna-Pick setting and gives rise to a general spectral factorization algorithm.

2. The Nevanlinna–Pick recursion. We begin with the following well-known lemma, which is simply an invariant formulation of Schwarz's lemma. This has provided an important tool in the theory of interpolation with complex analytic functions and was utilized in a masterful way in that context by Nevanlinna (see Garnett [13]).

LEMMA 2.1. *Let f_1 be in $\mathbb{U}$, and consider a sequence of points $(z_k \in \mathbb{D}, k = 1, 2, \cdots)$ and a sequence of parameters $(c_k \in \mathbb{D}, k = 1, 2, \cdots)$. Define*

$$w_k := f_k(z_k), \tag{2.2a}$$

$$\tilde{f}_{k+1} := \frac{1 - \bar{z}_k z}{z - z_k} \frac{f_k - w_k}{1 - \bar{w}_k f_k}, \tag{2.2b}$$

$$f_{k+1} := \frac{\tilde{f}_{k+1} - c_k}{1 - \bar{c}_k \tilde{f}_{k+1}}, \tag{2.2c}$$

for $k = 1, 2, \cdots$. Then, f_k, $k = 2, 3, \cdots$, is a sequence of $\mathbb{U}$-functions. In case $|w_n| = 1$ for a value $k = n$, then the above sequence terminates to a function $f_n \equiv w_n$. This last case *occurs only if* f_1 *is a finite Blaschke product*; *i.e.*, f_1 *is of the form*

$$f_1(z)=e^{i\varphi}\prod_{k=1}^{n}\frac{(z-\xi_k)}{(1-\bar{\xi}_k z)},$$

and consequently has modulus equal to one on the boundary of the disk.

The case where f_1 is a finite Blaschke product is of no interest to us because in this case the spectral factor of f_1 is the zero function, and the spectral factorization problem becomes trivial. Hence, in the sequel, even when not explicitly stated, we tacitly assume that this is not the case.

The sequence of the parameters c_k in the Nevanlinna recursion can be taken to be arbitrary constants in $\mathbb{D}$. However, we follow a standard and convenient normalization (see [4], [8]) described in the lemma below.

LEMMA 2.3. *Let* $f_1\in\mathbb{U}$ *(but not a finite Blaschke product), and let* $f_1(0)=0$. *Given a sequence of points* $(z_k\in\mathbb{D},\ k=1,2,\cdots)$, *and letting*

$$c_k:=\tilde{f}_{k+1}(0)=\begin{cases}\dfrac{w_k}{z_k} & \text{whenever } z_k\neq 0,\\[2mm] \lim\limits_{z\to 0}\dfrac{f_k(z)}{z} & \text{whenever } z_k=0,\end{cases}\tag{2.4}$$

the Nevanlinna recursion (2.2a–c) *produces a sequence* f_k, $k=2,3,\cdots$, *of* $\mathbb{U}$*-functions that satisfy* $f_k(0)=0$, *for all* k.

The Nevanlinna recursion in Lemma 2.1 can be used to provide a constructive approach to the Nevanlinna–Pick problem (see Garnett [13, p. 166]). It can also be used to generate, from a known function f_1 in $\mathbb{U}$, the associated sequence of the so-called Schur parameters/reflection coefficients w_k. This is the way we apply the Nevanlinna recursion. In fact, our objective in the next section is to *study the limiting behavior of the "by-product"* f_k *as* k *tends to* ∞.

3. Some convergence results. Let f_1 (different from a finite Blaschke product) be in $\mathbb{U}$ and assume that $f_1(0)=0$. This causes no loss of generality from our standpoint because the functions f in $\mathbb{U}$, and zf which is also in $\mathbb{U}$, have the same spectral factor. The assumption $f_1(0)=0$ simplifies the computations required in the sequel.

Compute now the sequence f_k, $k=2,3,\cdots$, of $\mathbb{U}$-functions from f_1 and the sequence of points $(z_k\in\mathbb{D},\ k=1,2,\cdots)$ via the Nevanlinna recursion (2.2) and (2.4). Recall that the choice (2.4) for the constants c_k readily implies that $f_k(0)=0$ for $k=2,3,\cdots$. This is very convenient as we will see shortly.

Using (2.2b) and (2.2c) it easily follows that

$$\frac{(1-|w_k|^2)(1-|f_k|^2)}{|1-\bar{w}_k f_k|^2}=\frac{(1-|c_k|^2)(1-|f_{k+1}|^2)}{|1+\bar{c}_k f_{k+1}|^2}\quad\text{for } z \text{ on } \mathbb{T},$$

and $k=1,2,\cdots,n-1$. Applying $\ln(\cdot)$ to both sides of the above equation, we obtain that

$$\begin{aligned}&\ln(1-|w_k|^2)+\ln(1-|f_k|^2)-\ln|1-\bar{w}_k f_k|^2\\ &\quad=\ln(1-|c_k|^2)+\ln(1-|f_{k+1}|^2)-\ln|1+\bar{c}_k f_{k+1}|^2\end{aligned}\tag{3.1}$$

for $z\in\mathbb{T}$. Note that $|\bar{w}_k f_k|<1$ in $\mathbb{D}$. Hence $1-\bar{w}_k f_k$ is an analytic function with no roots in $\mathbb{D}$. Therefore, (see Rudin [22, Thm. 13.12]) $u(z):=\ln|1-\bar{w}_k f_k|^2$ is a harmonic function in $\mathbb{D}$. Also, since $f_k(0)=0$, it follows that $u(0)=0$. Consequently, using the well-known mean value property of harmonic functions (see [22])

$$\int_{-\pi}^{\pi} \ln|1-\bar{w}_k f_k(e^{i\theta})|^2\,d\theta = u(0)=0.$$

A similar argument applies to $\ln|1+\bar{c}_k f_{k+1}|^2$ and yields

$$\int_{-\pi}^{\pi} \ln|1+\bar{c}_k f_{k+1}(e^{i\theta})|^2\,d\theta = 0.$$

Now, from (3.1), we integrate over the interval $[-\pi, \pi]$ and exponentiate both sides to obtain that

$$\exp\left\{(2\pi)^{-1}\int_{-\pi}^{\pi} \ln(1-|f_k(e^{i\theta})|^2)\,d\theta\right\} = \left(\frac{1-|c_k|^2}{1-|w_k|^2}\right)\exp\left\{(2\pi)^{-1}\int_{-\pi}^{\pi}\ln(1-|f_{k+1}|^2)\,d\theta\right\}$$

for all k. Finally by induction we conclude that

$$\exp\left\{(2\pi)^{-1}\int_{-\pi}^{\pi} \ln(1-|f_1(e^{i\theta})|^2)\,d\theta\right\} = \left(\prod_{k=1}^{n}\frac{1-|c_k|^2}{1-|w_k|^2}\right)\exp\left\{(2\pi)^{-1}\int_{-\pi}^{\pi}\ln(1-|f_{k+1}|^2)\,d\theta\right\}. \tag{3.2}$$

We are interested in the case where f_1 is a rational function in $\mathbb{U}$ but different from a finite Blaschke product. In this case the above integrals are different from zero and we can obtain the following theorem.

THEOREM 3.3. *Let f_1 be a rational function in $\mathbb{U}$ such that* $\ln(1-|f_1(e^{i\theta})|^2)$ *is in L^1 (hence not a Blaschke product), and $f_1(0)=0$. Let $(z_k,\ k=1,2,\cdots)$ be a sequence of points in $\mathbb{D}$ satisfying the property*

$$\sum_{k=1}^{\infty}(1-|z_k|)=\infty, \tag{3.4}$$

and obtain the corresponding sequence of w_k' s and c_k's from (2.2) *and* (2.4). *Then*

$$\exp\left\{(2\pi)^{-1}\int_{-\pi}^{\pi} \ln(1-|f_1(e^{i\theta})|^2)\,d\theta\right\} = \lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{1-|c_k|^2}{1-|w_k|^2}\right).$$

The proof of the above theorem is based on certain classical facts in function theory and some results obtained by Dewilde and Dym [8], [9] and Bultheel and Dewilde [4], and is given in § 6. Below we give an immediate corollary of Theorem 3.3 and relation (3.2).

COROLLARY 3.5. *Under the conditions of Theorem* 3.3,

$$\lim_{k\to\infty}\exp\left\{(2\pi)^{-1}\int_{-\pi}^{\pi}\ln(1-|f_{k+1}|^2)\,d\theta\right\}=1$$

and

$$\lim_{k\to\infty} f_{k+1}(z)=0 \text{ a.e. on } \mathbb{T}.$$

4. Invariance of spectral zeros–spectral factorization. Let f be a rational $\mathbb{U}$-function (but not a finite Blaschke product) and let it be represented as the ratio of two coprime polynomials $a(z)/b(z)$. Since f is in $\mathbb{U}$,

$$1-|f(z)|^2=\frac{|b(z)|^2-|a(z)|^2}{|b(z)|^2}\geqq 0 \quad \text{on } \mathbb{T},$$

and it admits a factorization

$$1-|f(z)|^2=\frac{|\mu\eta(z)|^2}{|b(z)|^2} \quad \text{for } z=e^{i\theta}, \tag{4.1}$$

where $\eta(z)$ is a polynomial that *can be taken to have no root inside* $\mathbb{D}$ *with* $\eta(0)=1$, *and* μ *a positive constant.* Under this normalization $\eta(z)$ is uniquely defined by f and will be called the **spectral numerator of** f. The canonical spectral factor of f is then given by

$$g(z)=\mu\frac{\eta(z)}{b(z)},$$

and is defined on the whole complex plane. Moreover, (4.1) *extends* to an equality of *meromorphic* functions

$$1-f(z)f(z^{-1})=\mu^2\frac{\eta(z)}{b(z)}\frac{\bar{\eta}(z^{-1})}{\bar{b}(z^{-1})} \tag{4.2}$$

valid throughout the complex plane.

Let now f_1 be a rational $\mathbb{U}$-function and f_k, $k=2,3,\cdots$, be the sequence of $\mathbb{U}$-functions obtained from f_1 and from a sequence of points (z_k, $k=1,2,\cdots$) via the Nevanlinna recursion. In view of (2.2), it is clear that f_{k+1} is also a rational function. Thus, the Nevanlinna recursion produces a sequence of rational functions f_k, $k=2,3,\cdots$.

We now express the recurrence formulas (2.2, 2.4) in terms of fractional representations a_k/b_k for the functions f_k, $k=1,2,\cdots$. First

$$w_k:=\frac{a_k(z_k)}{b_k(z_k)}. \tag{4.3}$$

By solving (2.2) for f_{k+1} in terms of f_k we obtain (see also Garnett [13, p. 167])

$$f_{k+1}=\frac{\alpha_k-\gamma_k f_k}{-\beta_k+\delta_k f_k}, \tag{4.4}$$

where

$$\begin{aligned}\alpha_k(z)&=w_k(1-\bar{z}_k z)+c_k(z-z_k),\\ \beta_k(z)&=\bar{c}_k w_k(1-\bar{z}_k z)+(z-z_k),\\ \gamma_k(z)&=(1-\bar{z}_k z)+c_k\bar{w}_k(z-z_k),\\ \delta_k(z)&=\bar{c}_k(1-\bar{z}_k z)+\bar{w}_k(z-z_k)\end{aligned} \tag{4.5}$$

and we use (2.4), which becomes

$$c_k=\begin{cases}\dfrac{w_k}{z_k} & \text{when } z_k\neq 0,\\[2mm] \displaystyle\lim_{z\to 0}\frac{a_k(z)}{zb_k(z)} & \text{when } z_k=0.\end{cases} \tag{4.6}$$

Starting from a coprime fraction for $f_1 = a_1/b_1$, i.e., a_1 and b_1 are polynomials with no common factor, define a sequence of pairs of functions (that will turn out to be polynomials) (a_k, b_k), for $k = 2, 3, \cdots$, via the following:

$$\begin{bmatrix} a_{kf1} \\ b_{k+1} \end{bmatrix} = \frac{1}{(z - z_k)(1 - |c_k|^2)} \begin{bmatrix} \gamma_k & -\alpha_k \\ -\delta_k & \beta_k \end{bmatrix} \begin{bmatrix} a_k \\ b_k \end{bmatrix} \tag{4.7}$$

for $k = 2, 3, \cdots$. We can now state the following proposition:

PROPOSITION 4.8. *Let f_1 be a rational $\mathbb{U}$-function that is not a finite Blaschke product, a_1/b_1 be a polynomial coprime fraction for f_1, and generate the sequence of (a_k, b_k) and of f_k via* (4.3)-(4.7). *Then the following hold.*

(4.8a) $f_k = a_k/b_k$ *for all k.*

(4.8b) (a_k, b_k) *for $k = 2, 3, \cdots$, are polynomials in z.*

(4.8c) *The maximum degree of the polynomials (a_k, b_k) never exceeds the maximum degree of the polynomials (a_1, b_1).*

(4.8d) *If $f_1(0) = 0$, and the polynomials a_1, b_1 have been normalized to satisfy $a_1(0) = 0$, and $b_1(0) = 1$, then*

$$a_k(0) = 0 \quad and \quad b_k(0) = 1 \quad for\ all\ k.$$

Proof.

(4.8a) It follows immediately by comparison of (4.4) with (4.7).

(4.8b) For $z = z_k$, the expressions $(\gamma_k a_k - \alpha_k b_k)$ and $(\delta_k a_k - \beta_k b_k)$ become equal to $[(1 - \bar{z}_k z_k) a_k(z_k) - w_k(1 - \bar{z}_k z_k) b_k(z_k)]$, and $[\bar{c}_k(1 - \bar{z}_k z_k) a_k(z_k) - \bar{c}_k w_k(1 - \bar{z}_k z_k) b_k(z_k)]$, respectively. But $a_k(z_k) = w_k b(z_k)$. Hence, both expressions become equal to zero. Therefore, $(\gamma_k a_k - \alpha_k b_k)$ and $(\delta_k a_k - \beta_k b_k)$ are polynomial expressions divisible by $(z - z_k)$. From (4.7) we now conclude that (a_k, b_k), for $k = 2, 3, \cdots$ are polynomials in z.

(4.8c) The polynomials α, β, γ and δ, have degree equal to one. Hence the maximum degree of $\{(\gamma_k a_k - \alpha_k b_k)/(z - z_k), (\delta_k a_k - \beta_k b_k)/(z - z_k)\}$ does not exceed the maximum degree of (a_k, b_k).

(4.8d) It follows by straightforward computation. □

However, $a_k(z)$ and $b_k(z)$ might have a common factor. The determinant of the transformation matrix in (4.7) is computed directly and is given below

$$\gamma_k \beta_k - \alpha_k \delta_k = (1 - |c_k|^2)(1 - |w_k|^2)(z - z_k)(1 - \bar{z}_k z) \tag{4.9}$$

for $k = 1, 2, \cdots$. (Note that $(z - z_k)$ cannot be a factor of a_{k+1} or b_{k+1} since it has been divided out in (4.7).) *Thus, the only possible common factor of a_{k+1} and b_{k+1}, in addition to common factors of a_k and b_k, is $(1 - \bar{z}_k z)$.* In fact a_{k+1} and b_{k+1} will have $(1 - \bar{z}_k z)$ as a common factor precisely when it is also a factor in $\eta_k(z)$. (Then, this becomes a common factor of every pair (a_{k+l}, b_{k+l}) for $l = 1, 2, \cdots$.) The following theorem addresses exactly this point.

THEOREM 4.10. *Let f_1 be a rational $\mathbb{U}$-function (different from a finite Blaschke product) and a_k/b_k be a coprime polynomial fraction description of f_1 satisfying* (4.8d). *Let (a_k, b_k), $k = 2, 3, \cdots$, be obtained from* (4.3), (4.5)-(4.7) *and a sequence of points (z_k in $\mathbb{D}$, for $k = 1, 2, \cdots$). Define d_k to be the greatest common divisor of (a_k, b_k)*

normalized by $d_k(0)=1$, *and let* η_k *denote the spectral numerator of the* $\mathbb{U}$*-function* $f_k := a_k/b_k$. *Then the following hold:*

(i) *If for* $k=n$, $(1-\bar{z}_n z)$ *is a factor of* $\eta_n(z)$, *then*

$$d_{n+1}=(1-\bar{z}_n z)\, d_n, \text{ whereas}$$

$$\eta_n=(1-\bar{z}_n z)\eta_{n+1};$$

(ii) *If for* $k=n$, $(1-\bar{z}_n z)$ *is* <u>*not*</u> *a factor of* $\eta_n(z)$, *then*

$$d_{n+1}=d_n, \text{ and } \eta_n=\eta_{n+1}.$$

Proof. We begin by recalling first a certain well-known property of the transformation matrix

$$M_k(z):=\frac{1}{(z-z_k)(1-|c_k|^2)}\begin{bmatrix}\gamma_k(z) & -\alpha_k(z)\\ -\delta_k(z) & \beta_k(z)\end{bmatrix}$$

used in (4.7). Define by

$$M_k(z)_* := M_k^*(z^{-1}),$$

where ( )* denotes complex conjugation of the coefficients and transposition of the matrix. Then,

$$M_k(z)_*\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}M_k(z)=\frac{(1-|w_k|^2)}{(1-|c_k|^2)}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}. \tag{4.11}$$

This property is known as *J-unitarity*; e.g., see [9]. (It follows directly by algebraic manipulations and the use of (4.5) and (4.9).)

We now compute

$$\begin{aligned}&\bar{b}_{k+1}(z^{-1})b_{k+1}(z)-\bar{a}_{k+1}(z^{-1})a_{k+1}(z)\\ &=[\bar{a}_{k+1}(z^{-1})\bar{b}_{k+1}(z^{-1})]\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}a_{k+1}(z)\\ b_{k+1}(z)\end{bmatrix},\end{aligned}$$

which because of (4.11)

$$\begin{aligned}&=\frac{(1-|w_k|^2)}{(1-|c_k|^2)}[\bar{a}_k(z^{-1})\bar{b}_k(z^{-1})]\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}a_k(z^{-1})\\ b_k(z^{-1})\end{bmatrix}\\ &=\frac{(1-|w_k|^2)}{(1-|c_k|^2)}(\bar{b}_k(z^{-1})b_k(z)-\bar{a}_k(z^{-1})a_k(z)).\end{aligned}$$

This last equality implies that

$$|d_{k+1}(z)\eta_{k+1}(z)|^2=|d_k(z)\eta_k(z)|^2, \qquad z=e^{i\theta}, \tag{4.12}$$

where μ_k is a nonzero constant that can be taken to be positive. Also note that, as it was argued before on the basis of (4.9) and (4.7),

$$d_{k+1}(z) \text{ is either equal to } d_k(z) \text{ or equal to } (1-\bar{z}_k z)d_k(z) \tag{4.13}$$

for all values of k. Consequently, $d_k(z)$ has no roots in $\mathbb{D}$. Since the same applies to $\eta_k(z)$, we now conclude from (4.12) that

$$d_{k+1}(z)\eta_{k+1}(z)=d_k(z)\eta_k(z) \quad \text{for } k=1,2,\cdots. \tag{4.14}$$

Now, if $(1-\bar{z}_k z)$ is not a factor of $\eta_k(z)$, then we conclude from (4.13) and (4.14) that

$$d_{k+1}(z)=d_k(z) \quad \text{and} \quad \eta_{k+1}(z)=\eta_k(z).$$

If $(1-\bar{z}_k z)$ is a factor of $\eta_k(z)$, we only need to consider the case where $z_k \neq 0$. In this case

$$\eta_k(\bar{z}_k^{-1})\bar{\eta}_k(\bar{z}_k)=0,$$

which implies that

$$1-f_k(\bar{z}_k^{-1})\bar{f}_k(\bar{z}_k)=0. \tag{4.15}$$

But $\bar{f}_k(\bar{z}_k)=\overline{a_k(z_k)/b_k(z_k)}=\bar{w}_k$; therefore (4.15) implies that

$$b_k(\bar{z}_k^{-1})=\bar{w}_k a_k(\bar{z}_k^{-1}). \tag{4.16}$$

From (4.5) we now have that

$$\begin{aligned} a_{k+1}(\bar{z}_k^{-1}) &= \frac{1}{(\bar{z}_k^{-1}-z_k)(1-|c_k|^2)}[\gamma_k(\bar{z}_k^{-1})a_k(\bar{z}_k^{-1})-\alpha_k(\bar{z}_k^{-1})b_k(\bar{z}_k^{-1})] \\ &= \frac{1}{(\bar{z}_k^{-1}-z_k)(1-|c_k|^2)}[c_k\bar{w}_k(\bar{z}_k^{-1}-z_k)a_k(\bar{z}_k^{-1})-c_k(\bar{z}_k^{-1}-z_k)b_k(\bar{z}_k^{-1})]. \end{aligned}$$

Because of (4.16), the above expression is equal to zero, hence

$$a_{k+1}(\bar{z}_k^{-1})=0,$$

and in fact $(1-\bar{z}_k z)$ is a factor of $a_{k+1}(z)$. Clearly, even if $(1-\bar{z}_k z)$ is already a factor of $d_k(z)$, the above can be used to show that $a_{k+1}(z)$ is divisible by $(1-\bar{z}_k z)d_k(z)$. In a similar way we conclude that $(1-\bar{z}_k z)d_k(z)$ divides $b_{k+1}(z)$. Therefore,

$$d_{k+1}(z)=(1-\bar{z}_k z)d_k(z),$$

and from (4.14) we deduce that

$$\eta_{k+1}(z)(1-\bar{z}_k z)=\eta_k(z).$$

This concludes the proof. □

An immediate consequence of the above is that *if for no point in the sequence* $(z_k,\ k=1,2,\cdots)$ *we have* $(1-\bar{z}_k z)$ *as a factor of* $\eta_1(z)$, *then*

$$\eta_k(z)=\eta_1(z) \quad \text{for } k=1,2,\cdots,$$

and also $d_k(z)\equiv 1$ for all values of k. Alternatively, *if all roots of* $\eta_1(z)$, *including multiplicities, have inverse complex conjugate values belonging to* $(z_k,\ k=1,2,\cdots)$, *then after a finite number of steps we will have that*

$$d_n(z)=d_{n+l}(z)=\eta_1(z) \quad \text{for } l=1,2,\cdots,$$

while $\eta_{n+l}(z)=1$.

However, regardless of how the points z_k, $k=1,2,\cdots$, are chosen (provided a mild condition on their distribution is met), the polynomials $a_k(z)$ and $b_k(z)$ as $k\to\infty$, tend to the zero polynomial and $\eta_1(z)$ respectively. This is the content of the next theorem.

THEOREM 4.17. *Let* $f_1(z)$ *be a rational* $\mathbb{U}$*-function* (*with* $f_1(0)=0$), $\eta_1(z)$ *be the associated spectral numerator, and* $f_1(z)=a_1(z)/b_1(z)$ *a representation of* f_1 *as the ratio of two coprime polynomials satisfying* $a_1(0)=0$ *and* $b_1(0)=1$. *Let* z_k, $k=1,2,\cdots$, *be a sequence of points in* $\mathbb{D}$ *satisfying*

$$\sum_{k=1}^{\infty}(1-|z_k|)=\infty, \tag{3.4}$$

and (a_k, b_k), $k=2,3,\cdots$, *be obtained from* (4.3), (4.5)-(4.7). *Then as* $k\to\infty$,

$$b_k(z)\to\eta_1(z), \qquad a_k(z)\to 0$$

coefficientwise.

Proof. Let

$$\frac{a_k(z)}{b_k(z)} = \rho_1 z + \rho_2 z^2 + \cdots + \rho_m z^m + \cdots \tag{4.18}$$

be a Taylor series for $f_k(z)$ around the origin. Since $f_k(z)$ is a rational function in $\mathbb{U}$, it is analytic in $\mathbb{D}$ and also continuous on the boundary (because of the rationality). Corollary 3.5 now implies that $f_k(z)$ tends uniformly to zero on compact subsets of $\mathbb{D}$ and in particular that $\rho_m \to 0$, for all m, as $k \to \infty$.

From Proposition 4.8, the polynomial $b_k(z)$ satisfies $b_k(0) = 1$ for all values of k and has no root in $\mathbb{D}$ (because f_k is in $\mathbb{U}$ and $d_k(z)$ has no root in $\mathbb{D}$ as discussed earlier). Therefore, the coefficients of b_k are all bounded by one. Also $\rho_m \to 0$, uniformly in m for $m = 1, \cdots, l$, when $k \to \infty$, and l being any finite integer. We conclude that $a_k(z) \to 0$ as a polynomial; i.e., its coefficients tend to zero.

From the proof of Theorem 4.10 we now have that

$$\begin{aligned} |b_k(z)|^2 - |a_k(z)|^2 &= |\mu_k|^2 |d_k(z)\eta_k(z)|^2 \\ &= |\mu_k \eta_1(z)|^2 \quad \text{on } \mathbb{T}. \end{aligned}$$

But $a_k(z) \to 0$, whereas $b_k(z)$ and $\eta_1(z)$ are polynomials that have no root in $\mathbb{D}$ and have value one at the origin. Therefore

$$b_k(z) \to \eta_1(z),$$

and $\mu_k \to 1$, when $k \to \infty$. □

Theorem 4.17 provides a general recursive scheme in the form of relations (4.3), (4.5)–(4.7), for obtaining the spectral factor of a rational $\mathbb{U}$-function $f_1(z) = a_1(z)/b_1(z)$ as summarized below:

1. Select a sequence of points (z_k in $\mathbb{D}$: $k = 1, 2, \cdots$) satisfying (3.4).

Iterate step 2 for $k = 1, 2, 3, \cdots$:

2. Given $(a_k(z), b_k(z))$ compute $(a_{k+1}(z), b_{k+1}(z))$ using (4.7), and λ_{k+1} using

$$\lambda_{k+1} = \frac{1 - |c_k|^2}{1 - |w_k|^2} \lambda_k,$$

with $\lambda_1 = 1$, and the parameters w_k and c_k obtained from (4.3) and (4.6) respectively.

3. Then as $k \to \infty$, $b_k(z)$ approaches the spectral numerator of $f_1(z)$ and $\lambda_k b_k(z)/b_1(z)$ approaches the canonical spectral factor of $f_1(z)$.

The choice of the sequence (z_k, $k = 1, 2, \cdots$) is arbitrary provided they do not converge too fast towards the boundary; i.e., condition (3.4) of the theorem is met. However, the choice of this sequence influences the speed of the convergence $b_k(z) \to \eta_1(z)$. But the convergence itself is guaranteed by the theorem. It appears that the choice of the z_k's in the vicinity of the roots of η_1 results in a relatively fast convergence. This may potentially be useful when η_1 has roots on or very near the boundary of $\mathbb{D}$. However, a thorough analysis of the numerical properties and speed of convergence will be pursued elsewhere.

5. Proof of Theorem 3.3. Define $\mathbb{C} := \{F(z)$ **analytic and with positive real part in** $\mathbb{D}\}$. ($\mathbb{C}$ for Caratheodory; also the class of positive real functions.) With any function f in $\mathbb{U}$ we associate the function

$$F(z) = \frac{1 - f(z)}{1 + f(z)} \quad \text{for } z \in \mathbb{D}. \tag{5.1}$$

It is well known that F is a $\mathbb{C}$-function. (In fact (5.1) sets up a bijective correspondence between $\mathbb{U}$ and $\mathbb{C}$.) Also, the condition $f(0)=0$, which appeared earlier, translates into $F(0)=1$.

The real part of $F(z)$ is defined almost everywhere on $\mathbb{T}$ and is equal to

$$\tau(\theta) := \operatorname{Re}\{F(e^{i\theta})\} = \frac{1-|f(e^{i\theta})|^2}{|1+f(e^{i\theta})|^2} \quad \text{a.e. on } \mathbb{T}. \tag{5.2}$$

The function $1+f$ has no root in $\mathbb{D}$. Hence, $\ln|1+f|$ is a harmonic function in $\mathbb{D}$. *Provided* $f(0)=0$, $\ln|1+f(z)|$ has value at the origin equal to zero. Therefore, the integral of $\ln|1+f(e^{i\theta})|$ in the interval $[-\pi, \pi]$ is equal to zero. Then, by applying $\ln(\cdot)$ to both sides in (5.2) and integrating we derive that

$$\int_{-\pi}^{\pi} \ln[\tau(\theta)]\, d\theta = \int_{-\pi}^{\pi} \ln[1-|f(e^{i\theta})|^2]\, d\theta. \tag{5.3}$$

The above integral plays a key role in an approximation problem for analytic functions (Szegö's Theorem—see Grenander and Szegö [16]):

$$\inf\left\{(2\pi)^{-1}\int_{-\pi}^{\pi} |p(e^{i\theta})|^2\tau(\theta)\, d\theta \colon p(z) \text{ polynomial with } p(0)=1\right\}$$
$$= \exp\left\{(2\pi)^{-1}\int_{-\pi}^{\pi} \ln[\tau(\theta)]\, d\theta\right\}.$$

(The left-hand side of the above equality can be seen as the error of approximating 1 with polynomials vanishing at the origin, in $L^2[\tau(\theta)d\theta]$.) In general *the infimum is attained for a function* $p_0(z)$ *in* H^2—the subspace of L^2 functions with analytic continuation inside $\mathbb{D}$. (This in general is not a polynomial and turns out to be a scalar multiple of the inverse of the canonical spectral factor corresponding to $\tau(\theta)$.) Hence,

$$(2\pi)^{-1}\int_{-\pi}^{\pi} |p_0(e^{i\theta})|^2\tau(\theta)\, d\theta = \exp\left\{(2\pi)^{-1}\int_{-\pi}^{\pi} \ln[\tau(\theta)]\, d\theta\right\}, \tag{5.4}$$

where $p_0(0)=1$.

Let now $(z_k, k=1,2,\cdots)$ be a sequence of points in $\mathbb{D}$. Define $K_n := (zB_n(z)H^2)^\perp$, where $B_n(z)$ denotes the Blaschke product corresponding to the first n "interpolation" points

$$B_n(z) := \prod_{k=1}^{n} \frac{(z-z_k)}{(1-\bar{z}_k z)},$$

and "$^\perp$" denotes the "orthogonal complement of." K_n is a finite dimensional linear space. Let $\tau(\theta)$ be the real part of a $\mathbb{C}$-function $F(z)$ for $z=e^{i\theta}$. Then $\tau(\theta)$ is defined a.e. in $[-\pi, \pi]$ and it is a nonnegative valued function. K_n can be endowed with an inner product defined by

$$\langle f, g\rangle_\tau := (2\pi)^{-1}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\tau(\theta)\, d\theta,$$

and then we will use the obvious notation $\|p(z)\|^2_{\tau(\theta)}$. Bultheel and Dewilde [4, Cor. 1] and Dewilde and Dym [8, Lemma 4.3] have considered approximation with functions in K_n and have shown that

$$\inf\{\|p(z)\|^2_{\tau(\theta)} \colon p(z) \text{ in } K_n \text{ and } p(0)=1\} = \prod_{k=1}^{n}\left(\frac{1-|c_k|^2}{1-|w_k|^2}\right). \tag{5.5}$$

Clearly,

$$\inf\{\|p(z)\|^2_{\tau(\theta)}: p(z) \text{ in } K_n \text{ and } p(0)=1\} \geqq \|p_0(z)\|^2_{\tau(\theta)}$$
$$= \exp\left\{(2\pi)^{-1}\int_{-\pi}^{\pi} \ln[\tau(\theta)]\, d\theta\right\} = \exp\left\{(2\pi)^{-1}\int_{-\pi}^{\pi} \ln[1-|f(e^{i\theta})|^2]\, d\theta\right\}.$$

In order to prove the theorem, we need to establish that the above holds with equality. To show this it suffices to show that

$$K := \bigcup_{n=1}^{\infty} K_n \text{ is dense in } H^2$$

with respect to $\langle\cdot,\cdot\rangle_{\tau(\theta)}$.

We now briefly indicate that it is sufficient to show that the aforementioned space is dense in H^2 with respect to the standard norm. Since f is a rational function, it can be readily shown that $\tau(\theta)=|g(e^{i\theta})|^2$ where $g(z)$ is a rational function with *no* poles on the unit circle (since $g(z)$ is an outer function). This implies that $\tau(\theta)$ is bounded from above for all $\theta\in[-\pi,\pi]$. Consequently, convergence in the standard norm implies convergence in $\|\cdot\|^2_{\tau(\theta)}$ and this establishes our claim.

Now we shall use a classical result of Blaschke, which states that

$$\sum_{k=1}^{\infty}(1-|z_k|)=\infty, \tag{5.6}$$

holds if and only if $B_n(z)$ tends to zero at every point in $\mathbb{D}$ as $n\to\infty$. Any function q in H^2 that is orthogonal to K belongs to zB_nH^2, for all n. Therefore, q must be the zero function. Hence the closure of K is in fact the whole of H^2. Therefore (5.6) implies that

$$\lim_{n\to\infty}\inf\{\|p(z)\|^2_{\tau(\theta)}: p\in K_n \text{ and } p(0)=1\}=\|p_0(z)\|^2_{\tau(\theta)} \tag{5.7}$$

and consequently that

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{1-|c_k|^2}{1-|w_k|^2}\right)=\exp\left\{(2\pi)^{-1}\int_{-\pi}^{\pi}\ln[1-|f(e^{i\theta})|^2]\,d\theta\right\}.$$

This completes the proof of the theorem. □

6. Remarks on spectral factorization of $\mathbb{C}$-functions. So far we have considered spectral factorization of rational $\mathbb{U}$-functions. In many cases one is given a $\mathbb{C}$-function $F(z)$ instead. So let

$$F(z)=\frac{\pi(z)}{\chi(z)} \text{ be in } \mathbb{C},$$

where $\pi(z)$ and $\chi(z)$ are polynomials in z. Then

$$\operatorname{Re}\{F(e^{i\theta})\}=\frac{\pi(z)\bar{\chi}(z^{-1})+\chi(z)\bar{\pi}(z^{-1})}{\chi(z)\bar{\chi}(z^{-1})}\geqq 0 \quad \text{for } z=e^{i\theta},$$

and assumes a factorization

$$\operatorname{Re}\{F(e^{i\theta})\}=\frac{|\kappa\eta(z)|^2}{|\chi(z)|^2} \quad \text{for } z=e^{i\theta},\quad \theta\in[-\pi,\pi],$$

where κ is a positive constant and $\eta(z)$ can be assumed to have no root in $\mathbb{D}$ and to have value equal to one at the origin. Then $\kappa\eta(z)/\chi(z)$ is called the *canonical spectral factor* of $F(z)$, and $\eta(z)$ will be said to be the *spectral numerator* of $F(z)$.

With no loss in generality we may assume that $F(0)=1$. From (6.1) we can obtain an associated function

$$f(z)=\frac{1-F(z)}{1+F(z)}=\frac{a(z)}{b(z)}$$

of class $\mathbb{U}$. Then $a(z):=\chi(z)-\pi(z)$ and $b(z):=\chi(z)+\pi(z)$ are also polynomials. Let $\eta(z)$ be the spectral numerator of $f(z)$, and

$$g_f(z)=\frac{\kappa\eta(z)}{b(z)}$$

be the canonical spectral factor of f. Then, from (6.1)-(6.2) we obtain that

$$\operatorname{Re}\{F(e^{i\theta})\}=\frac{|\kappa\eta(z)|^2}{|b(z)+a(z)|^2}=\frac{|\kappa\eta(z)|^2}{|\chi(z)|^2}\quad \text{for } z=e^{i\theta},\quad \theta\in[-\pi,\pi],$$

and the spectral factor of $F(z)$ is $\kappa\eta(z)/\chi(z)$. *Thus, the spectral numerator of both $F(z)$ and $f(z)$ is the same.* Therefore, when looking for the spectral factor of F, we may consider the corresponding $\mathbb{U}$-function $f(z)$. Then take

$$a(z)=\chi(z)-\pi(z)\quad \text{and}\quad b(z)=\chi(z)+\pi(z),$$

and apply the algorithm of Theorem 4.17 given by (4.3), (4.5)-(4.7) and an appropriate choice of points $(z_k\in\mathbb{D},\ k=1,2,\cdots)$ to obtain the spectral numerator $\eta(z)$. The algorithm expressed by (4.3), (4.5)-(4.7) can be written directly in terms of polynomial fractions of $\mathbb{C}$-functions. However, this offers no advantage over (4.3), (4.5)-(4.7), which seem to be simpler and thus preferable.

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