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A TOPOLOGICAL APPROACH TO NEVANLINNA-PICK INTERPOLATION*

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Abstract. We study the set of rational solutions of an $(N+1)$-point Nevanlinna-Pick problem, that has degree bounded by N. Based on the invariance of the topological degree of a certain mapping under deformation, we establish that when the $(N+1)$-point Nevanlinna-Pick problem is solvable, then for any dissipation polynomial of degree N or less, there corresponds an interpolating function with dimension at most N. Our results provide a novel topological proof for the sufficiency of Pick's criterion for the solvability of the Nevanlinna-Pick problem, and also give a solution to an extended interpolation problem.

Key words. Nevanlinna-Pick interpolation, bounded degree solution, topological degree theory

AMS(MOS) subject classifications. Primary 46; secondary 30, 55, 93

1. Introduction. The Nevanlinna-Pick interpolation theory has a long history in mathematics. Its origin can be traced back to the beginning of the century in the work of Pick [20] and Nevanlinna [19] and it has reached a high degree of achievement in the recent work of Adamjan, Arov and Krein [1], Sarason [21], Sz.-Nagy and Foias [25] and Ball and Helton [3].

In engineering, it was in a circuit theoretical context where interpolation theory found the first applications (see Belevitch [4] and Wohlers [28]). In recent years, renewal of interest in the Nevanlinna-Pick interpolation problem has been motivated by a multitude of applications to system theoretic problems. These have been in the area of robust control, circuit theory, approximation theory, filtering, and stochastic processes (see Zames and Francis [30], Khargonekar and Tannenbaum [15], Helton [11], Genin and Kung [7], Dewilde, Vieira and Kailath [6] and Delsarte, Genin and Kamp [5]).

This paper addresses certain questions that carry a significant interest from an engineering standpoint.

It is known that whenever an $(N+1)$-point Nevanlinna-Pick problem is solvable, there exist rational solutions of degree at most N. Generically, the solution is nonunique. In this paper we present a study of the solutions of the $(N+1)$-point Nevanlinna-Pick problem that are at most of degree N. We show in Theorem 5.3 that for any dissipation polynomial (for a definition, see § 4) of degree at most N there exists a corresponding solution of degree at most N. This provides a description of the set of degree N solutions. We must point out that the degree of the interpolating function is related to the dimension of a controller in a feedback system, to the dimension of a modeling filter of a stochastic process, or to the McMillan degree of a certain transfer function in a circuit theoretic context. We show the above by exploiting the invariance of the topological degree of a certain mapping under deformation.

This approach also provides an independent topological proof of the sufficiency of Pick's criterion for the solvability of the Nevanlinna-Pick problem.

Our results are applied to tackle the solvability of an extended interpolation problem (see § 5) where, in addition to the $N+1$ interpolating conditions of the

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standard problem, we require that the real part of the function satisfy extra N interpolating conditions on the boundary of the "stability" region. These N interpolating conditions are interpreted as attenuation zeros of an associated transmittance function.

This work follows the lines of an investigation on the Carathéodory problem [8], and a preliminary version was reported in [9].

A variety of different terminologies has appeared in connection with the Nevanlinna-Pick problem. For instance, the reflectance of a passive system is known also as a bounded real function or as a Schur function, etc. We have chosen to use a rather mathematical terminology as it appears in the classical references (e.g. Akhiezer [2]), although occasionally we indicate the "translation" of the various terms in the circuit theoretic or stochastic terminology.

2. Notation and terminology.

$\mathbb{C} = \{\text{complex numbers}\}$.

$\mathbb{R} = \{\text{real numbers}\}$.

$D =$ open unit disc
$\quad = \{z \in \mathbb{C}: |z| < 1\}$.

X^c, X^0, ∂X indicate the closure, the interior and the boundary of a set X, respectively.

$H(D) = \{\text{functions holomorphic in } D\}$.

$C =$ class C (for Carathéodory)
$\quad = \{f(z) \in H(D): \operatorname{Re}\{f(z)\} \geqq 0 \text{ for all } z \text{ in } D\}$.

$S =$ class S (for Schur)
$\quad = \{f(z) \in H(D): |f(z)| \leqq 1 \text{ for all } z \text{ in } D\}$.

$z^* =$ complex conjugate of $z \in \mathbb{C}$.

If $a(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots \in H(D)$, then
$a(z)_* = a^*(z^{-1})$
$\quad = a_0^* + a_1^* z^{-1} + \cdots + a_n^* z^{-n} + \cdots$ is analytic in $\mathbb{C} - D^c$.

L^2: the space of squarely integrable functions on ∂D.

H^2: the space of L^2-functions that have analytic continuation in D.

3. Nevanlinna–Pick interpolation. Consider two sets of $N+1$ points in $\mathbb{C}$,

$$z = \{z_\kappa: z_\kappa \in D \text{ for } \kappa = 0, 1, \cdots, N\} \quad \text{and} \quad w = \{w_\kappa: w_\kappa \in \mathbb{C} \text{ for } \kappa = 0, 1, \cdots, N\}.$$

For simplicity we will always assume that the points z_κ are all distinct. The Nevanlinna-Pick problem can be stated as follows.

PROBLEM NP (z, w). Construct, if possible, a function $f(z) \in C$ that satisfies the interpolation conditions

$$f(z_\kappa) = w_\kappa \quad \text{for } \kappa = 0, 1, \cdots, N. \tag{3.1}$$

In particular,

(NP_1) find necessary and sufficient conditions on the data (z, w) for the existence of a solution $f(z)$, and

(NP_2) give a complete description of *the set $C(z, w)$ of all C-functions satisfying* (3.1).

The solvability criterion was derived by Pick and a constructive algorithm was provided by Nevanlinna—we now outline these. For a more detailed exposition see Walsh [27].

Pick criterion. There exists a function $f(z) \in C$ that satisfies (3.1) if and only if the Pick matrix

$$P(z, w) := \left[\frac{w_k + w_l^*}{1 - z_\kappa z_l^*}\right]_{\kappa, l=0}^{N}$$

is nonnegative definite.

A similar interpolation problem can be stated in terms of functions of class S instead of C. Both formulations are equivalent. However, the Nevanlinna recursive scheme is simpler to describe in terms of S-functions. Define

$$\zeta_\kappa(z) := \frac{z - z_\kappa}{1 - z z_\kappa^*} \quad \text{for } \kappa = 0, 1, \cdots, N.$$

Nevanlinna recursive scheme. A function $f(z) \in C$ satisfies the interpolation conditions (3.1) if and only if

$$\text{(3.2a)} \qquad \operatorname{Re} w_0 \geqq 0$$

and

$$\text{(3.2b)} \qquad s_1(z) := \zeta_0(z)^{-1} \frac{f(z) - w_0}{f(z) + w_0^*}$$

belongs to the class S and satisfies the interpolation conditions

$$\text{(3.3)} \qquad s_1(z_\kappa) = v_{1,\kappa} := \zeta_0(z_\kappa)^{-1} \frac{f(z_\kappa) - w_0}{f(z_\kappa) + w_0^*} \quad \text{for } \kappa = 1, 2, \cdots, N.$$

Furthermore, a function $s_l(z) \in S$ such that

$$s_l(z_\kappa) = v_{l,\kappa} \quad \text{for } \kappa = l, l+1, \cdots, N$$

exists if and only if either

$$\text{(3.4a)} \qquad |v_{l,l}| < 1 \quad \text{and} \quad s_{l+1}(z) = \zeta_l(z)^{-1} \frac{s_l(z) - v_{l,l}}{1 - v_{l,l}^* s_l(z)} \text{ belongs to } S,$$

or,

$$\text{(3.4b)} \qquad |v_{l,l}| = 1 \quad \text{and} \quad s_l(z) = v_{l,l} = v_{l,l+1} = \cdots = v_{l,N}.$$

In case (3.4a) holds, $s_{l+1}(z)$ satisfies

$$s_{l+1}(z_\kappa) = v_{l+1,\kappa} := \zeta_\kappa(z_\kappa)^{-1} \frac{v_{l,\kappa}(z_\kappa) - v_{l,l}}{1 - v_{l,l}^* v_{l,\kappa}(z_\kappa)} \quad \text{for } \kappa = l+1, \cdots, N.$$

(For a proof see Walsh [27].)

Notice that at each step of this procedure the number of interpolation conditions is reduced. Thus, it leads to a recursive solution of NP (z, w). This is summarized below.

PROPOSITION 3.5. *The NP problem is solvable if and only if* $\operatorname{Re} w_0 > 0$ *and either*

$$\text{(3.5a)} \qquad |v_{\kappa,\kappa}| < 1 \quad \textit{for } \kappa = 1, \cdots, N$$

or

$$\text{(3.5b)} \qquad |v_{\kappa,\kappa}| < 1 \quad \textit{for } \kappa = 1, \cdots, m-1 \quad \textit{and} \quad |v_{m,m}| = 1, \quad v_{m,m} = \cdots = v_{m,N}.$$

In the later case the solution is unique, whereas in the former case the general solution is obtained using

$$f(z)=j \operatorname{Im} w_0+\operatorname{Re} w_0 \frac{1+\zeta_0(z)s_1(z)}{1-\zeta_0(z)s_1(z)} \tag{3.6a}$$

and

$$s_l(z)=\frac{v_{l,l}+\zeta_l(z)s_{l+1}(z)}{1+v_{l,l}^*\zeta_l(z)s_{l+1}(z)} \quad \text{for } l=n, n-1, \cdots, 1 \tag{3.6b}$$

from $v_{l,l}$, $l=1, \cdots, n$ *and an arbitrary* $s_{l+1}(z)\in S$.

We are interested in the "indeterminate" case when there is more than one solution. A necessary and sufficient condition for the problem to be indeterminate is (3.5a). This condition is equivalent to the positive definiteness of the associated Pick matrix. Hence, from now on, we will assume that *P is positive definite.*

In the indeterminate case, one particular solution of the NP problem is obtained by setting $s_{n+1}(z)=0$ in (3.6). We state some related facts in the following proposition.

PROPOSITION 3.7. *Let the Pick matrix P be positive definite and let* $f_0(z)$ *denote the solution of the NP problem that is obtained by setting* $s_{n+1}(z)\equiv 0$. *Then,* (a) $f_0(z)$ *is a rational function, and* (b) *if* $f_0(z)=\pi_0(z)/\chi_0(z)$ *with* $\pi_0(z)$ *and* $\chi_0(z)$ *coprime polynomials in z, then* $\max\{\deg \pi_0(z), \deg \chi_0(z)\}\leqq n$ *and* $\chi_0(z)\neq 0$ *for all* $z\in D^c$.

Proof of Proposition 3.7. From (3.6) it is easy to see that $f_0(z)$ is a rational function of degree less than or equal to n. (The *degree of a rational function* $\pi_0(z)/\chi_0(z)$ is defined to be the maximum of $\{\deg \pi_0(z), \deg \chi_0(z)\}$ where $\pi_0(z)$, $\chi_0(z)$ are polynomials in z.) Also using (3.6), one can derive that

$$\pi_0(z)\chi_0(z)_*+\chi_0(z)\pi_0(z)_*=k\prod_{\kappa=0}^{n-1}(z-z_\kappa)(z^{-1}-z_\kappa^*)$$

for some scalar $k>0$. Now, since $|z_\kappa|<1$ for all κ, $\chi_0(z)$ (and for that matter $\pi_0(z)$ also) cannot have a root on ∂D, otherwise, $\chi_{0*}(z)$ would have a root at the same point. This cannot happen because the right-hand side of the above has no root on ∂D. Finally, that $\chi_0(z)$ has no root outside D^c is a consequence of the fact that $f_0(z)$ is a C-function (Proposition 3.5). Q.E.D.

4. Rational C-functions. A well-known characterization of rational C-functions is given below (see Siljak [24]).

PROPOSITION 4.1. *Let* $\pi(z)$, $\chi(z)$ *be coprime polynomials in z. The rational function* $\pi(z)/\chi(z)$ *belongs to C if and only if*

$$\pi(z)+\chi(z)\neq 0 \quad \text{for all } z\in D^c \tag{4.1a}$$

and

$$d(z,z^{-1}):=\pi(z)\chi(z)_*+\chi(z)\pi(z)_*\geqq 0 \quad \text{for all } z\in\partial D. \tag{4.1b}$$

A polynomial $d(z,z^{-1})\in\mathbb{C}[z,z^{-1}]$ that satisfies (4.1b) will be called a *dissipation polynomial* (following Kalman [12]). The degree of the highest power of z will be called the *degree of* $d(z,z^{-1})$. (A necessary condition for $d(z,z^{-1})$ to be a dissipation polynomial is that $d(z,z^{-1})_*=d(z,z^{-1})$. Hence $\deg d(z,z^{-1})$ is also equal to the highest power of z^{-1}.)

Allowing the polynomials $\pi(z)$, $\chi(z)$ to have common factors, condition (4.1a) can be somewhat relaxed. The following modification of (4.1) will be utilized in the sequel.

PROPOSITION 4.2. *Let $\pi(z)$, $\chi(z)$ be polynomials in z (not necessarily coprime). If*

(4.2a) $$\pi(z)+\chi(z)\neq 0 \quad \textit{for all } z\in D \textit{ (and not necessarily in } D^c\textit{)}$$

and

(4.2b) $$d(z, z^{-1})\geqq 0 \quad \textit{for all } z\in \partial D,$$

then $\pi(z)/\chi(z)$ is a C-function.

Proof. From

$$|\pi(z)+\chi(z)|^2-|\pi(z)-\chi(z)|^2=2d(z, z^{-1})\geqq 0 \quad \text{for } z\in\partial D,$$

it follows that any root of $\pi(z)+\chi(z)$ on ∂D is also a root of $\pi(z)-\chi(z)$ and hence, of both $\pi(z)$ and $\chi(z)$. After extracting all common roots of $\pi(z)$ and $\chi(z)$ that lie on ∂D, we are left with a pair of polynomials $(\tilde{\pi}(z), \tilde{\chi}(z))$ that satisfy (4.1a), (4.1b) and also $\pi(z)/\chi(z)=\tilde{\pi}(z)/\tilde{\chi}(z)$. Therefore, $\pi(z)/\chi(z)$ is in C. Q.E.D.

5. Interpolation with rational *C*-functions of degree *N*: Main results.

PROBLEM 5.1: $I(z, w, \xi)$. Let (z, w) be a set of $N+1$ interpolating conditions, and let $\xi=\{\xi_\kappa: \xi_\kappa\in\partial D, \kappa=1,2,\cdots,N\}$. Find necessary and sufficient conditions for the existence of a rational function $f(z)=\pi(z)/\chi(z)\in C$ that satisfies the interpolation conditions

(5.1a) $$f(z_\kappa)=w_\kappa \quad \text{for } \kappa=0,1,\cdots,N,$$

and also

(5.1b) $$\pi(\xi_\kappa)\chi(\xi_\kappa)_*+\chi(\xi_\kappa)\pi(\xi_\kappa)_*=0 \quad \text{for } \kappa=1,\cdots,N.$$

Note that in case $\pi(z)$, $\chi(z)$ have no common factor, conditions (5.1b) can be written as

(5.1c) $$\operatorname{Re} f(\xi_\kappa)=0 \quad \text{for } \kappa=1,\cdots,N$$

(which represent Löwner-type interpolation conditions). In a circuit theoretic context the points $\xi_\kappa\in\partial D$ correspond to attenuation zeros for the corresponding Schur-bounded real transmittance function $s(z)$. That is, if $s(z)=1/z[f(z)-f(0)]/\ [f(z)+f(0)^*]$, then (5.1c) implies that $|s(\xi_\kappa)|=1$ and hence the attenuation $\log|s(\xi_\kappa)|=0$ for $\kappa=1,2,\cdots,N$.

Although we have two sets of interpolation conditions it turns out that the solvability depends again on the positive definiteness of the Pick matrix.

THEOREM 5.2. *Problem $I(z, w, \xi)$ is solvable if and only if the Pick matrix associated with (z, w) is positive definite. Moreover, in this case, there always exists a solution of degree less than or equal to N.*

Theorem 5.2 is a direct corollary of the following more general one.

MAIN THEOREM 5.3. *Let (z, w) be a set of $N+1$ interpolating conditions such that the associated Pick matrix is positive definite, and also let $d(z, z^{-1})$ be an arbitrary dissipation polynomial of degree at most N. Then, there exists a pair of polynomials $(\pi(z), \chi(z))$ such that*

(5.3a) $$f(z)=\frac{\pi(z)}{\chi(z)}\in C \textit{ and satisfies } f(z_\kappa)=w_\kappa \textit{ for } \kappa=0,1,\cdots,N,$$

(5.3b) $$\pi(z)\chi(z)_*+\chi(z)\pi(z)_*=k\,d(z, z^{-1}) \quad \textit{for some } k>0,$$

(5.3c) $$\deg f(z)\leqq N.$$

The proof of the above makes use of Topological Degree Theory (see § 6) and thus provides a novel approach to establish the sufficiency of Pick's criterion for the solvability of the NP problem which follows.

COROLLARY. *If the Pick matrix $P(z, w)$ is positive definite, then the* NP (z, w) *problem is solvable.*

Proof. This is a direct consequence of Theorem 5.3. Q.E.D.

Naturally, one would like to know whether NP (z, w) (or $I(z, w, \xi)$) has a solution of degree strictly less than N and for that matter, to determine the minimal degree (see Youla and Saito [29] and Kalman [13]). Unfortunately, this question seems to be tractable only by methods of decision theory. In fact, the problem of finding the minimal degree of a rational solution $f(z)$ when NP (z, w) is solvable, is a decidable one. The reason for that is that both the set of interpolation conditions and the conditions that guarantee that $f(z)$ belongs to C (see Proposition 4.1 and also Siljak [24]) can be phrased in terms of the solvability of a finite set of equations that depend polynomially on the coefficients of $f(z)$. For the existence of a solution the theory of Tarski [26] and Seidenberg [23] can be used. However, using the tools developed for the proof of Theorem 5.2 we obtain the following.

PROPOSITION 5.4. *The set of $N+1$-pairs (z, w) for which NP (z, w) is solvable but has no solution of degree strictly less than N, is open and nonempty for all N.*

Below we demonstrate the implications of our results to a particular case.

Example 5.5. Consider the problem NP (z, w) where $z=\{0, \frac{1}{2}\}$ and $w=\{1, 2\}$. The associated Pick matrix

$$P=\begin{pmatrix} 2 & 3 \\ 3 & 16/3 \end{pmatrix}$$

is positive definite. Consequently, the NP (z, w) is solvable. The general solution is given by

$$f(z)=\frac{1+zs_1(z)}{1-zs_1(z)}$$

where $s_1(z)$ is an S-function that satisfies

$$s_1\left(\frac{1}{2}\right)=-\frac{2}{3}$$

and a general expression for it is given by (3.6b).

Let us restrict our attention to $f(z)$ of degree 1 or less. A rational function $f(z)$ that meets the interpolation data (z, w) is given by

$$f(z)=\frac{1+\alpha z}{1+\beta z} \quad \text{where } \alpha=2(1+\beta). \tag{5.6}$$

In order for $f(z)$ to belong to C it is necessary and sufficient that

$$|\alpha+\beta|+|\alpha-\beta|\leqq 2. \tag{5.7}$$

This follows easily from Proposition 4.1.

Let us now consider solutions of $I(z, w, \xi)$ for various points $\xi=\exp\{j\theta\}\in\partial D$. It can easily be verified that for all $\xi\neq+1$ there exists a degree 1 function $f(z)\in C$ as above, such that

$$\operatorname{Re} f(\xi)=0. \tag{5.8}$$

For instance, if $\xi=-1$, then the required $f(z)$ is

$$f(z)=\frac{1+z}{1-\frac{1}{2}z}.$$

(For all other choices of ξ, $f(z)$ has complex coefficients.) It is straightforward to verify (5.6)-(5.8). However, for $\xi=+1$, the function $f(z)$ sought in Theorem 5.1 is

$$f(z)=\frac{1}{1-z}.$$

This satisfies (5.6), (5.7) and also (5.1b). But $f(z)$ has a pole at $\xi=+1$ and, in this case,

$$\lim_{z\to\xi} \operatorname{Re} f(z)=\tfrac{1}{2}\neq 0.$$

In the general case, similar situations occur with probability zero. In other words, with generic data (z, w, ξ), the solutions to $I(z, w, \xi)$ have no poles on ∂D and in this case (5.1c) is equivalent to (5.1b).

6. Proof of the main results. First we recall certain tools of the geometric-functional theoretic approach of Sarason [21] to the interpolation problem.

Let $B(z)$ denote the finite Blaschke product (all-pass function) with simple zeros at z_κ, $\kappa=0, 1, \cdots, N$; i.e.,

$$B(z)=\prod_{\kappa=0}^{N}\frac{z-z_\kappa}{1-z_\kappa^* z}\cdot\frac{|z_\kappa|}{z_\kappa}$$

(where $|z_\kappa|/z_\kappa$ is replaced by 1 when $z_\kappa=0$). Let K denote the subspace of H^2

$$K:=H^2\ominus B(z)H^2.$$

The orthogonal projection in L^2 with range K is denoted by $[\ ]_K$, whereas $\langle\ ,\ \rangle$ denotes the inner product in L^2.

K is an $(N+1)$-dimensional vector space and a commonly used basis for K is

$$B=\left\{g_\kappa(z)=\frac{1}{1-z_\kappa^* z},\ \kappa=0, 1, \cdots, N\right\}.$$

Note that for all $q(z)$ in H^2, $\langle q(z), g_\kappa(z)\rangle=q(z_\kappa)$.

Any element $q(z)$ in K can be represented as the ratio

$$q(z)=\frac{\chi(z)}{r(z)},$$

where $\chi(z)$ is a polynomial of degree N and

$$r(z):=\prod_{\kappa=0}^{N}(1-z_\kappa^* z).$$

Let $C(z, w)$ *be the set of solutions of* NP (z, w). The Pick matrix is assumed to be positive definite. Consequently, by Proposition 3.7, there exists a solution $f(z)$ in $C(z, w)$ that also belongs to $H(D^c)$. Define the linear operator

$$T: K\to K: q(z)\to[f(z)q(z)]_K.$$

It turns out that T depends only on the interpolation data (z, w) and not on the particular solution $f(z)$ in $C(z, w)$. Moreover, as it will become clear below, T can be defined directly on the basis of (z, w) (and does not require the solvability of NP (z, w)).

LEMMA 6.1. *Let* $p(z)=[f(z)q(z)]_K$, *where* $q(z)\in K$, *and* $f(z)\in C(z,w)\cap H(D^c)$ *as above. Then* $p(z)$ *is independent of the particular* $f(z)$, *it depends only on* (z,w) *and it satisfies*

$$\frac{p(z_\kappa)}{q(z_\kappa)}=w_\kappa, \qquad \kappa=0,1,\cdots,N.$$

Proof. Let $q(z)=\sum_{\kappa=0}^N b_\kappa g_\kappa(z)$ and $p(z)=\sum_{\kappa=0}^N a_\kappa g_\kappa(z)$. Then

$$\begin{aligned}p(z_\kappa)&=\langle p(z),g_\kappa(z)\rangle=\langle [f(z)q(z)]_K,g_\kappa(z)\rangle\\&=w_\kappa q(z_\kappa).\end{aligned}$$

This is a set of $N+1$ equations in the $N+1$ coefficients of $p(z)$:

$$G\begin{bmatrix}a_0\\a_1\\\vdots\\a_N\end{bmatrix}=\begin{bmatrix}w_0q(z_0)\\w_1q(z_1)\\\vdots\\w_Nq(z_N)\end{bmatrix}=\begin{bmatrix}w_0&&&\\&w_1&&\\&&\ddots&\\&&&w_N\end{bmatrix}G\begin{bmatrix}b_0\\b_1\\\vdots\\b_N\end{bmatrix}$$

where G is the Gram matrix

$$G=[\langle g_k(z),g_l(z)\rangle]_{\kappa,l=D}^N=\left[\frac{1}{1-z_\kappa z_l^*}\right]_{\kappa,l=0}^N.$$

Since the $g_\kappa(z)$, $\kappa=0,1,\cdots,N$ are linearly independent, G is nonsingular. (This can also be shown directly by computing the determinant of G. G is related in a simple way to the so-called Hilbert matrix and a formula for the determinant of a Hilbert matrix can be found in Knuth [16, p. 36].) Thus, T is the linear transformation specified by $p(z)=Tq(z)$ where

$$\begin{bmatrix}a_0\\a_1\\\vdots\\a_N\end{bmatrix}=G^{-1}\begin{bmatrix}w_0&&&\\&w_1&&\\&&\ddots&\\&&&w_N\end{bmatrix}G\begin{bmatrix}b_0\\b_1\\\vdots\\b_N\end{bmatrix}. \tag{6.1a}$$

Q.E.D.

PROPOSITION 6.2 (Sarason [21]). *The Pick matrix P is the real part of T.*

Proof. Let $q(z)=\sum_{\kappa=0}^N b_\kappa g_\kappa(z)$. Then,

$$\begin{aligned}2\,\mathrm{Re}\,\langle [f(z)q(z)]_K,q(z)\rangle&=\langle [f(z)q(z)]_K,q(z)\rangle+\langle q(z),[f(z)q(z)]_K\rangle\\&=\sum_{\kappa,l=0}^N b_l\frac{w_\kappa+w_l}{1-z_\kappa z_l^*}b_\kappa.\end{aligned}\tag{6.1b}$$

Q.E.D.

Let $f(z)\in C(z,w)\cap H(D^c)$ as before. Define the following linear map:

$$\Psi: K\to K: q(z)\to u(z)=[(1+f(z))q(z)]_K.$$

Since $f(z)\in C$, then $1+f(z)$ has an inverse in $H(D^c)$ and Ψ is invertible. Define

$$\psi:=\Psi^{-1}:K\to K:u(z)\to q(z)=[(1+f(z))^{-1}q(z)]_K.$$

Note that ψ (and also Ψ) depends only on (z,w) and not on our choice of $f(z)\in C(z,w)$. This can be readily established as in Lemma 6.1 and, in point of fact, if $u(z)=\Sigma u_\kappa g_\kappa$, then $\psi[u(z)]=q(z)=\Sigma b_\kappa q_\kappa$ where

$$\begin{bmatrix}b_0\\b_1\\\vdots\\b_N\end{bmatrix}=G^{-1}\begin{bmatrix}1/(1+w_0)&&&\\&1/(1+w_1)&&\\&&\ddots&\\&&&1/(1+w_N)\end{bmatrix}G\begin{bmatrix}u_0\\u_1\\\vdots\\u_N\end{bmatrix}. \tag{6.2a}$$

We also note that if $p(z)=T[q(z)]$ then $u(z)=p(z)+q(z)$. Using these facts we proceed to the proof of Theorem 5.3.

Proof of Theorem 5.3. Given $u(z)\in K$ we can readily obtain a pair $(p(z), q(z))\in K\times K$, by $q(z)=\psi[u(z)]$ and $p(z)=T[q(z)]$ using (6.1a) and (6.2a), such that

$$\frac{p(z)}{q(z)} \text{ is a rational function of degree at most } n,$$

$$\frac{p(z_i)}{q(z_i)}=w_i \qquad \text{(Lemma 6.1)}, \tag{6.2b}$$

$$p(z)+q(z)=u(z).$$

Let $q(z)=\chi(z)/r(z)$ and $p(z)=\pi(z)/r(z)$ where $\pi(z)$, $\chi(z)$ are polynomials of degree n and $r(z)$ is as earlier. Consider the function

$$\begin{aligned} e(z) &:= p(z)q(z)_* + q(z)p(z)_* \\ &= \frac{\pi(z)\chi(z)_* + \chi(z)\pi(z)_*}{r(z)r(z)_*} \\ &= d(z, z^{-1})\rho(z) \quad \text{with } z\in\partial D \end{aligned}$$

where $\rho(z)=(r(z), r(z)_*)^{-1}$ ($e(z)$ can be considered as an L^1-function).

In order for $p(z)/q(z)=\pi(z)/\chi(z)$ to be a C-function it is sufficient (by Proposition 4.2) that

$$u(z)\neq 0 \quad \text{for all } z\in D^c \tag{6.3a}$$

and

$$d(z, z^{-1})\geqq 0 \quad \text{for all } z\in\partial D. \tag{6.3b}$$

We will establish the theorem by studying the correspondence

$$u(z)\to d(z, z^{-1}),$$

and showing that the image of $\{u(z)\colon u(z)\in K$, such that (6.3a) holds$\}$ contains the set of all polynomials $d(z, z^{-1})$ of degree at most n that satisfy (6.3b) (and are properly normalized). We now proceed to consider a normalization of $u(z)$ and $d(z, z^{-1})$ so that their correspondence becomes a continuous map between smooth manifolds.

Both $e(z)$ and $\rho(z)$ can be easily seen to be in L^2. Let e_k (resp., ρ_k) with $\kappa\in Z$ denote the Fourier coefficients of $e(z)$ (resp., $\rho(z)$). The polynomial $d(z, z^{-1})$ is of degree N (in both z and z^{-1}) and it holds that

$$e_0=\sum_{\kappa=-N}^{N} d_\kappa\rho_{-\kappa}.$$

On the other hand, using Proposition 6.2, we have that if $q(z)=\sum_{\kappa=0}^{N} b_\kappa g_\kappa(z)$ then

$$e_0=(b_0^*\cdots b_N^*)P\begin{bmatrix} b_0\\ \vdots \\ b_N\end{bmatrix}.$$

Since P is positive definite, $e_0\neq 0$ (unless $q(z)\equiv 0$).

We now define the sets:

$$Y := \left\{ \tilde{d}(z, z^{-1}) = \sum_{\kappa=-N}^{N} \tilde{d}_\kappa z^\kappa \text{ such that } \sum_{k=-N}^{N} \tilde{d}_\kappa \rho_{-\kappa} = 1 \right\},$$

$$Y_+ := \{\tilde{d}(z, z^{-1}) \in Y \text{ such that } \tilde{d}(z, z^{-1}) \geqq 0 \text{ for all } z \in \partial D\},$$

$$X := \{u(z) \in K \text{ such that } u(0) = 1\},$$

$$X_+ := \{u(z) \in X \text{ such that } u(z) \neq 0 \text{ for all } z \in D^c\},$$

$$W := \{\text{rational functions } \pi(z)/\chi(z) \text{ of dimension at most } N \text{ that satisfy } \pi(z_k)/\chi(z_\kappa) = w_\kappa \text{ for } \kappa = 0, 1, \cdots, N\}.$$

Also we consider the following mappings:

$$\theta : K - \{0\} \to Y : q(z) \to \tilde{d}(z, z^{-1}) = \frac{1}{e_0} d(z, z^{-1}),$$

where e_0, $d(z, z^{-1})$ and $p(z)$ are computed from $q(z)$ as before, and

$$\omega := \theta \circ \psi|_X : X \to Y : u(z) \to \tilde{d}(z, z^{-1}).$$

Both mappings are completely specified by the interpolation data, and since $0 \in \chi(X)$, it is easy to see that ω is a continuous map. Now, X and Y are (smooth) linear manifolds of real dimension $2N$, and X_1 and Y_1 are compact subsets of X and Y respectively. On the other hand the mapping

$$X \to W : u(z) \to \frac{p(z)}{q(z)} = \frac{\pi(z)}{\chi(z)}$$

where $q(z) = \psi(u(z))$ and $p(z) = T(q(z))$, is clearly surjective. Hence, in order to establish the theorem we only need to show that

$$\omega(X_+^c) \supseteq Y_+.$$

To show this we will exploit the dependence of ω on the interpolation data w.

Consider z being fixed and define the set of w that render $P(z, w)$ positive definite:

$$B := \{w \in \mathbb{C}^{N+1} \text{ such that } P(z, w) \text{ is positive definite}\}.$$

We want to establish that B is a pathwise connected set. This follows immediately from the continuous dependence of w on the parameter $v_{\kappa\kappa}$, $\kappa = 1, 2, \cdots, N$ and the fact that the positive definiteness of P is equivalent to the conditions

$$\operatorname{Re} w_0 > 0 \quad \text{and} \quad |v_{\kappa\kappa}| < 1 \quad \text{for } \kappa = 1, 2, \cdots, N.$$

Now, provided the Pick matrix is positive definite (nonsingular would suffice), ω is a continuous map. Also, ω depends continuously on the parameters w. We shall indicate this by writing ω_w.

Since B is pathwise connected, we can construct a (continuous) homotopy H from $\omega_{w_{in}}$ to ω_w; by following a continuous path from an initial $w_{in} := \{w_\kappa = 1, \kappa = 0, 1, \cdots, N\}$ to any other point w in B, i.e.,

$$H : X \times [0, 1] \to Y,$$

such that $H(u(z), 0) = \omega_{w_{in}}(u(z))$ and $H(u(z), 1) = \omega_w(u(z))$.

We now proceed as follows: we first show that

$$H(X_+^c, t) \supseteq Y_+,$$

for $t=0$, and then that the same property holds for all $t \in [0,1]$; i.e., it remains invariant under the homotopy.

We first note that $f_0(z) \equiv 1 \in C(z, w_{in})$. Then, the map $H(\cdot, 0) = \omega_{w_{in}}(\cdot)$ assumes a simple form where

$$q(z) = \tfrac{1}{2}u(z) = p(z).$$

If $q(z) = \chi(z)/r(z)$ as before,

$$\omega_{w_{in}} : X \to Y : \frac{2\chi(z)}{r(z)} \to k\chi(z)\chi(z)_*,$$

where k is a positive scalar making $k\chi(z)\chi(z)_*$ an element of Y. By the Riesz–Fejer Theorem ([10, p. 21]) any element of Y_+ assumes a unique representation

$$\tilde{d}(z, z^{-1}) = k\chi(z)\chi(z)_*,$$

with $\chi(z)$ a polynomial in z, devoid of zeros in D.

From the above it readily follows that

$$\omega_{w_{in}}(X_+^c) = Y_+.$$

Moreover, the correspondence

$$\omega_{w_{in}}|_{X_+^c} : X_+^c \to Y_c \quad \text{is bijective.} \tag{6.4}$$

Now let $d(X_+, \omega, \tilde{d})$ denote the *topological degree of the map* ω *at* $\tilde{d}$ *relative to the set* X_+. The topological degree is a "measure" of the number of preimages in X_+ of the point $\tilde{d}$ under the mapping ω. In particular (6.4) implies that

$$d(X_+, H(\cdot, 0), \tilde{d}) = 1 \quad \text{for all } \tilde{d} \in Y_+^0, \tag{6.5}$$

where Y_+^0 indicates the interior of Y_+. For a comprehensive exposition of various aspects of degree theory see Lloyd [17] and Milnor [18].

We now show that

$$d(X_+, H(\cdot, t), \tilde{d}) = 1 \quad \text{for all } \tilde{d} \in Y_+^0 \text{ and } t \in [0,1]. \tag{6.6}$$

This follows from a very powerful theorem on the invariance of the degree under homotopy (Lloyd [17, p. 23]) after we prove that the image of the boundary of X_+ never intersects the interior of Y_+:

$$H(\partial X_+, t) \cap Y_+^0 = \varnothing \quad \text{for all } t \in [0,1] \tag{6.7}$$

(see also Lloyd [17, p. 32], Milnor [18] and Schwartz [22] for the case of continuous deformations of maps between smooth manifolds).

We now prove (6.7). Assume that the above intersection was not empty and let $\tilde{d}(z, z^{-1}) = H(u(z), t) \in Y_+^0$, where $u(z) \in \partial X_+$, and $t \in [0,1]$. Hence, $u(a) = p(a) + q(a) = 0$ for some value $z = a \in \partial D$. Also

$$\begin{aligned}\tilde{d}(z, z^{-1}) &= k[p(z)q(z)_* + q(z)p(z)_*] \\ &= \frac{k}{2}[|p(z)+q(z)|^2 - |p(z)-q(z)|^2] \geqq 0,\end{aligned}$$

for all $z \in \partial D$, while k is a positive scalar. Hence,

$$d(a, a^{-1}) = 0,$$

and $\tilde{d}(z, z^{-1})$ is not in the interior of Y_+. This is a contradiction.

Consequently, (6.7) is valid. Then, (6.6) follows from (6.5) and (6.7). Finally, (6.6) implies that

$$Y_+^0 \subseteq H(X_+, 1) = \omega_w(X_+),$$

which in turn, due to the compactness of X_+^c and the continuity of $\omega_w(\cdot)$, implies that

$$Y_+ \subseteq \omega_w(X_+^c).$$

This establishes the theorem. Q.E.D.

Proof of Theorem 5.2. From the Pick criterion it follows that $P(z, w)$ being nonnegative definite is a necessary condition for $I(z, w, \xi)$ to be solvable. To show the sufficiency part, let

$$d(z, z^{-1}) = \prod_{\kappa=1}^{N} (z - \xi_\kappa)(z^{-1} - \xi_\kappa^*)$$

and apply Theorem 5.3. Q.E.D.

Proof of Proposition 5.4. Consider the interpolation data (z, w) where

$$w_0 = 1 + a, \quad a \in \mathbb{C} \quad \text{and} \quad w_\kappa = 1 \quad \text{for } \kappa = 1, 2, \cdots, N.$$

The Pick matrix depends continuously on the parameters w and, consequently, it also depends continuously on the parameter a. For $a = 0$ the Pick matrix is positive definite. Hence, for $a \neq 0$ but is sufficiently small, the Pick matrix is still positive definite and NP(z, w) admits a solution. However, there is no rational function of degree *strictly* less than N that interpolates (z, w) unless $a = 0$. To see this, assume that such a function $f(z)$ exists, and let $f(z) = \pi(z)/\chi(z)$, where $\pi(z)$, $\chi(z)$ are polynomials of degree less than N. Then,

$$f(z) - 1 = \frac{\pi(z) - \chi(z)}{\chi(z)} = 0$$

for N different values of z. Therefore, $\pi(z) - \chi(z)$, being of degree at most $N - 1$, is identically zero. Hence, $f(z) \equiv 1$, which is a contradiction. This establishes the proposition. Q.E.D.

7. Concluding remarks. Theorems 5.2 and 5.3 provide existence-type results to an inherently nonlinear problem. However, the homotopy used in the derivation can be used to provide an algorithmic way to find the sought solutions. For a study of homotopy methods as they relate to deriving numerical algorithms see the work of Kellogg, Li and Yorke [14].

In this paper we have presented a description of the set of interpolating functions of degree N to the $(N+1)$-point NP problem. However, it is not known at the moment whether the correspondence in Theorem 5.3 represents in fact a parametrization of this set; i.e., whether the correspondence between interpolating functions of degree N and dissipation polynomials as in Theorem 5.3 is in fact bijective.

Finally, a simple criterion to determine whether there exists an $f(z)$ in $C(z, w)$ of degree strictly less than N is still lacking. Such a criterion seems necessary for a thorough understanding of minimal degree solutions to NP(z, w) as considered in Youla and Saito [29] and Kalman [13].

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