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ROBUST STABILITY OF FEEDBACK SYSTEMS: A GEOMETRIC APPROACH USING THE GAP METRIC*

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Abstract. A geometric framework for robust stabilization of infinite-dimensional time-varying linear systems is presented. The uncertainty of a system is described by perturbations of its graph and is measured in the gap metric. Necessary and sufficient conditions for robust stability are generalized from the time-invariant case. An example is given to highlight an important difference between the obstructions, which limit the size of a stabilizable gap ball, in the time-varying and time-invariant cases. Several results on the gap metric and the gap topology are established that are central in a geometric treatment of the robust stabilizability problem in the gap. In particular, the concept of a "graphable" subspace is introduced in the paper. Subspaces that fail to be graphable are characterized by an index condition on a certain semi-Fredholm operator.

Key words. Robust stabilization, gap metric, graph topology, graphability, stabilizability

AMS subject classifications. 47A53, 47N70, 93B27, 93B28, 93B36, 93C25, 93C50, 93D25

1. Introduction. In this paper we develop a geometric framework for robust stabilization of feedback systems using operator-theoretic methods. The theory is based on a description of the uncertainty of a system as a perturbation of its graph and is measured by the gap metric.

The gap metric has its origin in functional analysis [20], [13], where it was used in perturbation theory of linear operators. It was introduced into control theory in [25], [1] as being appropriate for the study of uncertainty in feedback systems. For shift-invariant systems it was shown in [5] that the gap metric was computable exactly in terms of two standard "2-block" H_∞ optimization problems. Building on this result and the work of [23], [24], [26], and [7], it was shown in [6] that robust stabilization in the gap metric is equivalent to robust stabilization for perturbations of the normalized coprime factors of the transfer function.

The simplicity of the robustness bounds obtained in [6] for the time-invariant case, which were expressed solely in terms of the plant and controller system operators, strongly suggests potential generalization. However, the techniques used in [6] are mostly function theoretic, relying on a specific representation for the graph of a time-invariant dynamical system as a shift-invariant subspace of $L_2[0,\infty]$, and do not admit immediate generalization to the shift-varying case. This motivated the search for a different approach, which does not rely on representations for the subspaces involved, and which elucidates the apparent geometric structure underlying the robust stabilization problem. It became apparent that substantially new techniques were needed, beyond those developed in [6], to meet this objective. The present paper is a continuation of work begun in [3], [4]. We note that some independent work on a geometric approach to robust stability in the gap metric has been presented in [15], [16], [19]. A generalization of the results of [5] has been presented in [2].

This paper is organized as follows. In §2 we present some basic material on

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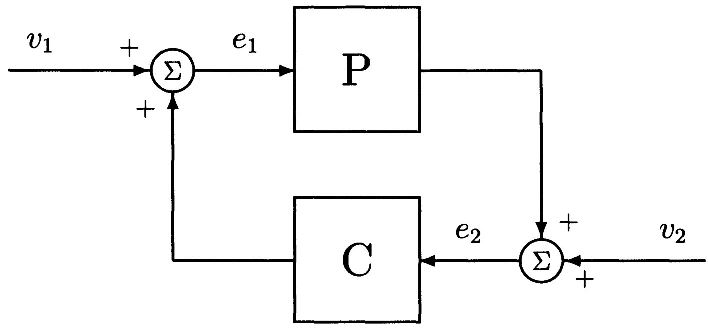

FIG. 1. *Standard feedback configuration*

graphs and stabilizability for linear systems. In §3 we establish several results on the gap metric that are used in the later development. Section 4 introduces the concept of graphability and proves a necessary and sufficient condition for a subspace to be graphable. Section 5 presents and proves the main robustness theorem for plant uncertainty in the gap metric. In §6 an example is presented to clarify the need for the uniform boundedness condition in the main robustness theorem. Section 7 uses the machinery of the previous sections to generalize an elegant result of Qiu and Davison [16] on combined plant-controller uncertainty to the time-varying case.

2. Graphs and stabilizability of linear systems.

We consider a linear system to be a (possibly unbounded) linear operator $\mathbf{P} : \mathcal{D}_\mathbf{P} \subset \mathcal{U} \to \mathcal{Y}$, where $\mathcal{U}$, $\mathcal{Y}$ are Hilbert spaces and $\mathcal{D}_\mathbf{P}$ is the *domain* of $\mathbf{P}$. We denote by $\mathcal{P}_{\mathcal{U},\mathcal{Y}}$ the class of all linear systems from $\mathcal{U}$ to $\mathcal{Y}$. A typical choice for the input and output spaces is $\mathcal{U} = \ell_2^m[0,\infty)$ and $\mathcal{Y} = \ell_2^p[0,\infty)$, or the corresponding continuous-time Lebesgue spaces. (Note: This paper does not impose the constraints of causality or time-invariance on the systems considered.)

Consider the feedback configuration of Fig. 1, where the *plant* $\mathbf{P} \in \mathcal{P}_{\mathcal{U},\mathcal{Y}}$ and the *controller* $\mathbf{C} \in \mathcal{P}_{\mathcal{Y},\mathcal{U}}$. This configuration, denoted by $[\mathbf{P}, \mathbf{C}]$, provides a pictorial representation of the following set of equations:

$$e_1 = v_1 + Ce_2,$$
$$e_2 = Pe_1 + v_2.$$

Define the *graph* of a system $\mathbf{P} \in \mathcal{P}_{\mathcal{U},\mathcal{Y}}$ as the linear manifold of bounded input-output pairs of $\mathbf{P}$

$$\mathcal{G}_\mathbf{P} := \begin{pmatrix} \mathbf{I}_\mathcal{U} \\ \mathbf{P} \end{pmatrix} \mathcal{D}_\mathbf{P} \subset \mathcal{L} := \mathcal{U} \oplus \mathcal{Y},$$

where $\mathbf{I}_\mathcal{U}$ denotes the identity operator on $\mathcal{U}$. Similarly, define the *inverse graph* of the controller $\mathbf{C}$ by

$$\mathcal{G}'_\mathbf{C} := \begin{pmatrix} \mathbf{C} \\ \mathbf{I}_\mathcal{Y} \end{pmatrix} \mathcal{D}_\mathbf{C} \subset \mathcal{L}.$$

The feedback configuration $[\mathbf{P}, \mathbf{C}]$ is said to be *stable* if the operators mapping $v_i \to e_j$ for $i, j = 1, 2$ are bounded. This is equivalent to the operator

$$\mathbf{F}_{\mathbf{P},\mathbf{C}} := \begin{pmatrix} \mathbf{I}_\mathcal{U} & \mathbf{C} \\ \mathbf{P} & \mathbf{I}_\mathcal{Y} \end{pmatrix} : \mathcal{D}_\mathbf{P} \times \mathcal{D}_\mathbf{C} \to \mathcal{G}_\mathbf{P} + \mathcal{G}'_\mathbf{C} : \begin{pmatrix} e_1 \\ -e_2 \end{pmatrix} \to \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}$$

having a bounded inverse defined on $\mathcal{L}$. In case an inverse exists it is denoted by $\mathbf{H}_{\mathbf{P},\mathbf{C}} := \mathbf{F}_{\mathbf{P},\mathbf{C}}^{-1}$.

A system $\mathbf{P}$ is said to be *stabilizable* if and only if there exists a controller $\mathbf{C}$ such that $[\mathbf{P},\ \mathbf{C}]$ is stable.

PROPOSITION 1. *If $\mathbf{P} \in \mathcal{P}_{\mathcal{U},\mathcal{Y}}$ is stabilizable then the graph of $\mathbf{P}$ is closed.*

Proof. Let $\binom{u_i}{y_i} \in \mathcal{G}_{\mathbf{P}}$ for $i = 1, 2, \ldots$ be a Cauchy sequence with limit point $\binom{u}{y}$, and let $\mathbf{C}$ be such that $[\mathbf{P},\ \mathbf{C}]$ is stable. Since

$$\begin{pmatrix} u_i \\ 0 \end{pmatrix} := \mathbf{H}_{\mathbf{P},\mathbf{C}} \begin{pmatrix} u_i \\ y_i \end{pmatrix},$$

then

$$\begin{pmatrix} u \\ 0 \end{pmatrix} := \lim_{i\to\infty} \mathbf{H}_{\mathbf{P},\mathbf{C}} \begin{pmatrix} u_i \\ y_i \end{pmatrix} = \mathbf{H}_{\mathbf{P},\mathbf{C}} \begin{pmatrix} u \\ y \end{pmatrix},$$

which means that $u \in \mathcal{D}_{\mathbf{P}}$ and $\binom{u}{y} = \binom{\mathbf{I}_{\mathcal{U}}}{\mathbf{P}} u \in \mathcal{G}_{\mathbf{P}}$. □

Thus, a necessary condition for $[\mathbf{P},\ \mathbf{C}]$ to be stable is that both $\mathbf{P}$ and $\mathbf{C}$ have closed graphs. A similar statement has been made in [27] for quotients of bounded operators. The idea in [27] gives Proposition 1 in the following way: Closed-loop stability ensures that $\mathbf{P}(\mathbf{I} + \mathbf{CP})^{-1}$ is bounded and that $(\mathbf{I} + \mathbf{CP})$ has closed graph and domain equal to $\mathcal{D}_{\mathbf{P}}$. This gives that $\mathbf{P} = \mathbf{P}(\mathbf{I} + \mathbf{CP})^{-1}(\mathbf{I} + \mathbf{CP})$ has closed graph.

The following proposition presents a geometric characterization of stability of $[\mathbf{P},\ \mathbf{C}]$.

PROPOSITION 2. *Let $\mathbf{P} \in \mathcal{P}_{\mathcal{U},\mathcal{Y}}$ and $\mathbf{C} \in \mathcal{P}_{\mathcal{Y},\mathcal{U}}$. Then the following are equivalent:*

(a) $[\mathbf{P},\ \mathbf{C}]$ *is stable,*

(b) $\mathcal{G}_{\mathbf{P}}$, $\mathcal{G}'_{\mathbf{C}}$ *are closed,*

$$\mathcal{G}_{\mathbf{P}} \cap \mathcal{G}'_{\mathbf{C}} = \{0\} \tag{1}$$

and

$$\mathcal{G}_{\mathbf{P}} + \mathcal{G}'_{\mathbf{C}} = \mathcal{L}. \tag{2}$$

Proof. (a)⇒(b). If $[\mathbf{P},\ \mathbf{C}]$ is stable then, by Proposition 1, $\mathcal{G}_{\mathbf{P}}$, $\mathcal{G}'_{\mathbf{C}}$ are closed. Moreover, both (1) and (2) are necessary for $\mathbf{F}_{\mathbf{P},\mathbf{C}}$ to be a one-to-one mapping onto $\mathcal{L}$.

(b)⇒(a). First note that (1) and (2) are sufficient to guarantee the existence of a set-theoretic inverse for $\mathbf{F}_{\mathbf{P},\mathbf{C}}$ defined on $\mathcal{L}$. We need to show that the inverse is also bounded. We first observe that the graph of $\mathbf{F}_{\mathbf{P},\mathbf{C}}$ is closed. To see this note that

$$\begin{aligned}
\mathcal{G}_{\mathbf{F}} &= \left\{ \begin{pmatrix} e_1 \\ -e_2 \\ e_1 - \mathbf{C}e_2 \\ \mathbf{P}e_1 - e_2 \end{pmatrix} : \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \in \mathcal{D}_{\mathbf{P}} \times \mathcal{D}_{\mathbf{C}} \right\} \\
&= \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} e_1 \\ \mathbf{P}e_1 \\ \mathbf{C}e_2 \\ e_2 \end{pmatrix} : \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \in \mathcal{D}_{\mathbf{P}} \times \mathcal{D}_{\mathbf{C}} \right\} \\
&\cong \left\{ \begin{pmatrix} e_1 \\ \mathbf{P}e_1 \\ \mathbf{C}e_2 \\ e_2 \end{pmatrix} : \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \in \mathcal{D}_{\mathbf{P}} \times \mathcal{D}_{\mathbf{C}} \right\} = \begin{pmatrix} \mathcal{G}_{\mathbf{P}} \\ \mathcal{G}'_{\mathbf{C}} \end{pmatrix},
\end{aligned}$$

where $\cong$ denotes a Hilbert space isomorphism. Consequently, the graph of $\mathbf{H}_{\mathbf{P},\mathbf{C}}$ $(= \mathbf{F}_{\mathbf{P},\mathbf{C}}^{-1})$ is also closed. The result now follows from the closed graph theorem. $\square$

Similar geometric concepts for expressing stability have been employed in earlier studies, notably in the context of nonlinear control systems [18], [22], and in the recent works [3], [14], [16], [19].

3. Preliminaries on the gap metric. In light of Proposition 2 we will restrict our attention in the rest of the paper to linear systems that have closed graphs. We will identify $\mathbf{P}$ (through its graph) and $\mathbf{C}$ (through its inverse graph) with elements of

$$\mathrm{S}_{\mathcal{L}} := \{\mathcal{K} \ : \ \mathcal{K} \text{ is a closed subpace of } \mathcal{L}\}.$$

For $\mathcal{K} \in \mathrm{S}_{\mathcal{L}}$ denote by $\mathbf{\Pi}_{\mathcal{K}}$ the orthogonal projection with range $\mathcal{K}$. The *gap* between $\mathcal{K}_1, \mathcal{K}_2 \in \mathrm{S}_{\mathcal{L}}$ is the metric defined as

$$\delta(\mathcal{K}_1, \mathcal{K}_2) := \|\mathbf{\Pi}_{\mathcal{K}_1} - \mathbf{\Pi}_{\mathcal{K}_2}\|$$

(see [11] and [12]), and $\mathrm{S}_{\mathcal{L}}$ is equiped with the natural topology induced by the gap metric. Thus the *gap* between two systems $\mathbf{P}_i$, $i = 1, 2$, is defined to be the gap between their respective graphs $\mathcal{G}_{\mathbf{P}_i}$, $i = 1, 2$ ([25]).

Let $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ denote the space of bounded operators between two Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$. For $\mathbf{X} \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ define $\tau(\mathbf{X}) := \inf_{x \in \mathcal{K}_1, \|x\|=1} \|\mathbf{X}x\|$.

PROPOSITION 3. *Let $\mathcal{K}_0 \in \mathrm{S}_{\mathcal{L}}$. The following are equivalent:*

(a) *$\mathcal{K} \in \mathrm{S}_{\mathcal{L}}$ and $\delta(\mathcal{K}_0, \mathcal{K}) < 1$;*

(b) *$\mathbf{\Pi}_{\mathcal{K}_0}|_{\mathcal{K}}$ is invertible;*

(c) *There exists an $\mathbf{X} \in \mathcal{B}(\mathcal{K}_0, \mathcal{K}_0^{\perp})$ such that $\mathcal{K} = (\mathbf{I}_{\mathcal{K}_0} + \mathbf{X})\mathcal{K}_0$.*

Furthermore, if $\mathcal{K} \in \mathrm{S}_{\mathcal{L}}$ and $\delta(\mathcal{K}_0, \mathcal{K}) < 1$, then

$$\delta(\mathcal{K}_0, \mathcal{K}) = \sqrt{1 - \tau^2(\mathbf{\Pi}_{\mathcal{K}_0}|_{\mathcal{K}})} = \sqrt{1 - \tau^2(\mathbf{\Pi}_{\mathcal{K}}|_{\mathcal{K}_0})} \tag{3}$$

$$= \|\mathbf{X}(\mathbf{I} + \mathbf{X}^*\mathbf{X})^{-1/2}\| = \frac{\|\mathbf{X}\|}{\sqrt{1 + \|\mathbf{X}\|^2}}. \tag{4}$$

Proof. The equivalence (a)$\Leftrightarrow$(b), as well as a proof of (3), can be found in [12, Lemma 15.1].

(b)$\Rightarrow$(c). Since $\delta(\mathcal{K}_0, \mathcal{K}) = \delta(\mathcal{K}, \mathcal{K}_0)$ both $\mathbf{\Pi}_{\mathcal{K}_0}|_{\mathcal{K}}$ and $\mathbf{\Pi}_{\mathcal{K}}|_{\mathcal{K}_0}$ are invertible. Hence

$$\begin{aligned}\mathcal{K} = \mathbf{\Pi}_{\mathcal{K}}|_{\mathcal{K}_0}\mathcal{K}_0 &= \left(\mathbf{\Pi}_{\mathcal{K}_0}\mathbf{\Pi}_{\mathcal{K}} + \mathbf{\Pi}_{\mathcal{K}_0^{\perp}}\mathbf{\Pi}_{\mathcal{K}}\right)\mathcal{K}_0 \\ &= \left(\mathbf{I}_{\mathcal{K}_0} + \mathbf{\Pi}_{\mathcal{K}_0^{\perp}}\mathbf{\Pi}_{\mathcal{K}}(\mathbf{\Pi}_{\mathcal{K}_0}|_{\mathcal{K}})^{-1}\right)\mathcal{K}_0.\end{aligned}$$

Thus, (c) holds for $\mathbf{X} = \mathbf{\Pi}_{\mathcal{K}_0^{\perp}}(\mathbf{\Pi}_{\mathcal{K}_0}|_{\mathcal{K}})^{-1}$.

(c)$\Rightarrow$(b). From the equality $k_0 = \mathbf{\Pi}_{\mathcal{K}_0}|_{\mathcal{K}}(\mathbf{I}+\mathbf{X})k_0$, for $k_0 \in \mathcal{K}_0$, we see that $\mathbf{\Pi}_{\mathcal{K}_0}|_{\mathcal{K}}$ is onto and from $\mathcal{K} = (\mathbf{I}_{\mathcal{K}_0} + \mathbf{X})\mathcal{K}_0$ that it is one-to-one.

We defer the derivation of (4) to Proposition 4 below, where we prove a slightly more general statement. $\square$

From the definition of the gap metric it follows that

$$\begin{aligned}\delta(\mathcal{K}_1, \mathcal{K}_2) &= \left\|\begin{pmatrix}\mathbf{\Pi}_{\mathcal{K}_1} \\ \mathbf{\Pi}_{\mathcal{K}_1^{\perp}}\end{pmatrix}(\mathbf{\Pi}_{\mathcal{K}_1} - \mathbf{\Pi}_{\mathcal{K}_2})(\mathbf{\Pi}_{\mathcal{K}_2^{\perp}}, \ \mathbf{\Pi}_{\mathcal{K}_2})\right\| \\ &= \left\|\begin{pmatrix}\mathbf{\Pi}_{\mathcal{K}_1}\mathbf{\Pi}_{\mathcal{K}_2^{\perp}} & \mathbf{0} \\ \mathbf{0} & -\mathbf{\Pi}_{\mathcal{K}_1^{\perp}}\mathbf{\Pi}_{\mathcal{K}_2}\end{pmatrix}\right\| \\ &= \max\{\|\mathbf{\Pi}_{\mathcal{K}_1^{\perp}}\mathbf{\Pi}_{\mathcal{K}_2}\|, \ \|\mathbf{\Pi}_{\mathcal{K}_2^{\perp}}\mathbf{\Pi}_{\mathcal{K}_1}\|\}\end{aligned}$$

(see [12]). Note that

$$\|\Pi_{\mathcal{K}_1^\perp}\Pi_{\mathcal{K}_2}\| = \sup_{x\in\mathcal{K}_2,\ \|x\|=1} \operatorname{dist}(x,\mathcal{K}_1),$$

where $\operatorname{dist}(x,\mathcal{K}_1) := \inf_{y\in\mathcal{K}_1}\|x-y\|$.

PROPOSITION 4. *Let* $\mathbf{X}_i \in \mathcal{B}(\mathcal{K}_0, {\mathcal{K}_0}^\perp)$, *for* $i = 1,2$, *and* $\mathcal{K}_i = (\mathbf{I}_{\mathcal{K}_0} + \mathbf{X}_i)\mathcal{K}_0$. *Then*

$$\delta(\mathcal{K}_1,\mathcal{K}_2) = \max\Big\{\|(\mathbf{I}+\mathbf{X}_1\mathbf{X}_1^*)^{-1/2}(\mathbf{X}_2-\mathbf{X}_1)(\mathbf{I}+\mathbf{X}_2^*\mathbf{X}_2)^{-1/2}\|,$$

$$\|(\mathbf{I}+\mathbf{X}_2\mathbf{X}_2^*)^{-1/2}(\mathbf{X}_2-\mathbf{X}_1)(\mathbf{I}+\mathbf{X}_1^*\mathbf{X}_1)^{-1/2}\|\Big\} \tag{5}$$

$$= \sqrt{1-\rho^2}, \tag{6}$$

where

$$\rho = \min\Big\{\tau\left((\mathbf{I}+\mathbf{X}_1^*\mathbf{X}_1)^{-1/2}(\mathbf{I}+\mathbf{X}_1^*\mathbf{X}_2)(\mathbf{I}+\mathbf{X}_2^*\mathbf{X}_2)^{-1/2}\right),$$
$$\tau\left((\mathbf{I}+\mathbf{X}_1\mathbf{X}_1^*)^{-1/2}(\mathbf{I}+\mathbf{X}_1\mathbf{X}_2^*)(\mathbf{I}+\mathbf{X}_2\mathbf{X}_2^*)^{-1/2}\right)\Big\}.$$

Proof. We compute

$$\|\Pi_{\mathcal{K}_1^\perp}\Pi_{\mathcal{K}_2}\| = \left\|\begin{pmatrix} -\mathbf{X}_1^* \\ \mathbf{I}\end{pmatrix}(\mathbf{I}+\mathbf{X}_1\mathbf{X}_1^*)^{-1}(-\mathbf{X}_1,\ \mathbf{I})\begin{pmatrix}\mathbf{I}\\ \mathbf{X}_2\end{pmatrix}(\mathbf{I}+\mathbf{X}_2^*\mathbf{X}_2)^{-1}(\mathbf{I},\ \mathbf{X}_2^*)\right\|$$

$$= \|(\mathbf{I}+\mathbf{X}_1\mathbf{X}_1^*)^{-1/2}(\mathbf{X}_2-\mathbf{X}_1)(\mathbf{I}+\mathbf{X}_2^*\mathbf{X}_2)^{-1/2}\| \tag{7}$$

since

$$\begin{pmatrix} -\mathbf{X}_1^* \\ \mathbf{I}\end{pmatrix}(\mathbf{I}+\mathbf{X}_1\mathbf{X}_1^*)^{-1/2}$$

is an isometry and $(\mathbf{I}+\mathbf{X}_2^*\mathbf{X}_2)^{-1/2}(\mathbf{I},\ \mathbf{X}_2^*)$ is a co-isometry. By symmetry $\|\Pi_{\mathcal{K}_2^\perp}\Pi_{\mathcal{K}_1}\|$ is given by the dual expression. This completes the proof of (5).

To prove (6) consider the unitary operators

$$\mathbf{Y}_i := \begin{pmatrix} (\mathbf{I}+\mathbf{X}_i^*\mathbf{X}_i)^{-1/2} & -\mathbf{X}_i^*(\mathbf{I}+\mathbf{X}_i\mathbf{X}_i^*)^{-1/2} \\ \mathbf{X}_i(\mathbf{I}+\mathbf{X}_i^*\mathbf{X}_i)^{-1/2} & (\mathbf{I}+\mathbf{X}_i\mathbf{X}_i^*)^{-1/2}\end{pmatrix},$$

for $i = 1,2$, and define

$$\mathbf{Y} := \mathbf{Y}_1^*\mathbf{Y}_2 = \begin{pmatrix} (\mathbf{Y})_{1,1} & (\mathbf{Y})_{1,2} \\ (\mathbf{Y})_{2,1} & (\mathbf{Y})_{2,2}\end{pmatrix}, \tag{8}$$

where $(\mathbf{Y})_{i,j}$ denotes the (i,j)-block entry of $\mathbf{Y}$. Since

$$\begin{pmatrix} (\mathbf{Y})_{1,1} \\ (\mathbf{Y})_{2,1}\end{pmatrix}$$

is an isometry, it follows that $\|(\mathbf{Y})_{2,1}\|^2 + \tau^2((\mathbf{Y})_{1,1}) = 1$. Using (7) it follows that

$$\|\Pi_{\mathcal{K}_1^\perp}\Pi_{\mathcal{K}_2}\| = \|(\mathbf{Y})_{2,1}\|$$
$$= \sqrt{1-\tau^2((\mathbf{Y})_{1,1})},$$

where $(\mathbf{Y})_{1,1} = (\mathbf{I}+\mathbf{X}_1^*\mathbf{X}_1)^{-1/2}(\mathbf{I}+\mathbf{X}_1^*\mathbf{X}_2)(\mathbf{I}+\mathbf{X}_2^*\mathbf{X}_2)^{-1/2}$. Similarly, $\|\mathbf{\Pi}_{\mathcal{K}_1^\perp}\mathbf{\Pi}_{\mathcal{K}_2}\| = \sqrt{1-\tau^2((\mathbf{Y})_{2,2})}$, where $(\mathbf{Y})_{2,2} = (\mathbf{I}+\mathbf{X}_1\mathbf{X}_1^*)^{-1/2}(\mathbf{I}+\mathbf{X}_1\mathbf{X}_2^*)(\mathbf{I}+\mathbf{X}_2\mathbf{X}_2^*)^{-1/2}$. This completes the proof. □

Consider two subspaces $\mathcal{K}_0$ and $\mathcal{K}_1$ at a distance $\delta(\mathcal{K}_0,\mathcal{K}_1) < 1$, where $\mathcal{K}_1 = (\mathbf{I}+\mathbf{X})\mathcal{K}$ and $\mathbf{X} \in \mathcal{B}(\mathcal{K}_0, \mathcal{K}_0{}^\perp)$. Define

$$\mathcal{K}_\lambda = (\mathbf{I}+\lambda\mathbf{X})\mathcal{K}_0 \tag{9}$$

for $\lambda \in \mathbb{R}$.

COROLLARY 1. *The family $\mathcal{K}_\lambda$, $\lambda \in [0,1]$, defines a path, continuous in the gap metric, between $\mathcal{K}_0$ and $\mathcal{K}_1$. Moreover, for $\lambda \in \mathbb{R}$,*

$$\delta(\mathcal{K}_0,\mathcal{K}_\lambda) = |\lambda|\|\mathbf{X}\|(1+\lambda^2\|\mathbf{X}\|^2)^{-1/2}, \tag{10}$$

$$\delta(\mathcal{K}_1,\mathcal{K}_\lambda) = |1-\lambda|\|\mathbf{X}^*\mathbf{X}(1+\mathbf{X}^*\mathbf{X})^{-1}(1+\lambda^2\mathbf{X}^*\mathbf{X})^{-1}\|^{1/2} \tag{11}$$

$$\leq \frac{|1-\lambda|}{|1+\lambda|}. \tag{12}$$

Proof. To establish that the family $\mathcal{K}_\lambda$, $\lambda \in [0,1]$, defines a path it suffices to prove (10)–(12). Equation (10) follows from Proposition 4 by identifying $\mathcal{K}_0,\mathcal{K}_1,\mathcal{K}_2$ with $\mathcal{K}_0,\mathcal{K}_0,\mathcal{K}_\lambda$, respectively, and $\mathbf{X}_1,\mathbf{X}_2$ with $\mathbf{0},\lambda\mathbf{X}$. If, in Proposition 4, we identify $\mathcal{K}_0,\mathcal{K}_1,\mathcal{K}_2$ with $\mathcal{K}_0,\mathcal{K}_1,\mathcal{K}_\lambda$ and $\mathbf{X}_1,\mathbf{X}_2$ with $\mathbf{X},\lambda\mathbf{X}$, then we obtain

$$\begin{aligned}\delta(\mathcal{K}_1,\mathcal{K}_\lambda) = \max\Big\{&|1-\lambda|\|(1+\mathbf{X}\mathbf{X}^*)^{-1/2}\mathbf{X}(1+\lambda^2\mathbf{X}^*\mathbf{X})^{-1/2}\|,\\ &|1-\lambda|\|(1+\mathbf{X}\mathbf{X}^*)^{-1/2}\mathbf{X}(1+\lambda^2\mathbf{X}^*\mathbf{X})^{-1/2}\|\Big\}.\end{aligned} \tag{13}$$

Both expressions are equal and can be seen to equal the right-hand side of (11). Since $\mathbf{X}^*\mathbf{X}$ is a positive, bounded, selfadjoint operator, by invoking the spectral mapping theorem, it follows that

$$\delta(\mathcal{K}_1,\mathcal{K}_\lambda) = \sup\{\frac{|1-\lambda|\sqrt{x}}{\sqrt{(1+x)(1+\lambda^2x)}} \ : \ x \in \text{Spectrum}(\mathbf{X}^*\mathbf{X})\}.$$

The supremum of the function in the interval $[0,\infty)$ occurs at $x = 1/\lambda$ and equals $|1-\lambda|/|1+\lambda|$. (Note that $\text{Spectrum}(\mathbf{X}^*\mathbf{X}) \subset [0,\infty)$.) □

COROLLARY 2. *Let $\mathcal{K}_0,\mathcal{K}_1,\mathcal{K}_2 \in \mathrm{S}_\mathcal{L}$ be such that $\delta^2(\mathcal{K}_0,\mathcal{K}_1) + \delta^2(\mathcal{K}_0,\mathcal{K}_2) < 1$. Then*

$$\delta(\mathcal{K}_1,\mathcal{K}_2) \leq \delta(\mathcal{K}_0,\mathcal{K}_1)\sqrt{1-\delta^2(\mathcal{K}_0,\mathcal{K}_2)} + \delta(\mathcal{K}_0,\mathcal{K}_2)\sqrt{1-\delta^2(\mathcal{K}_0,\mathcal{K}_1)}. \tag{14}$$

Proof. Since $\delta(\mathcal{K}_0,\mathcal{K}_1) < 1$ and $\delta(\mathcal{K}_0,\mathcal{K}_2) < 1$, by Proposition 3 there exist bounded operators $\mathbf{X}_i : \mathcal{K}_0 \to \mathcal{K}_0^\perp$ such that $\mathcal{K}_i = (\mathbf{I}_{\mathcal{K}_0} + \mathbf{X}_i)\mathcal{K}_0$, for $i = 1,2$. We observe that

$$\begin{aligned}\delta^2(\mathcal{K}_0,\mathcal{K}_1) + \delta^2(\mathcal{K}_0,\mathcal{K}_2) < 1 &\Rightarrow \frac{\|\mathbf{X}_1\|^2}{1+\|\mathbf{X}_1\|^2} + \frac{\|\mathbf{X}_2\|^2}{1+\|\mathbf{X}_2\|^2} < 1\\ &\Rightarrow \|\mathbf{X}_1\|\|\mathbf{X}_2\| < 1.\end{aligned}$$

Since $\|\mathbf{X}_1^*\mathbf{X}_2\| \le \|\mathbf{X}_1\|\|\mathbf{X}_2\| < 1$, it follows that $\tau(\mathbf{I}+\mathbf{X}_1^*\mathbf{X}_2) = \tau(\mathbf{I}+\mathbf{X}_2^*\mathbf{X}_1) > 0$. Consequently $\delta(\mathcal{K}_1,\mathcal{K}_2) = \sqrt{1-\rho^2} < 1$, where

$$\begin{aligned}\rho &= \tau\left((\mathbf{I}+\mathbf{X}_1^*\mathbf{X}_1)^{-1/2}(\mathbf{I}+\mathbf{X}_1^*\mathbf{X}_2)(\mathbf{I}+\mathbf{X}_2^*\mathbf{X}_2)^{-1/2}\right)\\ &\ge \tau\left((\mathbf{I}+\mathbf{X}_1^*\mathbf{X}_1)^{-1/2}\right)\tau(\mathbf{I}+\mathbf{X}_1^*\mathbf{X}_2)\tau\left((\mathbf{I}+\mathbf{X}_2^*\mathbf{X}_2)^{-1/2}\right)\\ &\ge \frac{1-\|\mathbf{X}_1\|\|\mathbf{X}_2\|}{\sqrt{1+\|\mathbf{X}_1\|^2}\sqrt{1+\|\mathbf{X}_2\|^2}}.\end{aligned}$$

Thus,

$$\begin{aligned}\delta(\mathcal{K}_1,\mathcal{K}_2) &\le \sqrt{1-\frac{(1-\|\mathbf{X}_1\|\|\mathbf{X}_2\|)^2}{(1+\|\mathbf{X}_1\|^2)(1+\|\mathbf{X}_2\|^2)}}\\ &= \frac{\|\mathbf{X}_1\|}{\sqrt{1+\|\mathbf{X}_1\|^2}}\frac{1}{\sqrt{1+\|\mathbf{X}_2\|^2}} + \frac{\|\mathbf{X}_2\|}{\sqrt{1+\|\mathbf{X}_2\|^2}}\frac{1}{\sqrt{1+\|\mathbf{X}_1\|^2}}.\end{aligned}$$

This completes the proof. □

The arcsine of the gap metric can be thought of as the *maximal angle* between two subspaces, denoted by $\theta_{\max}(\mathcal{K}_1,\mathcal{K}_2) := \arcsin\delta(\mathcal{K}_1,\mathcal{K}_2)$. Corollary 2 is given in [16]. It was also observed in [16] that (14) can be rewritten in the form

$$\theta_{\max}(\mathcal{K}_1,\mathcal{K}_2) \le \theta_{\max}(\mathcal{K}_1,\mathcal{K}) + \theta_{\max}(\mathcal{K},\mathcal{K}_2) \tag{15}$$

so that $\theta_{\max}$ defines a metric in $\mathrm{S}_{\mathcal{L}}$.

Given any subspace $\mathcal{K}_0 \in \mathrm{S}_{\mathcal{L}}$ and a positive number b let

$$\mathrm{Ball}(\mathcal{K}_0,b) := \{\mathcal{K} \,:\, \delta(\mathcal{K}_0,\mathcal{K}) < b\}$$

denote the gap-ball about $\mathcal{K}_0$ of radius b, and let $\overline{\mathrm{Ball}}(\mathcal{K}_0,b)$ denote the closure of $\mathrm{Ball}(\mathcal{K}_0,b)$.

PROPOSITION 5. *If* $b<1$, *then* $\overline{\mathrm{Ball}}(\mathcal{K}_0,b) = \{\mathcal{K} \,:\, \delta(\mathcal{K}_0,\mathcal{K}) \le b\}$.

Proof. Let $\mathcal{K}_1$ be such that $\delta(\mathcal{K}_0,\mathcal{K}_1)=b$ and consider $\mathcal{K}_\lambda$, $\lambda\in\mathbb{R}$, constructed as in Corollary 1. It follows that

$$\{\mathcal{K}_\lambda :\ \lambda\in[0,1)\} \subset \mathrm{Ball}(\mathcal{K}_0,b).$$

Thus, any neighbourhood of $\mathcal{K}_1$ contains a subspace $\mathcal{K}_\lambda$ for some $\lambda\in[0,1)$. Hence, $\mathcal{K}_1 \in \overline{\mathrm{Ball}}(\mathcal{K}_0,b)$. Conversely, take $\mathcal{K}_1$ such that for all $\epsilon>0$, $\mathrm{Ball}(\mathcal{K}_0,b)\cap\mathrm{Ball}(\mathcal{K}_1,\epsilon)\neq\emptyset$. Then take $\mathcal{K}\in\mathrm{Ball}(\mathcal{K}_0,b)\cap\mathrm{Ball}(\mathcal{K}_1,\epsilon)$. Using the triangular inequality it follows that $\delta(\mathcal{K}_0,\mathcal{K}_1) \le \delta(\mathcal{K}_0,\mathcal{K}) + \delta(\mathcal{K},\mathcal{K}_1) < b+\epsilon$. Since this is valid for any $\epsilon>0$, $\delta(\mathcal{K}_0,\mathcal{K}_1)\le b$. □

It should be noted that when $b=1$, then $\overline{\mathrm{Ball}}(\mathcal{K}_0,b) \neq \{\mathcal{K} \,:\, \delta(\mathcal{K}_0,\mathcal{K}) \le b\}$.

THEOREM 1. *Let* $b,\zeta\in\mathbb{R}$, *with* $0<b<1$ *and* $0<\zeta<1$. *There exists* $\epsilon>0$ *such that*

$$\mathrm{Ball}(\mathcal{K}_0,b+\epsilon) \subseteq \bigcup_{\mathcal{K}\in\overline{\mathrm{Ball}}(\mathcal{K}_0,b)} \mathrm{Ball}(\mathcal{K},\zeta). \tag{16}$$

Proof. Consider any $\mathcal{K}_a$ such that $1>\delta(\mathcal{K}_0,\mathcal{K}_a)=a>b$. We will construct a $\mathcal{K}_b$ with $\delta(\mathcal{K}_0,\mathcal{K}_b)=b$ and $\delta(\mathcal{K}_b,\mathcal{K}_a) = a\sqrt{1-b^2}-b\sqrt{1-a^2}$. This establishes (16) for any ϵ such that $(b+\epsilon)\sqrt{1-b^2}-b\sqrt{1-(b+\epsilon)^2}<\zeta$.

Write $\mathcal{K}_a = (\mathbf{I}+\mathbf{X})\mathcal{K}_0$ with $\mathbf{X} \in \mathcal{B}(\mathcal{K}_0, {\mathcal{K}_0}^{\perp})$. Let $\mathbf{X} = \mathbf{U}\mathbf{R}$ be a polar decomposition for $\mathbf{X}$ (i.e., with $\mathbf{R} = (\mathbf{X}^*\mathbf{X})^{1/2}$ a positive selfadjoint operator and $\mathbf{U} : \overline{\text{range}\mathbf{R}} \to \overline{\text{range}\mathbf{X}}$ a partial isometry), denote by $\mathbf{E}_\lambda$ the spectral family of projections corresponding to $\mathbf{R}$, define

$$\mathbf{\Lambda} := \int_{0-}^{\infty} g(\lambda) d\mathbf{E}_\lambda,$$

where

$$g(\lambda) = \begin{cases} 1 & \text{if } 0 \le \lambda < \frac{b}{\sqrt{1-b^2}} \\ \frac{1}{\lambda}\frac{b}{\sqrt{1-b^2}} & \text{if } \frac{b}{\sqrt{1-b^2}} \le \lambda \end{cases},$$

and define

$$\mathcal{K}_b := (\mathbf{I}+\mathbf{X}\mathbf{\Lambda})\mathcal{K}_0. \tag{17}$$

In the rest of the proof we verify that $\delta(\mathcal{K}_0, \mathcal{K}_b) = b$ and $\delta(\mathcal{K}_b, \mathcal{K}_a) = a\sqrt{1-b^2} - b\sqrt{1-a^2}$.

From Proposition 3, $\|\mathbf{X}\| = a/\sqrt{1-a^2}$. Also,

$$\begin{aligned} \|\mathbf{X}\mathbf{\Lambda}\| &= \|\mathbf{R}\mathbf{\Lambda}\| \\ &= \|\int_{0-}^{\infty} \lambda g(\lambda) d\mathbf{E}_\lambda\| \\ &= \sup\{\lambda g(\lambda) \ : \ \lambda \in \text{Spectrum}(\mathbf{R})\} \\ &= \lambda g(\lambda)\Big|_{\frac{a}{\sqrt{1-a^2}}} \\ &= \frac{b}{\sqrt{1-b^2}} \end{aligned}$$

since $\lambda g(\lambda)$ is nondecreasing on $[0,\infty)$, $\text{Spectrum}(\mathbf{R}) \subseteq [0, \|\mathbf{R}\|]$, and $(b/\sqrt{1-b^2}) < (a/\sqrt{1-a^2}) = \|\mathbf{R}\|$. From Proposition 3, we conclude that

$$\delta(\mathcal{K}_0, \mathcal{K}_b) = \frac{\|\mathbf{X}\mathbf{\Lambda}\|}{\sqrt{1+\|\mathbf{X}\mathbf{\Lambda}\|^2}} = b.$$

From Proposition 4, we have that

$$\delta(\mathcal{K}_a, \mathcal{K}_b) = \max\{\delta_1, \delta_2\},$$

where the two expressions δ_1, δ_2 are computed below. First,

$$\begin{aligned} \delta_1 &:= \|(\mathbf{I}+\mathbf{X}\mathbf{X}^*)^{-1/2}(\mathbf{X}\mathbf{\Lambda}-\mathbf{X})(\mathbf{I}+\mathbf{\Lambda}\mathbf{X}^*\mathbf{X}\mathbf{\Lambda})^{-1/2}\| \\ &= \|\mathbf{U}\mathbf{R}(\mathbf{I}+\mathbf{R}^2)^{-1/2}(\mathbf{\Lambda}-\mathbf{I})(\mathbf{I}+\mathbf{\Lambda}\mathbf{R}^2\mathbf{\Lambda})^{-1/2}\| \\ &= \|\mathbf{R}(\mathbf{I}+\mathbf{R}^2)^{-1/2}(\mathbf{\Lambda}-\mathbf{I})(\mathbf{I}+\mathbf{\Lambda}\mathbf{R}^2\mathbf{\Lambda})^{-1/2}\| \\ &= \sup\{\frac{\lambda|g(\lambda)-1|}{\sqrt{1+\lambda^2}\sqrt{1+\lambda^2 g(\lambda)^2}} \ : \ \lambda \in \text{Spectrum}(\mathbf{R})\} \qquad (18) \\ &= \frac{\lambda|g(\lambda)-1|}{\sqrt{1+\lambda^2}\sqrt{1+\lambda^2 g(\lambda)^2}} \text{ evaluated at } \lambda = \frac{a}{\sqrt{1-a^2}} \qquad (19) \\ &= a\sqrt{1-b^2} - b\sqrt{1-a^2}. \end{aligned}$$

The step (18)⇒(19) follows because $\lambda|g(\lambda)-1|/\sqrt{1+\lambda^2}\sqrt{1+\lambda^2 g(\lambda)^2}$ is monotonically nondecreasing in Spectrum($\mathbf{R}$) $\subseteq [0, a/\sqrt{1-a^2}]$, while $\|\mathbf{R}\| = a/\sqrt{1-a^2}$. Next,

$$\begin{aligned}
\delta_2 &:= \|(\mathbf{I}+\mathbf{X}\mathbf{\Lambda}^2\mathbf{X}^*)^{-1/2}(\mathbf{X}\mathbf{\Lambda}-\mathbf{X})(\mathbf{I}+\mathbf{X}^*\mathbf{X})^{-1/2}\| \\
&= \|(\mathbf{I}+\mathbf{U}\mathbf{R}\mathbf{\Lambda}^2\mathbf{R}\mathbf{U}^*)^{-1/2}\mathbf{U}\mathbf{R}(\mathbf{\Lambda}-\mathbf{I})(\mathbf{I}+\mathbf{R}^2)^{-1/2}\| \qquad (20)\\
&= \|\mathbf{U}(\mathbf{I}+\mathbf{R}\mathbf{\Lambda}^2\mathbf{R})^{-1/2}\mathbf{R}(\mathbf{\Lambda}-\mathbf{I})(\mathbf{I}+\mathbf{R}^2)^{-1/2}\| \qquad (21)\\
&= \delta_1
\end{aligned}$$

since $\mathbf{R}$ and $\mathbf{\Lambda}$ commute. The step (20)⇒(21) is based on the fact that $\mathbf{R}\mathbf{U}^*\mathbf{U} = \mathbf{R}$ and $\mathbf{U}^*\mathbf{U}\mathbf{R} = \mathbf{R}$. □

It is interesting to note that for arbitrary $\mathcal{K}_0, \mathcal{K}_a \in \mathrm{S}_{\mathcal{L}}$ with $0 < b < a = \delta(\mathcal{K}_0, \mathcal{K}_a) < 1$ and $\mathcal{K}_b$ as in (17), we have $\delta(\mathcal{K}_0, \mathcal{K}_b) = b$ and $\theta_{\max}(\mathcal{K}_0, \mathcal{K}_b) + \theta_{\max}(\mathcal{K}_b, \mathcal{K}_a) = \theta_{\max}(\mathcal{K}_0, \mathcal{K}_a)$.

4. Graphability. Let $\mathcal{K}, \mathcal{W} \in \mathrm{S}_{\mathcal{L}}$. We say that $\mathcal{K}$ is a *graph with respect to* $\mathcal{W}$ if $\mathcal{K} \cap \mathcal{W}^\perp = \{0\}$. For any such $\mathcal{K}$ we can define a linear operator $\mathbf{K}$ by the relation $\mathbf{K}(\mathbf{\Pi}_{\mathcal{W}} k) = \mathbf{\Pi}_{\mathcal{W}^\perp} k$ for all $k \in \mathcal{K}$. For convenience we will identify each $u \in \mathcal{U}$ with $\binom{u}{0} \in \mathcal{L}$ and, similarly, every $y \in \mathcal{Y}$ with $\binom{0}{y} \in \mathcal{L}$, i.e., $\mathcal{U}$ is identified with the subspace $\mathcal{U} \oplus \{0\}$ of $\mathcal{L}$ and $\mathcal{Y}$ with $\{0\} \oplus \mathcal{Y}$. We say $\mathcal{K}$ is a *graph* if $\mathcal{K}$ is a graph with respect to $\mathcal{U}$. Similarly, we say $\mathcal{K}$ is an *inverse graph* if $\mathcal{K}$ is a graph with respect to $\mathcal{Y}$. Denote

$$\mathrm{Graph}_{\mathcal{W}} := \{\mathcal{K} \in \mathrm{S}_{\mathcal{L}} \; : \; \mathcal{K} \cap \mathcal{W}^\perp = \{0\}\}.$$

For $\mathcal{K}, \mathcal{W} \in \mathrm{S}_{\mathcal{L}}$ let $\mathbf{X}_{\mathcal{K}} := \mathbf{\Pi}_{\mathcal{W}}|_{\mathcal{K}}$ and define $\mathrm{S}_{\mathrm{npi}}$ to be the *complement* in $\mathrm{S}_{\mathcal{L}}$ of the set

$$\mathrm{S}_{\mathrm{pi}} := \{\mathcal{K} \in \mathrm{S}_{\mathcal{L}} \; : \; \mathbf{X}_{\mathcal{K}} \text{ is semi-Fredholm and } \operatorname{ind} \mathbf{X}_{\mathcal{K}} > 0\}.$$

An operator $\mathbf{X}$ is said to be *semi-Fredholm* if its range is closed and if at least one of $\dim\ker \mathbf{X}$, $\dim\ker \mathbf{X}^*$ is finite. In this case the *Fredholm index* is defined as $\operatorname{ind} \mathbf{X} := \dim\ker \mathbf{X} - \dim\ker \mathbf{X}^*$.

LEMMA 1. $\mathrm{S}_{\mathrm{npi}}$ *is closed in* $\mathrm{S}_{\mathcal{L}}$.

Proof. Let $\mathcal{K} \in \mathrm{S}_{\mathrm{pi}}$ and let $\mathcal{K}_i \in \mathrm{S}_{\mathcal{L}}$ satisfy $\delta(\mathcal{K}, \mathcal{K}_i) \to 0$ for $i = 1, 2 \ldots$. We have

$$\begin{aligned}
\|\mathbf{X}_{\mathcal{K}}^* - \mathbf{\Pi}_{\mathcal{K}}\mathbf{X}_{\mathcal{K}_i}^*\| &= \|(\mathbf{\Pi}_{\mathcal{K}} - \mathbf{\Pi}_{\mathcal{K}}\mathbf{\Pi}_{\mathcal{K}_i})|_{\mathcal{W}}\| \\
&\le \|\mathbf{\Pi}_{\mathcal{K}} - \mathbf{\Pi}_{\mathcal{K}_i}\| = \delta(\mathcal{K}, \mathcal{K}_i) \to 0.
\end{aligned}$$

Since the set of semi-Fredholm operators from $\mathcal{W}$ to $\mathcal{K}$ with a given index is open in the space of all bounded operators from $\mathcal{W}$ to $\mathcal{K}$ (see [11, Thm. 5.17]), it follows that there exists an N such that for $i \ge N$, $\mathbf{\Pi}_{\mathcal{K}}\mathbf{X}_{\mathcal{K}_i}^*$ is semi-Fredholm and

$$\operatorname{ind} \mathbf{\Pi}_{\mathcal{K}}\mathbf{X}_{\mathcal{K}_i}^* = \operatorname{ind} \mathbf{X}_{\mathcal{K}}^* = -\operatorname{ind} \mathbf{X}_{\mathcal{K}}.$$

On the other hand, if $\delta(\mathcal{K}, \mathcal{K}_i) < 1$ then $\mathbf{Y} := \mathbf{\Pi}_{\mathcal{K}}|_{\mathcal{K}_i}$ is an invertible operator from $\mathcal{K}_i$ to $\mathcal{K}$ and therefore $\mathbf{X}_{\mathcal{K}_i}^* = \mathbf{Y}^{-1}\mathbf{\Pi}_{\mathcal{K}}\mathbf{X}_{\mathcal{K}_i}^*$ is semi-Fredholm and $\operatorname{ind} \mathbf{X}_{\mathcal{K}_i}^* = \operatorname{ind} \mathbf{\Pi}_{\mathcal{K}}\mathbf{X}_{\mathcal{K}_i}^*$ for $i \ge N$. It follows that for i large enough $\mathbf{X}_{\mathcal{K}_i}^*$ is also semi-Fredholm and

$$\operatorname{ind} \mathbf{X}_{\mathcal{K}_i} = -\operatorname{ind} \mathbf{X}_{\mathcal{K}_i}^* = \operatorname{ind} \mathbf{X}_{\mathcal{K}} > 0.$$

Thus S_{pi} is open and hence $\mathrm{S}_{\mathrm{npi}}$ is closed in $\mathrm{S}_{\mathcal{L}}$. □

The following result characterizes the closure of $\mathrm{Graph}_{\mathcal{W}}$ in $\mathrm{S}_{\mathcal{L}}$. Any $\mathcal{K} \in \overline{\mathrm{Graph}_{\mathcal{W}}}$ is said to be *graphable* with respect to $\mathcal{W}$.

THEOREM 2. $\overline{\text{Graph}_\mathcal{W}} = \text{S}_{\text{npi}}$.

Before we proceed with the proof of the theorem we provide a characterization of S_{npi}. By $\mathbf{E}_\lambda$ (respectively, $\mathbf{E}_{*,\lambda}$), $\lambda \in \mathbb{R}$, we denote the spectral family associated with $\mathbf{X}_\mathcal{K}^*\mathbf{X}_\mathcal{K}$ (respectively, $\mathbf{X}_\mathcal{K}\mathbf{X}_\mathcal{K}^*$). Then

$$\mathbf{X}_\mathcal{K}^*\mathbf{X}_\mathcal{K} = \int_{0_-}^{\infty} \lambda d\mathbf{E}_\lambda, \text{ and } \mathbf{X}_\mathcal{K}\mathbf{X}_\mathcal{K}^* = \int_{0_-}^{\infty} \lambda d\mathbf{E}_{*,\lambda}$$

and the projections are chosen (strongly) continuous from the right (see [17]).

LEMMA 2. $\text{S}_{\text{npi}} = \{\mathcal{K} : \dim \mathbf{E}_\lambda \mathcal{K} \le \dim \mathbf{E}_{*,\lambda}\mathcal{W},$ *for all* $\lambda > 0$ *and* λ *small enough*$\}$.

Proof. (Inclusion $\supset$). It is a standard fact that $\mathbf{E}_0\mathcal{K} = \ker \mathbf{X}_\mathcal{K}$, $\mathbf{E}_{*,0}\,\mathcal{W} = \ker \mathbf{X}_\mathcal{K}^*$. If $\mathcal{K} \in \text{S}_{\text{pi}}$ then we must have $\dim \mathbf{E}_0\mathcal{K} > \dim \mathbf{E}_{*,0}\,\mathcal{W}$. Since $\mathbf{X}_\mathcal{K}$ is semi-Fredholm then $\dim \mathbf{E}_\lambda\mathcal{K} \to \dim \mathbf{E}_0\mathcal{K}$ and $\dim \mathbf{E}_{*,\lambda}\,\mathcal{W} \to \dim \mathbf{E}_{*,0}\,\mathcal{W}$ as $\lambda \to 0$. Hence, $\dim \mathbf{E}_\lambda\mathcal{K} > \dim \mathbf{E}_{*,\lambda}\mathcal{W}$ for sufficiently small $\lambda > 0$.

(Inclusion $\subset$). Assume $\mathcal{K} \in \text{S}_{\text{npi}}$ and that there exists a sequence $\lambda_i > 0$, $i = 1, 2, \ldots$, $\lambda_i \to 0$, for which $\dim \mathbf{E}_{\lambda_i}\mathcal{K} > \dim \mathbf{E}_{*,\lambda_i}\mathcal{W}$. Since $\dim \mathbf{E}_\lambda\mathcal{K}$, $\dim \mathbf{E}_{*,\lambda}\mathcal{W}$ are nondecreasing in λ, $\dim \mathbf{E}_{*,\lambda_i}\mathcal{W}$ must be constant and finite—say equal to d—for i large enough. Thus 0 is isolated in the spectrum of $\mathbf{X}_\mathcal{K}\mathbf{X}_\mathcal{K}^*$ and $\dim\ker \mathbf{X}_\mathcal{K}^* = \dim\ker \mathbf{X}_\mathcal{K}\mathbf{X}_\mathcal{K}^* = d$. It follows that $\mathbf{X}_\mathcal{K}$ is semi-Fredholm and that for λ_i small enough $\text{ind}\, \mathbf{X}_\mathcal{K} = \dim\ker \mathbf{X}_\mathcal{K} - \dim\ker \mathbf{X}_\mathcal{K}^* = \dim \mathbf{E}_{\lambda_i}\mathcal{K} - d > 0$. Thus $\mathcal{K} \in \text{S}_{\text{pi}}$, which is a contradiction. □

Proof of Theorem 2. We first show that $\overline{\text{Graph}_\mathcal{W}} \subseteq \text{S}_{\text{npi}}$. Let $\mathcal{K} \notin \text{S}_{\text{npi}}$. Then

$$\begin{aligned} \text{ind}\, \mathbf{X}_\mathcal{K} > 0 &\Rightarrow \dim\ker \mathbf{X}_\mathcal{K} > 0 \\ &\Rightarrow \mathcal{K} \cap \mathcal{W}^\perp \neq \{0\} \\ &\Rightarrow \mathcal{K} \notin \text{Graph}_\mathcal{W}. \end{aligned}$$

Thus $\text{Graph}_\mathcal{W} \subseteq \text{S}_{\text{npi}}$ and, since S_{npi} is closed in $\text{S}_\mathcal{L}$ in the topology induced by the gap metric by Lemma 1, this implies that $\mathcal{K} \notin \overline{\text{Graph}_\mathcal{W}}$ and completes the proof of the first part.

We now show that $\overline{\text{Graph}_\mathcal{W}} \supseteq \text{S}_{\text{npi}}$. Let $\mathcal{K} \in \text{S}_{\text{npi}}$. For $\epsilon > 0$ small enough $\dim \mathbf{E}_\epsilon\mathcal{K} \le \dim \mathbf{E}_{*,\epsilon}\mathcal{W}$ and therefore there exists an isometry $\mathbf{V}_\epsilon : \mathbf{E}_\epsilon\mathcal{K} \to \mathbf{E}_{*,\epsilon}\mathcal{W}$. Set $\mathcal{H}_\epsilon := \mathbf{E}_\epsilon\mathcal{K}$, $\mathcal{K}_\epsilon := (\mathcal{K} \ominus \mathcal{H}_\epsilon) + (\mathbf{I} + \sqrt{2\epsilon}\mathbf{V}_\epsilon)\mathcal{H}_\epsilon$ and note that

$$\begin{aligned} \delta(\mathcal{K}, \mathcal{K}_\epsilon) &= \delta\left((\mathcal{K} \ominus \mathcal{H}_\epsilon) \oplus \mathcal{H}_\epsilon, (\mathcal{K} \ominus \mathcal{H}_\epsilon) + (\mathbf{I} + \sqrt{2\epsilon}\mathbf{V}_\epsilon)\mathcal{H}_\epsilon\right) \\ &= \delta\left(\mathcal{H}_\epsilon, (\mathbf{I} + \sqrt{2\epsilon}\mathbf{\Pi}_{(\mathcal{K}\ominus\mathcal{H}_\epsilon)^\perp}\mathbf{V}_\epsilon)\mathcal{H}_\epsilon\right). \end{aligned}$$

Therefore, $\lim_{\epsilon\to 0} \delta(\mathcal{K}, \mathcal{K}_\epsilon) = 0$ by Proposition 3. We will now show that $\mathcal{K}_\epsilon \in \text{Graph}_\mathcal{W}$. Let $y \in \mathcal{K}_\epsilon \cap \mathcal{W}^\perp$. Then $y = k + h + \sqrt{2\epsilon}\mathbf{V}_\epsilon h$ with $k \in \mathcal{K} \ominus \mathcal{H}_\epsilon$, $h \in \mathcal{H}_\epsilon$. It follows that

$$\begin{aligned} 0 = \mathbf{\Pi}_\mathcal{W} y &= \mathbf{\Pi}_\mathcal{W} k + \mathbf{\Pi}_\mathcal{W} h + \sqrt{2\epsilon}\mathbf{V}_\epsilon h \\ &= \mathbf{X}_\mathcal{K} k + \mathbf{X}_\mathcal{K} h + \sqrt{2\epsilon}\mathbf{V}_\epsilon h, \end{aligned}$$

and thus

$$-\mathbf{X}_\mathcal{K}^*\mathbf{X}_\mathcal{K} k = \mathbf{X}_\mathcal{K}^*\mathbf{X}_\mathcal{K} h + \sqrt{2\epsilon}\mathbf{X}_\mathcal{K}^*\mathbf{V}_\epsilon h.$$

Since $\mathbf{E}_\epsilon\mathbf{X}_\mathcal{K}^*\mathbf{X}_\mathcal{K} = \mathbf{X}_\mathcal{K}^*\mathbf{X}_\mathcal{K}\mathbf{E}_\epsilon$, $\mathbf{E}_\epsilon\mathbf{X}_\mathcal{K}^* = \mathbf{X}_\mathcal{K}^*\mathbf{E}_{*,\epsilon}$ and $\mathbf{E}_{*,\epsilon}\mathbf{V}_\epsilon = \mathbf{V}_\epsilon$, we see that

$$0 = -\mathbf{E}_\epsilon\mathbf{X}_\mathcal{K}^*\mathbf{X}_\mathcal{K} k$$

$$\begin{aligned} &= \mathbf{E}_\epsilon(\mathbf{X}_\mathcal{K}^*\mathbf{X}_\mathcal{K}h + \sqrt{2\epsilon}\mathbf{X}_\mathcal{K}^*\mathbf{V}_\epsilon h) \\ &= \mathbf{X}_\mathcal{K}^*\mathbf{X}_\mathcal{K}h + \sqrt{2\epsilon}\mathbf{X}_\mathcal{K}^*\mathbf{V}_\epsilon h \\ &= -\mathbf{X}_\mathcal{K}^*\mathbf{X}_\mathcal{K}k. \end{aligned}$$

Since $\|\mathbf{X}_\mathcal{K}^*\mathbf{X}_\mathcal{K}k\| \geq \epsilon\|k\|$, it follows that $k = 0$. Thus $y = h + \sqrt{2\epsilon}\mathbf{V}_\epsilon h$ and therefore

$$2\epsilon\|h\|^2 = \|\sqrt{2\epsilon}\mathbf{V}_\epsilon h\|^2 = \|\mathbf{X}_\mathcal{K}h\|^2 \leq \epsilon\|h\|^2$$

because $h \in \mathcal{H}_\epsilon$. This implies that $h = 0$, that is $y = 0$. □

Remark. It is easy to see that the complement of $\overline{\text{Graph}_\mathcal{W}}$ can be characterized in the following way: $\mathcal{K} \notin \overline{\text{Graph}_\mathcal{W}} \Leftrightarrow$ there exists $\epsilon > 0$ such that $\mathcal{K}' \cap \mathcal{W}^\perp \neq \{0\}$ for all $\mathcal{K}' \in \text{Ball}(\mathcal{K}, \epsilon)$. From Theorem 2 the complement of $\overline{\text{Graph}_\mathcal{W}}$ is the set S_{pi}. A geometric characterization of S_{pi} is as follows:

$$\text{S}_{\text{pi}} = \{\mathcal{K} \in \text{S}_\mathcal{L} \,:\, \mathcal{K} + \mathcal{W}^\perp \text{ is closed, } \dim(\mathcal{K} \cap \mathcal{W}^\perp) > \dim\left(\mathcal{L} \ominus (\mathcal{K} + \mathcal{W}^\perp)\right)\}.$$

To see the equality, note that $\mathcal{K}+\mathcal{W}^\perp = \mathbf{\Pi}_\mathcal{W}\mathcal{K}+\mathcal{W}^\perp$. Hence $\mathcal{K}+\mathcal{W}^\perp$ is closed $\Leftrightarrow \mathbf{\Pi}_\mathcal{W}\mathcal{K}$ is closed. Also $\ker(\mathbf{\Pi}_\mathcal{W}|_\mathcal{K}) = \mathcal{K} \cap \mathcal{W}^\perp$ and $\ker(\mathbf{\Pi}_\mathcal{K}|_\mathcal{W}) = \mathcal{W} \cap \mathcal{K}^\perp = \mathcal{L} \ominus (\mathcal{K} + \mathcal{W}^\perp)$.

5. Robust stabilization. Consider the feedback interconnection $[\mathbf{P},\ \mathbf{C}]$ and let $\mathcal{M} = \mathcal{G}_\mathbf{P}$, $\mathcal{N} = \mathcal{G}'_\mathbf{C} \in \text{S}_\mathcal{L}$. Define $\mathbf{A}_{\mathcal{M},\mathcal{N}} := \mathbf{\Pi}_{\mathcal{N}^\perp}|_\mathcal{M}$. The following is a standard result in operator theory.

PROPOSITION 6. *Let $\mathcal{M}, \mathcal{N} \in \text{S}_\mathcal{L}$. The following are equivalent:*

(a) $\mathcal{M} \cap \mathcal{N} = \{0\}$ *and* $\mathcal{M} + \mathcal{N} = \mathcal{L}$;

(b) $\mathbf{A}_{\mathcal{M},\mathcal{N}}$ *is invertible.*

Proof. (a) $\Rightarrow$ (b). Note that

$$\begin{aligned} \mathbf{\Pi}_{\mathcal{N}^\perp}\mathcal{M} &= \mathbf{\Pi}_{\mathcal{N}^\perp}(\mathcal{M} + \mathcal{N}) \\ &= \mathbf{\Pi}_{\mathcal{N}^\perp}\mathcal{L} \\ &= \mathcal{N}^\perp. \end{aligned}$$

Thus $\mathbf{A}_{\mathcal{M},\mathcal{N}}$ maps $\mathcal{M}$ onto $\mathcal{N}^\perp$. Since $\mathcal{M} \cap \mathcal{N} = \{0\}$ then $\mathbf{A}_{\mathcal{M},\mathcal{N}}$ is one-to-one. Thus $\mathbf{A}_{\mathcal{M},\mathcal{N}}$ is invertible (see [9, Prob. 52]).

(b) $\Rightarrow$ (a). For any $x \in \mathcal{M} \cap \mathcal{N}$ it clearly holds that $\mathbf{A}_{\mathcal{M},\mathcal{N}}x = 0$. Since $\mathbf{A}_{\mathcal{M},\mathcal{N}}$ is invertible, then $\mathcal{M} \cap \mathcal{N} = \{0\}$. Also, for any $x \in \mathcal{L}$ we can write

$$x = \mathbf{A}_{\mathcal{M},\mathcal{N}}^{-1}\mathbf{\Pi}_{\mathcal{N}^\perp}x + (\mathbf{I}_\mathcal{L} - \mathbf{A}_{\mathcal{M},\mathcal{N}}^{-1}\mathbf{\Pi}_{\mathcal{N}^\perp})x =: m + n. \tag{22}$$

Clearly $m = \mathbf{A}_{\mathcal{M},\mathcal{N}}^{-1}\mathbf{\Pi}_{\mathcal{N}^\perp}x \in \mathcal{M}$. We claim that $n \in \mathcal{N}$. To see this note that $\mathbf{\Pi}_{\mathcal{N}^\perp}n = \mathbf{\Pi}_{\mathcal{N}^\perp}x - \mathbf{\Pi}_{\mathcal{N}^\perp}\mathbf{A}_{\mathcal{M},\mathcal{N}}^{-1}\mathbf{\Pi}_{\mathcal{N}^\perp}x = 0$. Thus, $\mathcal{L} = \mathcal{M} + \mathcal{N}$. □

It follows from Proposition 2 that $[\mathbf{P},\ \mathbf{C}]$ is a stable feedback configuration if and only if $\mathbf{A}_{\mathcal{M},\mathcal{N}}$ is invertible. When $[\mathbf{P},\ \mathbf{C}]$ is stable we define the operator

$$\mathbf{Q}_{\mathcal{M},\mathcal{N}} := \mathbf{A}_{\mathcal{M},\mathcal{N}}^{-1}\mathbf{\Pi}_{\mathcal{N}^\perp}.$$

This is the *parallel projection onto $\mathcal{M}$ along $\mathcal{N}$*. Note that $\mathbf{Q}_{\mathcal{M},\mathcal{N}}$ can be expressed directly in terms of $\mathbf{P}$ and $\mathbf{C}$ as follows:

$$\mathbf{Q}_{\mathcal{M},\mathcal{N}} = \begin{pmatrix} \mathbf{I}_\mathcal{U} \\ \mathbf{P} \end{pmatrix} \left((\mathbf{I}_\mathcal{U} - \mathbf{CP})^{-1}, -\mathbf{C}(\mathbf{I}_\mathcal{Y} - \mathbf{PC})^{-1}\right) \tag{23}$$

and in terms of the input-to-error operator $\mathbf{H}_{\mathbf{P},\mathbf{C}}$:

$$\mathbf{Q}_{\mathcal{M},\mathcal{N}} = \begin{pmatrix} \mathbf{I}_{\mathcal{U}} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{\mathcal{Y}} \end{pmatrix} \mathbf{H}_{\mathbf{P},\mathbf{C}} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\mathcal{Y}} \end{pmatrix}. \tag{24}$$

When [**P**, **C**] is stable we also define

$$\begin{aligned} b_{\mathcal{M},\mathcal{N}} &:= \|\mathbf{Q}_{\mathcal{M},\mathcal{N}}\|^{-1} \\ &= \tau(\mathbf{\Pi}_{\mathcal{N}^\perp}|_{\mathcal{M}}) = \sqrt{1-\delta(\mathcal{M},\mathcal{N}^\perp)^2} \\ &= \inf\{\|\mathbf{A}_{\mathcal{M},\mathcal{N}}x\| : x \in \mathcal{M} \text{ and } \|x\| = 1\} \\ &= \inf\{\text{dist}\,(x,\mathcal{N}) \;:\; x \in \mathcal{M} \text{ and } \|x\| = 1\} \\ &= \inf\{\sin\theta(x,y) \;:\; 0 \neq x \in \mathcal{M}, 0 \neq y \in \mathcal{N}\}, \end{aligned} \tag{25}$$

where

$$\theta(x,y) := \arccos \frac{|<x,y>|}{\|x\|\|y\|}$$

denotes the *angle* between two nonzero vectors $x, y \in \mathcal{L}$. When [**P**, **C**] is not stable we set $b_{\mathcal{M},\mathcal{N}} := 0$. Equation (25) follows from (3). The quantity $b_{\mathcal{M},\mathcal{N}}$ is the sine of the *minimal angle*

$$\begin{aligned} \theta_{\min}(\mathcal{M},\mathcal{N}) &:= \inf\{\theta(x,y) \;:\; 0 \neq x \in \mathcal{M}, 0 \neq y \in \mathcal{N}\} \\ &= \arcsin b_{\mathcal{M},\mathcal{N}} \\ &= \arccos \delta(\mathcal{M},\mathcal{N}^\perp) \end{aligned}$$

(e.g., see [8]). Since

$$\begin{aligned} \delta(\mathcal{M},\mathcal{N}^\perp) &= \delta(\mathcal{M}^\perp,\mathcal{N}) \\ &= \delta(\mathcal{N},\mathcal{M}^\perp) \\ &= \delta(\mathcal{N}^\perp,\mathcal{M}), \end{aligned}$$

it follows that

$$\begin{aligned} b_{\mathcal{M},\mathcal{N}} &= b_{\mathcal{M}^\perp,\mathcal{N}^\perp} \\ &= b_{\mathcal{N},\mathcal{M}} \\ &= b_{\mathcal{N}^\perp,\mathcal{M}^\perp}. \end{aligned} \tag{26}$$

Equation (26) was shown in [3] (cf. [21, Lemma 1.1, p. 341]), [6]. It also follows from [10, Lemma 4] after noting that $b_{\mathcal{M},\mathcal{N}}$ is the inverse of the norm of a parallel projection.

Conditions for stability of a feedback configuration can be expressed in a number of equivalent ways (cf. [3], [14]).

COROLLARY 3. *The following are equivalent:*

(a) [**P**, **C**] *is stable*;
(b) $\mathbf{A}_{\mathcal{M},\mathcal{N}}$ *is invertible*;
(b) $\delta(\mathcal{M},\mathcal{N}^\perp) < 1$;
(c) $\theta_{\max}(\mathcal{M},\mathcal{N}^\perp) < \pi/2$;
(d) $\theta_{\min}(\mathcal{M},\mathcal{N}) > 0$.

Proof. The proof follows from Propositions 3 and 6. □

THEOREM 3. *The following are equivalent:*

(a) $[\mathbf{P},\ \mathbf{C}]$ *is stable and* $b < b_{\mathcal{M},\mathcal{N}}$;

(b) $[\mathbf{P}',\mathbf{C}]$ *is stable and* $\mathbf{Q}_{\mathcal{M}',\mathcal{N}}$ *is uniformly bounded for all* $\mathbf{P}'$ *so that, with* $\mathcal{M}' := \mathcal{G}_{\mathbf{P}'}$, $\delta(\mathcal{M},\mathcal{M}') \leq b$.

Before presenting the proof of the theorem we will establish the following lemma. For any $0 \neq h \in \mathcal{L}$ we denote $\hat{h} := h/\|h\|$.

LEMMA 3. *Let* $\mathcal{K}$, $\mathcal{W} \in \mathrm{S}_{\mathcal{L}}$. *Take any* $0 \neq h \in \mathcal{K}$ *and* $0 \neq h_1 \notin \mathcal{K}$. *Define* $\mathcal{K}_- = \mathcal{K} \ominus \mathbb{C}\, h$ *and* $\mathcal{K}_1 = \mathcal{K}_- + \mathbb{C}\, h_1$. *Then*

(a) $\delta(\mathcal{K},\mathcal{K}_1) = \sqrt{1 - |<\hat{h},\hat{h}_2>|^2}$, *where* $h_2 = \mathbf{\Pi}_{\mathcal{K}_-^\perp} h_1$;

(b) *if* $\mathcal{K} \in \mathrm{Graph}_{\mathcal{W}}$ *then* $\mathcal{K}_1 \in \overline{\mathrm{Graph}}_{\mathcal{W}}$.

Proof. (a) Since $\mathcal{K}_1 = \mathcal{K}_- \oplus \mathbb{C}\, h_2$ and $\mathcal{K}^\perp = \mathcal{K}_-^\perp \ominus \mathbb{C}\, h$ we obtain

$$\begin{aligned}\|\mathbf{\Pi}_{\mathcal{K}^\perp}\mathbf{\Pi}_{\mathcal{K}_1}\| &= \|(\mathbf{\Pi}_{\mathcal{K}_-^\perp} - \mathbf{\Pi}_{\mathbb{C}\,h})(\mathbf{\Pi}_{\mathcal{K}_-} + \mathbf{\Pi}_{\mathbb{C}\,h_2})\| \\ &= \|\,(\mathbf{\Pi}_{\mathbb{C}\,h_2} - \mathbf{\Pi}_{\mathbb{C}\,h})\,\mathbf{\Pi}_{\mathbb{C}\,h_2}\| \\ &= \sqrt{1 - |<\hat{h},\hat{h}_2>|^2}.\end{aligned}$$

Similarly, $\|\mathbf{\Pi}_{\mathcal{K}_1^\perp}\mathbf{\Pi}_{\mathcal{K}}\| = \|\,(\mathbf{\Pi}_{\mathbb{C}\,h_2} - \mathbf{\Pi}_{\mathbb{C}\,h})\,\mathbf{\Pi}_{\mathbb{C}\,h}\| = \sqrt{1 - |<\hat{h},\hat{h}_2>|^2}$. Therefore, $\delta(\mathcal{K},\mathcal{K}_1) = \sqrt{1 - |<\hat{h},\hat{h}_2>|^2}$.

(b) Define $\mathbf{X} := \mathbf{\Pi}_{\mathcal{W}}|_{\mathcal{K}}$, $\mathbf{X}_1 := \mathbf{\Pi}_{\mathcal{W}}|_{\mathcal{K}_1}$, $\hat{\mathbf{X}} := \mathbf{X}\mathbf{\Pi}_{\mathcal{K}}|_{(\mathcal{K}+\mathcal{K}_1)}$, and $\hat{\mathbf{X}}_1 := \mathbf{X}_1\mathbf{\Pi}_{\mathcal{K}_1}|_{(\mathcal{K}+\mathcal{K}_1)}$. If $\mathcal{K}_1 \notin \overline{\mathrm{Graph}}_{\mathcal{W}} = \mathrm{S}_{\mathrm{npi}}$, then $\mathbf{X}_1$ is semi-Fredholm and then it is obvious that $\mathbf{X}, \hat{\mathbf{X}}, \hat{\mathbf{X}}_1$ are also semi-Fredholm. Since $\mathcal{K} \in \mathrm{Graph}_{\mathcal{W}}$, $\ker \mathbf{X} = 0$ and $\mathrm{ind}\, \mathbf{X} \leq 0$. Because $h_1 \notin \mathcal{K}$, it can be seen that $\dim\ker \hat{\mathbf{X}} = \dim\ker \mathbf{X} + 1$ and similarly that $\dim\ker \hat{\mathbf{X}}_1 = \dim\ker \mathbf{X}_1 + 1$. It follows that $\mathrm{ind}\, \hat{\mathbf{X}} = \mathrm{ind}\, \mathbf{X} + 1$ and $\mathrm{ind}\, \hat{\mathbf{X}}_1 = \mathrm{ind}\, \mathbf{X}_1 + 1$. However,

$$\hat{\mathbf{X}}_1 = \mathbf{\Pi}_{\mathcal{W}}\mathbf{\Pi}_{\mathcal{K}}|_{(\mathcal{K}+\mathcal{K}_1)} + \mathbf{\Pi}_{\mathcal{W}}(\mathbf{\Pi}_{\mathbb{C}\,h_2} - \mathbf{\Pi}_{\mathbb{C}\,h})|_{(\mathcal{K}+\mathcal{K}_1)} = \hat{\mathbf{X}} + \text{ finite-rank operator.}$$

Therefore, $\mathrm{ind}\, \hat{\mathbf{X}}_1 = \mathrm{ind}\, \hat{\mathbf{X}}$ and hence $\mathrm{ind}\, \mathbf{X}_1 = \mathrm{ind}\, \mathbf{X}$. Thus, $\mathrm{ind}\, \mathbf{X}_1 \leq 0$, and consequently, $\mathcal{K}_1 \notin \mathrm{S}_{\mathrm{pi}}$; that is, $\mathcal{K}_1 \in \mathrm{S}_{\mathrm{npi}}$, a contradiction. This proves that $\mathcal{K}_1 \in \overline{\mathrm{Graph}}_{\mathcal{W}}$. □

Proof of Theorem 3. (a)⇒(b). Assume (b) fails. We will show that, if $[\mathbf{P},\ \mathbf{C}]$ is stable, then $b \geq b_{\mathcal{M},\mathcal{N}}$. Since (b) fails then, either $\|(\mathbf{\Pi}_{\mathcal{N}^\perp}|_{\mathcal{M}'})^{-1}\|$ is not bounded above in $\overline{\mathrm{Ball}}(\mathcal{M},b) \cap \mathrm{Graph}_{\mathcal{U}}$ or $\mathbf{\Pi}_{\mathcal{N}^\perp}|_{\mathcal{M}'}$ is not invertible for some $\mathcal{M}' \in \overline{\mathrm{Ball}}(\mathcal{M},b) \cap \mathrm{Graph}_{\mathcal{U}}$. This means that one of the following two possibilities holds:

(i) $\tau(\mathbf{\Pi}_{\mathcal{N}^\perp}|_{\mathcal{M}'})$ is not bounded below in $\overline{\mathrm{Ball}}(\mathcal{M},b) \cap \mathrm{Graph}_{\mathcal{U}}$;

(ii) there exists $\mathcal{M}' \in \overline{\mathrm{Ball}}(\mathcal{M},b) \cap \mathrm{Graph}_{\mathcal{U}}$ and $0 \neq y \in \mathcal{N}^\perp$ such that $\mathbf{\Pi}_{\mathcal{M}'}y = 0$.

In case (i), for all $\epsilon > 0$ there exists $\mathcal{M}'$ and $x \in \mathcal{M}'$ of unit norm such that $\|\mathbf{\Pi}_{\mathcal{N}^\perp}x\| = \mathrm{dist}(x,\mathcal{N}) < \epsilon$. Note that $b \geq \delta(\mathcal{M},\mathcal{M}') \geq \sup_{\xi\in\mathcal{M}',\|\xi\|=1}\mathrm{dist}(\xi,\mathcal{M}) \geq \mathrm{dist}(x,\mathcal{M}) = \|\mathbf{\Pi}_{\mathcal{M}^\perp}x\|$. Also

$$\begin{aligned} b_{\mathcal{M},\mathcal{N}} = b_{\mathcal{N},\mathcal{M}} &\leq \left\|\mathbf{\Pi}_{\mathcal{M}^\perp}\frac{\mathbf{\Pi}_{\mathcal{N}}x}{\|\mathbf{\Pi}_{\mathcal{N}}x\|}\right\| \\ &\leq \frac{\|\mathbf{\Pi}_{\mathcal{M}^\perp}x\| + \|\mathbf{\Pi}_{\mathcal{M}^\perp}\mathbf{\Pi}_{\mathcal{N}^\perp}x\|}{\|\mathbf{\Pi}_{\mathcal{N}}x\|} \leq \frac{b+\epsilon}{\sqrt{1-\epsilon^2}}. \end{aligned} \tag{27}$$

Since (27) holds for all ϵ then $b_{\mathcal{M},\mathcal{N}} \leq b$. In case (ii), note that $y \in \mathcal{M}'^{\perp}$. Thus $b \geq \delta(\mathcal{M},\mathcal{M}') = \delta(\mathcal{M}^{\perp},\mathcal{M}'^{\perp}) \geq \operatorname{dist}(y,\mathcal{M}^{\perp}) = \|\mathbf{\Pi}_{\mathcal{M}}y\|$. Also $b_{\mathcal{M},\mathcal{N}} = b_{\mathcal{M}^{\perp},\mathcal{N}^{\perp}} \leq \|\mathbf{\Pi}_{\mathcal{M}}y\|$ since $y \in \mathcal{N}^{\perp}$. Therefore $b_{\mathcal{M},\mathcal{N}} \leq b$.

(b)⇒(a). Suppose that (b) holds for some $b \geq b_{\mathcal{M},\mathcal{N}}$. Then the same is true for some $b > b_{\mathcal{M},\mathcal{N}}$. To see this, first note that $b < 1$ necessarily, otherwise $[\mathbf{P}',\mathbf{C}]$ is stable for *any* system $\mathbf{P}'$, and there is an easy contradition. By assumption, there exists a c such that $\|\mathbf{Q}_{\mathcal{M}',\mathcal{N}}\| \leq c$ for all $\mathbf{P}'$ with $\delta(\mathcal{M},\mathcal{M}') \leq b$. From the identity $\mathbf{A}_{\mathcal{M}'',\mathcal{N}} = \mathbf{A}_{\mathcal{M}',\mathcal{N}}(\mathbf{I}_{\mathcal{M}'} + \mathbf{Q}_{\mathcal{M}',\mathcal{N}}(\mathbf{\Pi}_{\mathcal{M}''} - \mathbf{\Pi}_{\mathcal{M}'})|_{\mathcal{M}'})(\mathbf{\Pi}_{\mathcal{M}''}|_{\mathcal{M}'})^{-1}|_{\mathcal{M}''}$ (cf. [3]) we can see that $\mathbf{A}_{\mathcal{M}'',\mathcal{N}}$ is invertible for all $\mathbf{P}''$ with $\delta(\mathcal{M}',\mathcal{M}'') \leq 1/2c$ for some $\mathbf{P}'$ with $\delta(\mathcal{M},\mathcal{M}') \leq b$. Moreover

$$\begin{aligned}\|\mathbf{Q}_{\mathcal{M}'',\mathcal{N}}\| &= \|\mathbf{\Pi}_{\mathcal{M}''}|_{\mathcal{M}'}(\mathbf{I}_{\mathcal{M}'} + \mathbf{Q}_{\mathcal{M}',\mathcal{N}}(\mathbf{\Pi}_{\mathcal{M}''} - \mathbf{\Pi}_{\mathcal{M}'})|_{\mathcal{M}'})^{-1}\mathbf{Q}_{\mathcal{M}',\mathcal{N}}\| \\ &\leq 2c.\end{aligned}$$

From Theorem 1 the union of open balls, of radius $1/2c$, about all $\mathcal{M}'$ with $\delta(\mathcal{M},\mathcal{M}') \leq b$ includes an open ball about $\mathcal{M}$ of radius $b+\epsilon$ for some $\epsilon > 0$. It now follows that (b) holds for $\mathbf{P}''$ in a ball about $\mathbf{P}$ and radius strictly greater than $b_{\mathcal{M},\mathcal{N}}$. So from now on we assume that $b_{\mathcal{M},\mathcal{N}} < b < 1$.

Next we prove that there exists a subspace $\mathcal{M}' \in \overline{\mathrm{Graph}_{\mathcal{U}}}$ with $\delta(\mathcal{M},\mathcal{M}') < b$ such that $\mathbf{\Pi}_{\mathcal{N}^{\perp}}|_{\mathcal{M}'}$ is not invertible. Let $\mathbf{A} := \mathbf{A}_{\mathcal{M},\mathcal{N}}$, $\mathbf{E}_{\lambda}$ be the spectral family of $\mathbf{A}^*\mathbf{A}$, and $h \in \mathbf{E}_{\tau(\mathbf{A})+\epsilon}\mathcal{M}$ of unit norm, for some arbitrary $\epsilon > 0$. Then $\|(\mathbf{A}^*\mathbf{A} - b_{\mathcal{M},\mathcal{N}})h\| \leq \epsilon$. Define $\mathcal{M}_- := \mathcal{M} \ominus \mathbb{C}\,h$, $p_0 := \mathbf{\Pi}_{\mathcal{N}}h \in \mathcal{N}$, $q_0 := \mathbf{\Pi}_{\mathcal{M}_-^{\perp}}p_0$, and $\mathcal{M}' = \mathcal{M}_- + \mathbb{C}\,p_0$. Since $\mathcal{M}' \cap \mathcal{N} \neq \{0\}$ then $\mathbf{\Pi}_{\mathcal{M}'}|_{\mathcal{N}^{\perp}}$ is not invertible. From Lemma 3 we have

$$\delta(\mathcal{M},\mathcal{M}') = \sqrt{1 - |\langle h,\hat{q}_0\rangle|^2} \tag{28}$$

and $\mathcal{M}' \in \overline{\mathrm{Graph}_{\mathcal{U}}}$. To evaluate (28) we first note that

$$\begin{aligned}\langle h,q_0\rangle = \langle h,p_0\rangle &= \|p_0\|^2 \\ &= 1 - \|\mathbf{A}h\|^2.\end{aligned} \tag{29}$$

We also have

$$\begin{aligned}\|q_0\|^2 &= \|p_0\|^2 - \|\mathbf{\Pi}_{\mathcal{M}_-}\mathbf{\Pi}_{\mathcal{N}}h\|^2 \\ &= \|p_0\|^2 - \|\mathbf{\Pi}_{\mathcal{M}_-}\mathbf{\Pi}_{\mathcal{N}^{\perp}}h\|^2 \\ &= \|p_0\|^2 - \langle \mathbf{\Pi}_{\mathcal{N}^{\perp}}h, (\mathbf{\Pi}_{\mathcal{M}} - \mathbf{\Pi}_{\mathbb{C}h})\,\mathbf{\Pi}_{\mathcal{N}^{\perp}}h\rangle \\ &= \|p_0\|^2 - \langle \mathbf{\Pi}_{\mathcal{M}}\mathbf{\Pi}_{\mathcal{N}^{\perp}}h, \mathbf{\Pi}_{\mathcal{M}}\mathbf{\Pi}_{\mathcal{N}^{\perp}}h\rangle + |\langle \mathbf{\Pi}_{\mathcal{N}^{\perp}}h,h\rangle|^2 \\ &= 1 - \|\mathbf{A}h\|^2 - \|\mathbf{A}^*\mathbf{A}h\|^2 + \|\mathbf{A}h\|^4 \\ &= 1 - \|\mathbf{A}h\|^2 + O(\epsilon).\end{aligned} \tag{30}$$

From (28)–(30), we obtain $\delta(\mathcal{M},\mathcal{M}') = \|\mathbf{A}h\| + O(\epsilon)$. In particular, for sufficiently small ϵ we have $\delta(\mathcal{M},\mathcal{M}') < b$.

In case $\mathcal{M}' \in \mathrm{Graph}_{\mathcal{U}}$ then the hypothesis is violated and the proof is complete. If not, consider a sequence $\mathcal{M}'_i \in \mathrm{Graph}_{\mathcal{U}}$, $i = 1,2,\ldots$, converging to $\mathcal{M}'$. If there is a subsequence such that $\mathbf{\Pi}_{\mathcal{M}'_i}|_{\mathcal{N}^{\perp}}$ is not invertible then, again, there is a contradiction. Otherwise, since $\lim_{i\to\infty}\|\mathbf{\Pi}_{\mathcal{M}'}|_{\mathcal{N}^{\perp}} - \mathbf{\Pi}_{\mathcal{M}'_i}|_{\mathcal{N}^{\perp}}\| = 0$, we can find a subsequence such that $\lim_{i\to\infty}\|(\mathbf{\Pi}_{\mathcal{M}'_i}|_{\mathcal{N}^{\perp}})^{-1}\| = \infty$. This also violates the hypothesis. □

Remark 1. A similar result was established in [6, Thm. 5] for linear time-invariant causal systems. However, the result in [6] differs from the one above in that the <

and $\leq$ signs are interchanged, and the "uniform boundedness" is absent. We will show that the exact statement of [6, Thm. 5] is not valid in the case of time-varying systems so that the uniformity condition is in fact necessary. More precisely, in the next section we will present an example where $[\mathbf{P}_1, \mathbf{C}]$ is stable for all $\mathbf{P}_1$ such that $\delta(\mathcal{M}, \mathcal{M}_1) < 1$ while $b_{\mathcal{M},\mathcal{N}} = 1/\sqrt{2} < 1$.

Remark 2. The basic geometric ideas behind the proof of Theorem 3 can be simply expressed. The essence of the sufficiency part of Theorem 3 ((a)$\Rightarrow$(b)) can be seen from the identity $\mathbf{A}_{\mathcal{M}',\mathcal{N}} = \mathbf{A}_{\mathcal{M},\mathcal{N}}(\mathbf{I}_{\mathcal{M}} + \mathbf{Q}_{\mathcal{M},\mathcal{N}}(\mathbf{\Pi}_{\mathcal{M}'} - \mathbf{\Pi}_{\mathcal{M}})|_{\mathcal{M}})(\mathbf{\Pi}_{\mathcal{M}'}|_{\mathcal{M}})^{-1}|_{\mathcal{M}'}$, which implies that $\mathbf{A}_{\mathcal{M}',\mathcal{N}}$ is invertible for all $\mathbf{P}'$ if $\delta(\mathcal{M}, \mathcal{M}') \leq \|\mathbf{Q}_{\mathcal{M},\mathcal{N}}\|^{-1}$. The key idea in the necessity part is to find a subspace $\mathcal{M}'$ such that $\delta(\mathcal{M}, \mathcal{M}') = \|\mathbf{Q}_{\mathcal{M},\mathcal{N}}\|^{-1}$ and so that Proposition 6(a) is violated. The construction given in the proof is to remove a direction orthogonally from $\mathcal{M}$ and to replace it with a direction from $\mathcal{N}$. The vectors are chosen in such a way that the angle between them is equal (or arbitrarily close to) $\theta_{\min}(\mathcal{M}, \mathcal{N})$. This gives $\mathcal{M}' \cap \mathcal{N} \neq \{0\}$ with $\delta(\mathcal{M}, \mathcal{M}')$ having the required value. (The reader is referred to [19] for another version of this idea.) The additional ingredients in the proof deal with the uniform boundedness condition and the need to impose graphability on the perturbed subspaces.

Remark 3. In the theorem we do not impose any time-invariance and causality constraint on the systems considered. Certainly the implication (a)$\Rightarrow$(b) of Theorem 3 is still valid when the class of systems is restricted by a causality requirement, but the reverse implication requires a construction different from the one given here.

6. Clarification of uniform boundedness condition. We now present an example to show that, in the time-varying case, the obstruction that limits the largest perturbation ball in the gap metric may be due solely to the lack of uniform boundedness of the closed loop operator, as expressed in Theorem 3.

Let $\mathcal{U} = \mathcal{Y} = \ell_2[0, \infty) =: \mathcal{V}$, $\mathcal{L} = \mathcal{U} \oplus \mathcal{Y}$, and identify $\mathcal{U}$ and $\mathcal{Y}$ with the corresponding subspaces of $\mathcal{L}$. Consider $\mathbf{P}$ having the matrix representation

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

and let $\mathbf{C} = 0$. Then $\mathcal{M} = \{\binom{v}{(v)_0} : v \in \mathcal{V}\}$, where $(v)_0 = (v_0, 0, 0, \ldots)$ for any $v = (v_0, v_1, v_2, \ldots) \in \mathcal{V}$, and $\mathcal{N} = \{\binom{0}{v} : v \in \mathcal{V}\}$. For any $\mathcal{M}_1 \in \mathrm{Graph}_{\mathcal{U}}$ define $\mathbf{P}_1$ by $\mathcal{M}_1 = \mathcal{G}_{\mathbf{P}_1}$. Also define $\mathrm{Ball}_{\mathcal{U}}(\mathcal{M}, b) := \mathrm{Ball}(\mathcal{M}, b) \cap \mathrm{Graph}_{\mathcal{U}}$.

PROPOSITION 7. *For the example given above*

$$\sup\{b : \{\mathcal{M}_1 \in \mathrm{Ball}_{\mathcal{U}}(\mathcal{M}, b) \textit{ implies that } [\mathbf{P}_1, \mathbf{C}] \textit{ is stable}\}\} = 1.$$

This should be contrasted against the fact that for the particular $\mathbf{P}, \mathbf{C}$ given above

$$\begin{aligned} b_{\mathcal{M},\mathcal{N}} &= \left\| \begin{pmatrix} \mathbf{I}_{\mathcal{U}} \\ \mathbf{P} \end{pmatrix} \left((\mathbf{I}_{\mathcal{U}} - \mathbf{C}\mathbf{P})^{-1}, -\mathbf{C}(\mathbf{I}_{\mathcal{Y}} - \mathbf{P}\mathbf{C})^{-1} \right) \right\|^{-1} \\ &= \left\| \begin{pmatrix} \mathbf{I}_{\mathcal{U}} \\ \mathbf{P} \end{pmatrix} \right\|^{-1} = (1 + \|\mathbf{P}\|^2)^{-1/2} = \frac{1}{\sqrt{2}}. \end{aligned}$$

Proof of Proposition 7. Since $\mathcal{N} = \mathcal{U}^{\perp}$, for any $\mathcal{M}_1 \in \mathrm{Ball}_{\mathcal{U}}(\mathcal{M}, b)$, $\mathcal{M}_1 \cap \mathcal{N} = \{0\}$. Therefore, $[\mathbf{P}_1, \mathbf{C}]$ is stable $\Leftrightarrow$ $\mathcal{M}_1 + \mathcal{N} = \mathcal{L}$ $\Leftrightarrow$ $\mathbf{\Pi}_{\mathcal{U}}\mathcal{M}_1 = \mathcal{U}$. Next note that $\mathcal{M}^\perp = \{\binom{-(v)_0}{v} : v \in \mathcal{V}\}$. Proposition 3 also implies that any $\mathcal{M}_1$ such that $\delta(\mathcal{M}, \mathcal{M}_1) < 1$ can be written as

$$\mathcal{M}_1 = \left\{ \begin{pmatrix} v \\ (v)_0 \end{pmatrix} + \begin{pmatrix} -(\mathbf{X}v)_0 \\ \mathbf{X}v \end{pmatrix} : v \in \mathcal{V} \right\},$$

where $\mathbf{X} : \mathcal{V} \to \mathcal{V}$ is a bounded operator. However,

$$\begin{aligned} \mathcal{M}_1 \cap \mathcal{N} = \{0\} &\Leftrightarrow v - (\mathbf{X}v)_0 = 0 \text{ implies } (v)_0 + \mathbf{X}v = 0 \\ &\Leftrightarrow v = (v)_0 = (\mathbf{X}v)_0 \text{ implies } (v)_0 + \mathbf{X}v = 0 \\ &\Leftrightarrow (\mathbf{X})_{0,0} \neq 1, \end{aligned}$$

where $(\mathbf{X})_{0,0}$ denotes the $(0,0)$-entry in a matrix representation of $\mathbf{X}$ with respect to the standard basis of $\mathcal{V} = \ell_2[0, \infty)$. Take any $\mathcal{M}_1 \in \mathrm{Ball}_{\mathcal{U}}(\mathcal{M}, 1)$. Then

$$\mathbf{\Pi}_{\mathcal{U}} \mathcal{M}_1 = \{v - (\mathbf{X}v)_0 : v \in \mathcal{V}\} = \mathcal{U}$$

since $(\mathbf{X})_{0,0} \neq 1$. Hence, $\mathcal{M}_1 \in \mathrm{Ball}_{\mathcal{U}}(\mathcal{M}, 1)$ implies that $[\mathbf{P}_1, \mathbf{C}]$ is stable. So $b = 1$ is the supremal b. □

7. Combined plant and controller uncertainty. When both plant and controller are subject simultaneously to gap-ball uncertainty, there is a maximal amount for the combined uncertainty that can be tolerated. The following theorem is a generalization of an elegant result of Qiu and Davison [16] to the time-varying case.

THEOREM 4. *Let* $\mathbf{P} \in \mathcal{P}_{\mathcal{U},\mathcal{Y}}$, $\mathbf{C} \in \mathcal{P}_{\mathcal{Y},\mathcal{U}}$ *and let* b_1, b_2 *be fixed nonnegative numbers such that* $b_1^2 + b_2^2 < 1$. *Then the following are equivalent:*

(a) $[\mathbf{P}, \mathbf{C}]$ *is stable and* $b_1\sqrt{1 - b_2^2} + b_2\sqrt{1 - b_1^2} < b_{\mathcal{M},\mathcal{N}}$;

(b) $[\mathbf{P}', \mathbf{C}']$ *is stable and* $\mathbf{Q}_{\mathcal{M}',\mathcal{N}'}$ *is uniformly bounded for all* $\mathbf{P}'$, $\mathbf{C}'$ *with* $\mathcal{M}' := \mathcal{G}_{\mathbf{P}'}$, $\mathcal{N}' := \mathcal{G}'_{\mathbf{C}'}$, $\delta(\mathcal{M}, \mathcal{M}') \leq b_1$ *and* $\delta(\mathcal{N}, \mathcal{N}') \leq b_2$.

Proof. (a)⇒(b) Suppose (b) fails and that $[\mathbf{P}, \mathbf{C}]$ is stable. We will show that $b_1\sqrt{1 - b_2^2} + b_2\sqrt{1 - b_1^2} \geq b_{\mathcal{M},\mathcal{N}}$. As in the proof of Theorem 3 there are two possibilities:

(i) $\tau(\mathbf{\Pi}_{\mathcal{N}'^\perp}|_{\mathcal{M}'})$ is not bounded below for $\mathcal{M}' \in \overline{\mathrm{Ball}}(\mathcal{M}, b_1) \cap \mathrm{Graph}_{\mathcal{U}}$ and $\mathcal{N}' \in \overline{\mathrm{Ball}}(\mathcal{N}, b_2) \cap \mathrm{Graph}_{\mathcal{Y}}$,

(ii) there exists $\mathcal{M}' \in \overline{\mathrm{Ball}}(\mathcal{M}, b_1) \cap \mathrm{Graph}_{\mathcal{U}}$ and $\mathcal{N}' \in \overline{\mathrm{Ball}}(\mathcal{N}, b_2) \cap \mathrm{Graph}_{\mathcal{Y}}$ and $0 \neq z \in \mathcal{N}'^\perp \cap \mathcal{M}'^\perp$.

In case (i), for all $\epsilon > 0$ there exists $\mathcal{M}'$, $\mathcal{N}'$, and $x \in \mathcal{M}'$ of unit norm such that $\|\mathbf{\Pi}_{\mathcal{N}'^\perp} x\| < \epsilon$. Setting $y := \mathbf{\Pi}_{\mathcal{N}'} x \in \mathcal{N}'$ we have

$$\theta(x, y) := \arccos \frac{|<x, y>|}{\|x\|\|y\|} = \arccos \left(\frac{1 - \|\mathbf{\Pi}_{\mathcal{N}'^\perp} x\|^2}{\|y\|} \right) < \arcsin \epsilon.$$

Since $\delta(\mathcal{M}, \mathcal{M}') \leq b_1$ it follows that $\|\mathbf{\Pi}_{\mathcal{M}^\perp} x\| \leq b_1$. Thus, if $x_0 := \mathbf{\Pi}_{\mathcal{M}} x \in \mathcal{M}$, then

$$\theta(x_0, x) \leq \arcsin b_1.$$

Similarly, since $\delta(\mathcal{N}, \mathcal{N}') \leq b_2$,

$$\theta(y_0, y) \leq \arcsin b_2,$$

where $y_0 := \mathbf{\Pi}_{\mathcal{N}} y \in \mathcal{N}$. It follows from (15) that

$$\begin{aligned} \arcsin b_1 + \arcsin b_2 + \arcsin \epsilon &> \theta(x_0, x) + \theta(y, y_0) + \theta(x, y) \\ &\geq \theta(x_0, y_0) \\ &\geq \theta_{\min}(\mathcal{M}, \mathcal{N}) = \arcsin b_{\mathcal{M},\mathcal{N}}. \end{aligned} \tag{31}$$

Since (31) holds for all ϵ we have

$$\arcsin b_1 + \arcsin b_2 \geq \arcsin b_{\mathcal{M},\mathcal{N}}.$$

Therefore, $b_1\sqrt{1-b_2^2} + b_2\sqrt{1-b_1^2} \geq b_{\mathcal{M},\mathcal{N}}$ and so (a) fails. In case (ii) we proceed similarly. Set $z_1 := \mathbf{\Pi}_{\mathcal{M}^\perp} z$ and $z_2 := \mathbf{\Pi}_{\mathcal{N}^\perp} z$. Since $b_1 \geq \delta(\mathcal{M},\mathcal{M}') = \delta(\mathcal{M}^\perp,\mathcal{M}'^\perp)$ we have $\theta(z,z_1) \leq \arcsin b_1$. Also, $b_2 \geq \delta(\mathcal{N},\mathcal{N}') = \delta(\mathcal{N}^\perp,\mathcal{N}'^\perp)$ implies that $\theta(z,z_2) \leq \arcsin b_2$. Thus

$$\begin{aligned}\arcsin b_1 + \arcsin b_2 &\geq \theta(z_1,z_2)\\ &\geq \theta_{\min}(\mathcal{M}^\perp,\mathcal{N}^\perp)\\ &= \arcsin b_{\mathcal{M}^\perp,\mathcal{N}^\perp} = \arcsin b_{\mathcal{M},\mathcal{N}}\end{aligned}$$

and (a) fails once again.

(b)⇒(a) Suppose (b) holds for some b_1 and b_2 satisfying

$$b_1\sqrt{1-b_2^2} + b_2\sqrt{1-b_1^2} \geq b_{\mathcal{M},\mathcal{N}}. \tag{32}$$

Similar reasoning to the proof of Theorem 3 shows that we may take strict inequality in (32). In particular, if $\|\mathbf{Q}_{\mathcal{M}',\mathcal{N}'}\| \leq c$ in (b), then $\|\mathbf{Q}_{\mathcal{M}'',\mathcal{N}'}\| \leq 2c$ for all

$$\mathcal{M}'' \in \bigcup_{\mathcal{M}' \in \overline{\mathrm{Ball}}_{\mathcal{U}}(\mathcal{M},b_1)} \mathrm{Ball}_{\mathcal{U}}\left(\mathcal{M}', \frac{1}{2c}\right)$$

and all $\mathcal{N}' \in \mathrm{Ball}_{\mathcal{Y}}(\mathcal{N},b_2)$. Theorem 1 then shows that statement (b) holds with b_1 replaced by some $b_1+\epsilon$ for $\epsilon > 0$. Since the left-hand side of (32) is monotonically increasing in b_1, it follows that (b) holds for some b_1 and b_2 satisfying (32) with strict inequality. Henceforth we will assume that this is the case. We also note from Theorem 3 that $b_1, b_2 < b_{\mathcal{M},\mathcal{N}}$.

We now show that there are subspaces $\mathcal{M}' \in \overline{\mathrm{Graph}_{\mathcal{U}}}$ and $\mathcal{N}' \in \overline{\mathrm{Graph}_{\mathcal{Y}}}$ with $\delta(\mathcal{M},\mathcal{M}') < b_1$ and $\delta(\mathcal{N},\mathcal{N}') < b_2$ such that $\mathbf{\Pi}_{\mathcal{N}'^\perp}|_{\mathcal{M}'}$ is not invertible.

As in the proof of Theorem 3, let $\mathbf{A} := \mathbf{A}_{\mathcal{M},\mathcal{N}}$, $\mathbf{E}_\lambda$ be the spectral family of $\mathbf{A}^*\mathbf{A}$, and $h \in \mathbf{E}_{\tau(\mathbf{A})+\epsilon}\mathcal{M}$ of unit norm, for some arbitrary $\epsilon > 0$. Then $\|\mathbf{A}h\| < \tau(\mathbf{A})+\epsilon = b_{\mathcal{M},\mathcal{N}}+\epsilon$. Define $p_\lambda = \lambda h + (1-\lambda)\mathbf{\Pi}_{\mathcal{N}} h$ and write $\mathcal{M}_- := \mathcal{M} \ominus \mathbb{C}h$, $\mathcal{N}_- := \mathcal{N} \ominus \mathbb{C}\Pi_{\mathcal{N}} h$, $\mathcal{M}_\lambda := \mathcal{M}_- + \mathbb{C}p_\lambda$, and $\mathcal{N}_\lambda := \mathcal{N}_- + \mathbb{C}p_\lambda$. We also write $q_\lambda := \mathbf{\Pi}_{\mathcal{M}_-^\perp} p_\lambda$ and $r_\lambda := \mathbf{\Pi}_{\mathcal{N}_-^\perp} p_\lambda$. We first show that

$$\delta(\mathcal{M},\mathcal{M}_\lambda) = \frac{(1-\lambda)\|\mathbf{A}h\|\sqrt{1-\|\mathbf{A}h\|^2}}{\sqrt{1-(1-\lambda^2)\|\mathbf{A}h\|^2}} + O(\epsilon) =: c_\lambda + O(\epsilon). \tag{33}$$

From Lemma 3 we know that

$$\delta(\mathcal{M},\mathcal{M}_\lambda) = \sqrt{1-|\langle h,\hat{q}_\lambda\rangle|^2}. \tag{34}$$

To evaluate (34) we must compute $\langle h,q_\lambda\rangle$ and $\|q_\lambda\|$. First,

$$\begin{aligned}\langle h,q_\lambda\rangle &= \langle h,p_\lambda\rangle\\ &= \lambda + (1-\lambda)\|p_0\|^2\\ &= \lambda + (1-\lambda)(1-\|\mathbf{A}h\|^2)\\ &= 1-(1-\lambda)\|\mathbf{A}h\|^2.\end{aligned} \tag{35}$$

Next, using (30), we have

$$\begin{aligned}\langle q_\lambda, q_\lambda\rangle &= \lambda^2 + 2\lambda(1-\lambda)\|p_0\|^2 + (1-\lambda)^2\|q_0\|^2\\ &\simeq \lambda^2 + \left(2\lambda(1-\lambda) + (1-\lambda)^2\right)\left(1-\|\mathbf{A}h\|^2\right)\\ &= 1-(1-\lambda^2)\|\mathbf{A}h\|^2, \qquad (36)\end{aligned}$$

where $\simeq$ denotes equality to $O(\epsilon)$. Equations (35) and (36) together show that

$$|\langle h, \hat{q}_\lambda\rangle|^2 = \frac{\left(1-(1-\lambda)\|\mathbf{A}h\|^2\right)^2}{1-(1-\lambda^2)\|\mathbf{A}h\|^2} + O(\epsilon)$$

from which (33) follows by simple manipulation. Next we show that

$$\delta(\mathcal{N}, \mathcal{N}_\lambda) = \frac{\lambda\|\mathbf{A}h\|}{\sqrt{1-(1-\lambda^2)\|\mathbf{A}h\|^2}} =: d_\lambda. \qquad (37)$$

From Lemma 3 we know that

$$\delta(\mathcal{N}, \mathcal{N}_\lambda) = \sqrt{1-|\langle \hat{p}_0, \hat{r}_\lambda\rangle|^2}. \qquad (38)$$

To evaluate (34) we need the following computations:

$$\begin{aligned}\langle p_0, r_\lambda\rangle &= \langle \mathbf{\Pi}_{\mathcal{N}}h, (\mathbf{\Pi}_{\mathcal{N}^\perp} + \mathbf{\Pi}_{\mathbb{C}p_0})p_\lambda\rangle\\ &= \langle \mathbf{\Pi}_{\mathcal{N}}h, h\rangle\\ &= 1-\|\mathbf{A}h\|^2 \qquad (39)\\ &= \|p_0\|^2 \qquad (40)\end{aligned}$$

and

$$\begin{aligned}\langle r_\lambda, r_\lambda\rangle &= \langle (\mathbf{\Pi}_{\mathcal{N}^\perp} + \mathbf{\Pi}_{\mathbb{C}p_0})p_\lambda, (\mathbf{\Pi}_{\mathcal{N}^\perp} + \mathbf{\Pi}_{\mathbb{C}p_0})p_\lambda\rangle\\ &= \|\mathbf{\Pi}_{\mathcal{N}^\perp}p_\lambda\|^2 + \|\mathbf{\Pi}_{\mathbb{C}p_0}p_\lambda\|^2\\ &= \lambda^2\|\mathbf{\Pi}_{\mathcal{N}^\perp}h\|^2 + \|\mathbf{\Pi}_{\mathcal{N}}h\|^2\\ &= 1-(1-\lambda^2)\|\mathbf{A}h\|^2. \qquad (41)\end{aligned}$$

Equations (39)–(41) together show that

$$|\langle \hat{p}_0, \hat{r}_\lambda\rangle|^2 = \frac{1-\|\mathbf{A}h\|^2}{1-(1-\lambda^2)\|\mathbf{A}h\|^2}$$

from which (37) follows by simple manipulation. Next we observe from (33) and (37) that $c_\lambda\sqrt{1-d_\lambda^2} + d_\lambda\sqrt{1-c_\lambda^2} = \|\mathbf{A}h\|$. Since c_λ is monotonically decreasing in λ on the interval $[0,1]$ we can choose λ such that $c_\lambda = b_1 - \epsilon$. Then for sufficiently small ϵ, we must have $d_\lambda < b_2$; otherwise, we have a contradiction to (32) with strict inequality. For the above choice of λ and sufficiently small ϵ we set $\mathcal{M}' = \mathcal{M}_\lambda$ and $\mathcal{N}' = \mathcal{N}_\lambda$ which gives $\delta(\mathcal{M}, \mathcal{M}') < b_1$ and $\delta(\mathcal{N}, \mathcal{N}') < b_2$. Lemma 3 shows that $\mathcal{M}' \in \overline{\mathrm{Graph}_{\mathcal{U}}}$ and $\mathcal{N}' \in \overline{\mathrm{Graph}_{\mathcal{Y}}}$. Also $\mathbf{\Pi}_{\mathcal{N}'^\perp}|_{\mathcal{M}'}$ is not invertible since $\mathcal{M}' \cap \mathcal{N}' \neq \{0\}$.

Now consider a sequence $\mathcal{M}_i' \in \mathrm{Graph}_{\mathcal{U}}$, $i = 1, 2, \ldots$, converging to $\mathcal{M}'$ and a sequence $\mathcal{N}_i' \in \mathrm{Graph}_{\mathcal{Y}}$, $i = 1, 2, \ldots$, converging to $\mathcal{N}'$. If there is a subsequence such that $\mathbf{\Pi}_{\mathcal{N}_i'^\perp}|_{\mathcal{M}_i'}$ is not invertible, then there is a contradiction. Otherwise, we can find a subsequence so that $\mathbf{\Pi}_{\mathcal{N}_i'^\perp}|_{\mathcal{M}_i'}$ is invertible. First observe that $\lim_{i\to\infty} \|\mathbf{\Pi}_{\mathcal{N}'^\perp}\mathbf{\Pi}_{\mathcal{M}'} -$

$\Pi_{\mathcal{N}_i'^\perp}\Pi_{\mathcal{M}_i'}\| = 0$. Since $\Pi_{\mathcal{N}'^\perp}|_{\mathcal{M}'}$ is not invertible, for any ϵ we can find an $x \in \mathcal{M}'$ of unit norm such that $\|\Pi_{\mathcal{N}'^\perp}x\| < \epsilon$. Thus $\|\Pi_{\mathcal{N}_i'^\perp}y_i\| < \epsilon$ for sufficiently large i, where $y_i := \Pi_{\mathcal{M}_i'}x$. Since $\mathcal{M}_i' \to \mathcal{M}'$ we also have $\|\Pi_{\mathcal{N}_i'^\perp}\hat{y}_i\| < \epsilon$ for sufficiently large i. This means that $\lim_{i\to\infty}\|(\Pi_{\mathcal{N}_i'^\perp}|_{\mathcal{M}_i'})^{-1}\| = \infty$. This violates the hypothesis. □

Remark. The necessity part of the proof of Theorem 4 requires a simultaneous perturbation of $\mathcal{M}$ and $\mathcal{N}$. The construction removes orthogonally one-dimensional subspaces from each of $\mathcal{M}$ and $\mathcal{N}$ that are at an angle $\theta_{\min}(\mathcal{M},\mathcal{N})$ to each other, and replaces them by a convex combination of these directions. The subspaces are each perturbed through the required minimal angles and together violate the direct sum property of Proposition 6(a).

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