MOMENT PROBLEMS AND RELATIVE ENTROPY

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- System modeling
- Spectral analysis of time-series
- Radar, tomographic techniques
- Ultrasound spectroscopy/sensing, earth sciences
- Acoustic microscopy, echolocation
- Reflectivity in thin films, deposition

- Systems viewpoint:

$$\text{Data} \Rightarrow \{ \text{ Family of consistent spectra/models } \}$$

- ◦ particular elements $\Rightarrow$ algorithms
- ◦ size of family quantifies uncertainty
- ◦ integration of data from various sensors
- ◦ incorporation of prior information
- ◦ optimization of data collection techniques
- ◦ high resolution

- Basic “inverse problem”:

  data/statistics $\Rightarrow$ consistent models/distributions/
  power spectra/etc.

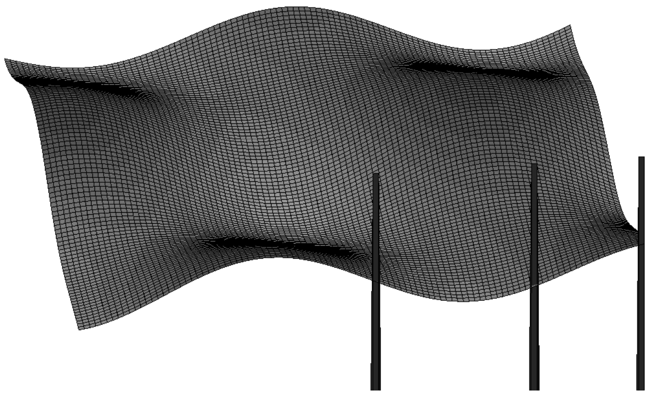
- Statistical ensemble averaging: $\boxed{r = \sum_k g(k)\rho(k)}$

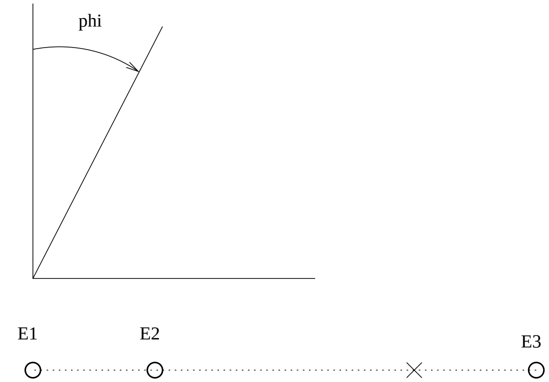
  Determine $\rho(k)$ based on r

- Quantum measurements: $\boxed{\rho^A = \mathrm{trace}_B(\rho^{AB}) = \sum_{k=1}^{2} G_k \rho^{AB} G_k^*}$

  Determine ρ^{AB} based on ρ^A,

  e.g., $\rho^{AB} = \left(\frac{|01\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right)$ and $\rho^A = \frac{1}{2}(|1\rangle\langle 1| + |0\rangle\langle 0|)$.





Sensor readings: $u_\ell(t) = \int A(\theta) e^{j(\omega t - p x_\ell \cos(\theta) + \phi(\theta))} d\theta$

Correlations: $\boxed{R_k = E\{u_{\ell_1} \bar{u}_{\ell_2}\} := \int \overbrace{e^{-jk\cos(\theta)}}^{g_k(\theta)} \overbrace{\rho(\theta) d\theta}^{d\mu}}$ with $\rho(\theta) = A(\theta)^2$,

$$k \in \{0, 1, \sqrt{2}, \sqrt{2} + 1\}.$$

Given R_0, R_1, $R_{\sqrt{2}}$, $R_{\sqrt{2}+1}$

(i) how can we tell they originate as above with $\rho > 0$?

(ii) how can we recover ρ?

(iii) how can we parametrize all admissible ρ's?

$$\int \left(\begin{bmatrix} 1 \\ e^{-j\tau} \\ e^{-j(\sqrt{2}+1)\tau} \end{bmatrix} \overbrace{\rho(\theta) d\theta}^{d\mu} \begin{bmatrix} 1 & e^{j\tau} & e^{j(\sqrt{2}+1)\tau} \end{bmatrix} \right) = \begin{bmatrix} R_0 & R_1 & R_{\sqrt{2}+1} \\ \bar{R}_1 & R_0 & R_{\sqrt{2}} \\ \bar{R}_{\sqrt{2}+1} & \bar{R}_{\sqrt{2}} & R_0 \end{bmatrix} \geq 0$$

necessary but not sufficient

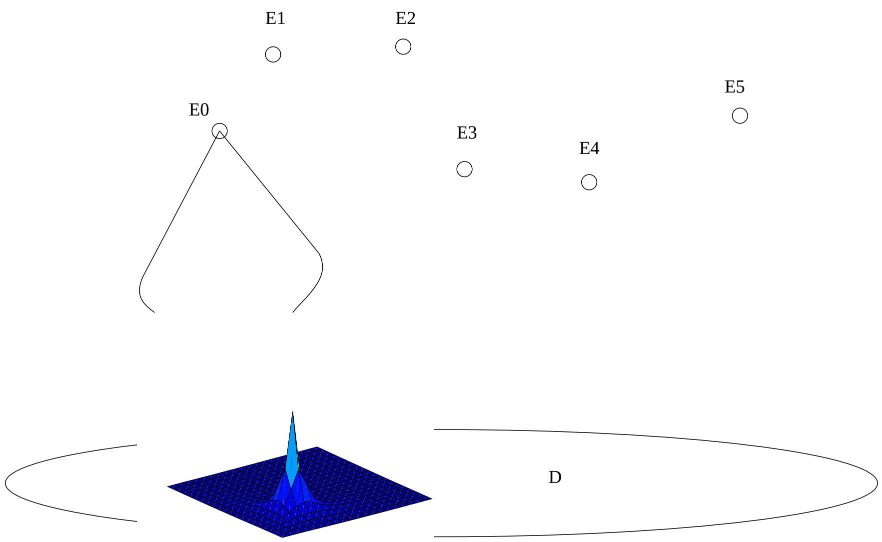
Given R_0, R_1, R_{100}, with $R_2, \ldots R_{99}$ missing,
$\exists$? values for the missing R's so that $T_{100} > 0$?

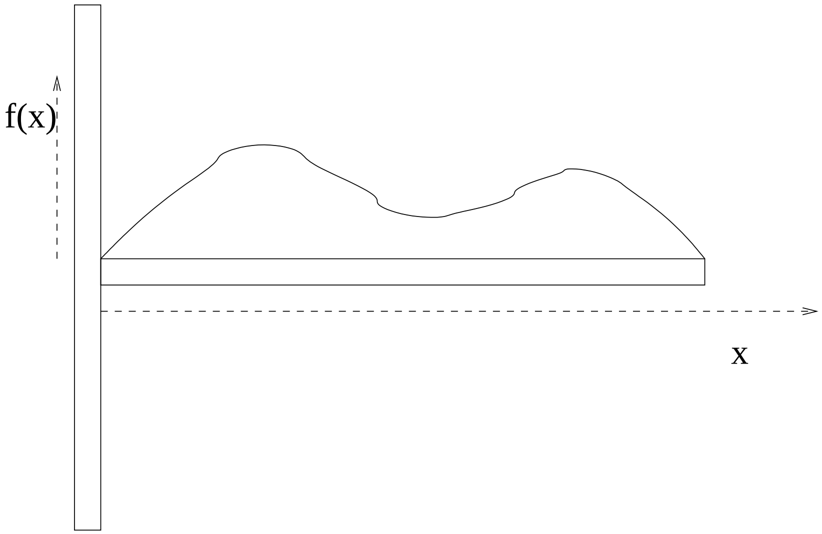
$$T_{100} := \begin{bmatrix} R_0 & R_1 & x_2? & x_3? & x_4? & \ldots & x_{98}? & R_{99} \\ R_1 & R_0 & R_1 & x_2? & x_3? & \ldots & x_{97}? & x_{98}? \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \end{bmatrix}$$

- Linear Matrix Inequalities: dominant in Systems, Control, Optimization
- convex optimization

- Scattered "sensors" $E_0, E_1, \ldots$
with Green's functions/transfer functions/etc. $g_k(\omega, \theta)$
- stochastic excitation with spectral measure $d\mu(\omega, \theta)$
- knowledge of correlations of sensor readings

$$r_{k,\ell} = \int g_k(\omega, \theta) \overbrace{\rho(\omega, \theta) d\omega d\theta}^{d\mu(\omega,\theta)} g_\ell(\omega, \theta)$$



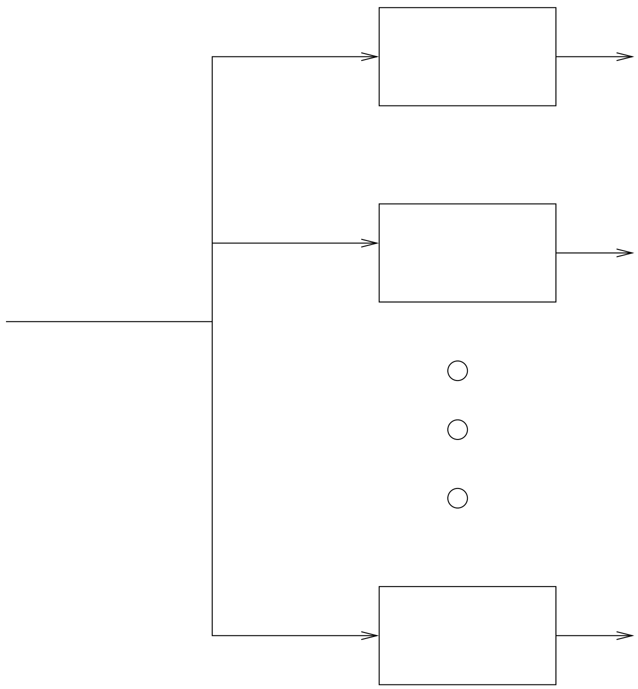


What can we infer about an unknown mass density $\rho(x)$ from a set of moments:

$$R_k := \int_0^\infty x^k \underbrace{\rho(x)dx}_{d\mu(x)}, \quad k = 0, 1, 2, \ldots ?$$

R_0 : total mass
R_1 : torque to hold the beam
etc.

- Low-pass “sensors” $H_k(i\omega) = 1/(1 + i\omega/\tau_k)$
- stochastic input with spectral density $\rho(\omega)$
- knowledge of output covariances

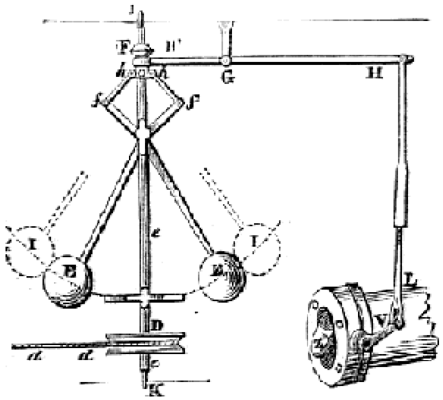


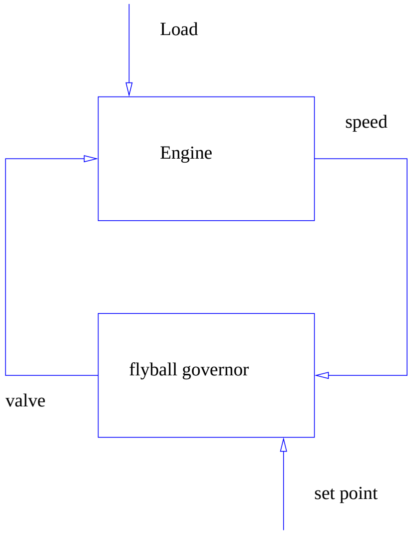
$\boxed{r_k = \int_{-\infty}^{\infty} g_k(\omega)\rho(\omega)d\omega}$, with $g_0 = 1$, $g_k(\omega) = \frac{1}{\omega^2/\tau_k^2+1}$, $k = 0, 1, 2, 3$.

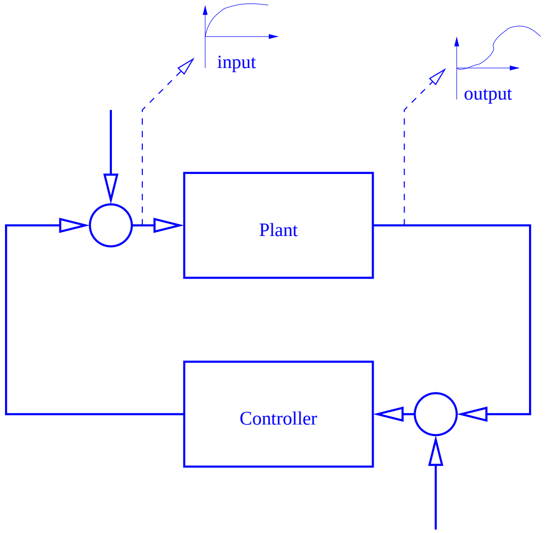
Control problems

- **Sensitivity minimization:** $\min\{\|w(1 - PC)^{-1}\|,$ over "stabilizing" $C\}$
  P stable $\Rightarrow (I - PC)^{-1} = I - PQ$ affine in a free parameter $Q \in H_\infty$

$$\Rightarrow w(I - PC)^{-1} = w - BQ$$







Interpolation problem: Determine, if possible, $Q \in H_\infty$ such that

$$\| \overbrace{w - BQ}^{s(z)} \| < 1$$

Example: $B(z) = z^n$, $B(z) = \frac{z - z_0}{1 - \overline{z_0} z}$, etc.

Re-cast as a moment problem: e.g., $B(z) = z$ and $w = w_0$

$$s(z) = w_0 - zQ(z) \in \mathcal{S}$$

$$\Leftrightarrow$$

$$F(z) := \frac{1 + zs(z)}{1 - zs(z)} = 1 + 2\underbrace{\frac{1 + w_0}{1 - w_0}}_{R_1} z + \ldots \in \mathcal{C}$$

In general: $F(z_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + z_0 e^{jx}}{1 - z_0 e^{jx}} \rho(\theta) d\theta$,

$F'(z_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2e^{jx}}{(1 - z_0 e^{jx})^2} \rho(\theta) d\theta$, etc. $\Rightarrow$ $\boxed{R = \int G \overbrace{\rho(\theta) d\theta}^{d\mu}}$

Interpolation problem:

$F(z) = \frac{1}{2\pi}\int_0^\pi \frac{1+e^{-j\theta}}{1-e^{-j\theta}}\rho(\theta)d\theta.$

Find $F(z)$ analytic with positive real part so that:

$F(0) = R_0$, $\frac{d}{dz}F(z)|_{z=0} = R_1$, $\frac{d^{1/2}}{dz^{1/2}}F(z)|_{z=0} = \hat{R}_{1/2}$,

or, e.g., more important,

$R_{\sqrt{2}}$, R_π, $R_{1.534}$, etc.

$$R = E\{xy\} = \int_{\mathcal{S}} g_{\text{left}} \overbrace{\rho(\theta)d\theta}^{d\mu(\theta)} g_{\text{right}}$$

- Characterize R
- Given R "find" $\rho(\theta)$
- Parametrize all $\rho(\theta)$'s = Uncertainty Set
- What is the effect of the g's

- Motivating examples & non-classical moment problems

  Measurements, Sensor arrays, Completion problems

  Control, modeling, quantum measurements, etc.

- The classical moment problems

  Existence (necessary conditions) & history

  Sufficiency - constructive-canonical equations, entropy principle

- General moment problems

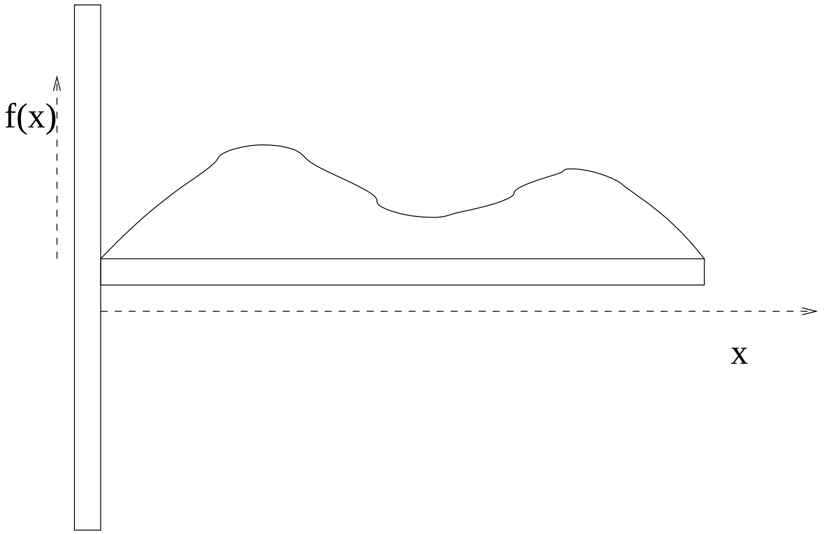
  Existence & parametrization of solutions

  —via minimizers of relative entropy

  —homotopy methods

  matricial/bi-tangential generalizations

- Applications



What can we infer about an unknown mass density $\rho(x)$ from a set of moments:

$$R_k := \int_0^\infty x^k \underbrace{\rho(x)dx}_{d\mu(x)}, \quad k = 0, 1, 2, \ldots ?$$

R_0 : total mass
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etc.

Existence question:

Fact: Polynomials in x, which are positive on $[0, \infty)$ include $(a_0 + a_1 x + \ldots)^2$ and $x(b_0 + b_1 x + \ldots)^2$.

Necessity condition:

$$\boxed{\int (a_0 + a_1 x + \ldots)^2 \rho(x) dx \geq 0 \text{ and } \int x(b_0 + b_1 x + \ldots)^2 \rho(x) dx \geq 0}$$

$$\Leftrightarrow$$

$$\begin{pmatrix} R_0 & R_1 & \ldots & R_n \\ R_1 & R_2 & \ldots & R_{n+1} \\ \ldots & & & \\ R_n & R_{n+1} & \ldots & R_{2n} \end{pmatrix} \geq 0, \ \underline{\text{and}} \ \begin{pmatrix} R_1 & R_2 & \ldots & R_{n+1} \\ R_2 & R_3 & \ldots & R_{n+2} \\ \ldots & & & \\ R_{n+1} & R_{n+2} & \ldots & R_{2n+1} \end{pmatrix} \geq 0$$

Sufficient condition: Same!

Variants of the problem

- Support of f:
  $[0, \infty)$ (Stieljes),
  $(-\infty, \infty)$ (Hamburger),
  $[0, 1]$ (1-D Hausdorff), etc.

- Moment kernels:
  $g_k(x) = x^k$, $x \in \mathbb{R}$ or $[0, 1]$
  g_k being trigonometric, or
  $g_k(\theta) = e^{jk\theta}$, $\theta \in [-\pi, \pi]$

- Index set: $0, 1, \ldots, n$, or $\mathbb{N}$

Solvability

Non-negativity of quadratic forms

e.g., non-negativity of a Pick or Toeplitz matrix, Sarason operator, etc.

Moment problem – late 1800's early 1900's

Chebysev, Markov, Stieljes, Shohat, Tamarkin, Achiezer, Krein, Nudelman, …

Analytic interpolation – early 1900's …

Caratheodory, Toeplitz, Schur, Nevanlinna, Pick…Krein, Arov, Sarason, Sz-Nagy, Foias, Ball, Helton, Gohberg, Dym…

Stochastic processes, Circuit theory, Control: 1950's on…

Levinson, Youla, Zames…

Entropy and relative entropy functionals …

Gibbs, Boltzmann, von Neumann, Shannon, Kullback, Leibler, Umegaki
Jaynes, Csiszar, Lewis, Lang & McClellan…

- Motivating examples & non-classical problems

  Sensor arrays etc.

  Control, completion, etc.

- The classical moment problem

  Existence & history

  Sufficiency: constructive-canonical equations, entropy principle

- General moment problems

  Existence & parametrization of solutions

  —via minimizers of relative entropy

  —homotopy methods

  matricial/bi-tangential generalizations

- Applications

Trigonometric moments (finite indexing set):

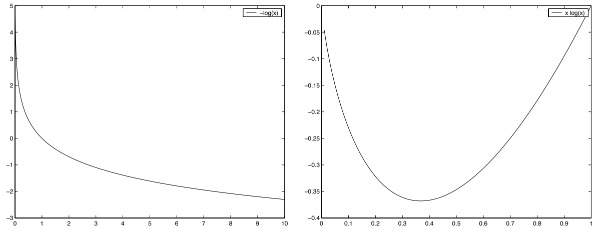
$$R_k = \int_{-\pi}^{\pi} e^{-jk\theta} \rho(\theta) d\theta,\ k = 0, \pm 1, \pm 2, \dots, \pm n.$$

L. Fejer and F. Riesz: Any non-negative trigonometric polynomial is of the form $p(e^{j\theta}) = |a_0 + a_1 e^{j\theta} + \dots a_n e^{nj\theta}|^2$

Solvability condition:

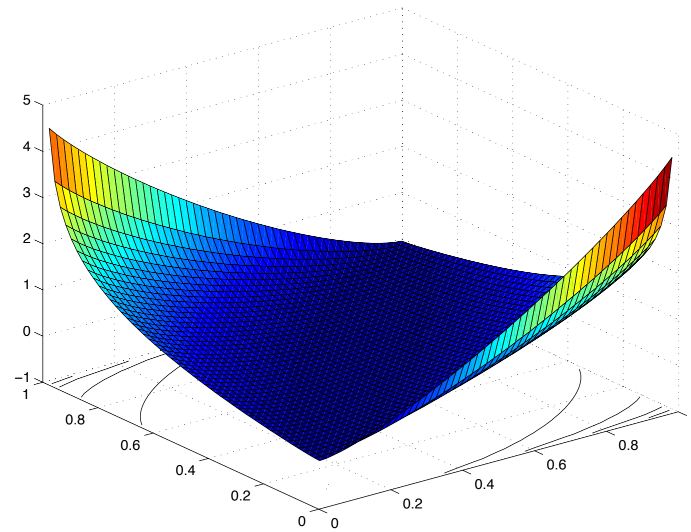
$$\int_{-\pi}^{\pi} p(e^{j\theta}) \rho(\theta) d\theta = \begin{bmatrix} \bar{a}_0 & \bar{a}_1 & \dots & \bar{a}_n \end{bmatrix} \begin{bmatrix} R_0 & R_1 & \dots & R_n \\ R_{-1} & R_0 & \dots & R_{n-1} \\ \dots & & & \\ R_{-n} & R_{-n+1} & \dots & R_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \dots \\ a_n \end{bmatrix} \geq 0, \forall a's$$

Distance between:



$\rho = \big(p,\ q\big)$ and $\hat{\rho} = \big(\hat{p},\ \hat{q}\big)$, with $p + q = \hat{p} + \hat{q} = 1$

$$\mathbb{S}(\rho||\hat{\rho}) = p \log \frac{p}{\hat{p}} + q \log \frac{q}{\hat{q}}$$



In general: $\mathbb{S}(\rho||\hat{\rho}) = \int \left(\rho \log(\rho) - \rho \log(\hat{\rho})\right)$, and
$\mathbb{S}(A||B) = \mathrm{trace}(A \log A - A \log B)$,
are jointly convex in their arguments

Example: Find $\rho = \big(p,\ q\big)$, $p, q > 0$ such that

$$\rho = \arg\min\{\log(p) + \log(q)\ :\ pa + qb = R\}$$

Answer:
$p = \frac{R}{2a}$ and $q = \frac{R}{2b}$

Parametrization of solutions via a choice of $\hat{\rho}$ in:
$\rho = \arg\min\{\mathbb{S}(\hat{\rho}||\rho) = \hat{p}\log(p) + \hat{q}\log(q)\ :\ pa + qb = R\}$

"Entropy rate" (concave)

$$\mathbb{I}_\rho := \int \log \rho(\theta) d\theta$$

"distance" to 1 (convex)

$$\mathbb{S}(1||\rho) := \int (1 \cdot \log(1) - 1 \cdot \log(\rho(\theta)) d\theta = -\mathbb{I}_\rho$$

Find $\text{argmax}(\mathbb{I}_\rho)$

subject to $\int G(e^{j\theta})\rho(\theta)d\theta = R$

where $G := \begin{bmatrix} e^{-jn\theta} \\ \vdots \\ 1 \\ \vdots \\ e^{jn\theta} \end{bmatrix}$ and $R := \begin{bmatrix} R_n \\ \vdots \\ R_0 \\ \vdots \\ R_{-n} \end{bmatrix}$.

Analysis: $L(\rho, \lambda) := \int \log \rho d\theta - \lambda(\int G\rho d\theta - R)$

$$\delta L(\rho, \lambda; \delta\rho) \equiv 0 \Rightarrow \int (\frac{1}{\rho} - \lambda G)\delta\rho d\theta \equiv 0$$
$$\Rightarrow \rho = \frac{1}{\lambda G}$$

Set

$$\lambda G = \sum_{-n}^{n} \lambda_k e^{jk\theta} =: |\sum_{0}^{n} a_k e^{jk\theta}|^2 = |a(e^{j\theta})|^2$$

Then

$$\int \frac{\bar{a}(e^{-j\theta})}{|a(e^{j\theta})|^2} = \int \frac{1}{a(e^{j\theta})} = \frac{1}{a_0}, \text{ and } \int \frac{e^{jk\theta}\bar{a}(e^{-j\theta})}{|a(e^{j\theta})|^2} = \int \frac{e^{jk\theta}}{a(e^{j\theta})} = 0, k \geq 1,$$

together with

$$\boxed{R_k = \int_{-\pi}^{\pi} e^{-jk\theta} \frac{1}{|a(e^{j\theta})|^2} d\theta, \text{ for } k = 0, \pm 1, \ldots,}$$

gives

$$\begin{bmatrix} R_0 & R_1 & \ldots & R_n \\ R_{-1} & R_0 & \ldots & R_{n-1} \\ \ldots & & & \\ R_{-n} & R_{-n+1} & \ldots & R_0 \end{bmatrix} \begin{bmatrix} \bar{a}_0 \\ \bar{a}_1 \\ \vdots \\ \bar{a}_n \end{bmatrix} = \begin{bmatrix} \frac{1}{a_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \boxed{a\text{'s !}}$$

- Motivating examples & non-classical problems
- The classical moment problem

- General moment problems

  Existence & parametrization of solutions

  —via minimizers of relative entropy

  —homotopy methods

  matricial/bi-tangential generalizations
- Applications

Relative entropy, Kullback-Leibler-von Neumann-Leib-Umegaki distance

Fact: Let ρ, $\hat{\rho}$ non-negative functions, then

$$\mathbb{S}(\rho||\hat{\rho}) := \mathrm{trace} \int \rho \log(\rho) - \rho \log(\hat{\rho})$$

is jointly convex

Idea:

> Choose "parameter" $\hat{\rho}$,
> then determine the minimizer ρ which agrees with the moments.

Similarly,

> repeat with the roles of ρ and $\hat{\rho}$ reversed.

$$\mathbb{S}(\rho\|\hat{\rho}) = \int \rho \log \rho - \rho \log \hat{\rho}$$

Given ψ find	$\text{argmin}(\mathbb{S}(\rho\|\|\psi))$
subject to	$\int G(e^{j\theta})\rho(\theta)d\theta = R$

Given ψ find	$\text{argmin}(\mathbb{S}(\psi\|\|\hat{\rho}))$
subject to	$\int G(e^{j\theta})\hat{\rho}(\theta)d\theta = R$

If $\exists$ a solution ρ, then it belongs to:

$\mathfrak{F}_{\text{exp}} := \{\psi(\theta)e^{-<\lambda, G(\theta)>}\}$ or, respectively, $\mathfrak{F}_{\text{rat}} := \{\frac{\psi(\theta)}{< \lambda, G(\theta) >}$: with $\lambda G > 0\}$

for any $\psi(\theta) > 0$.

Need to solve $\int G(e^{j\theta})\rho(\theta)d\theta = R$

- For general G, there exists no representation for positive elements $\lambda G = \sum_{\text{index set}} \lambda_k g_k$ hence, no canonical equations, ...

Observation:

If

$$h \;:\; \lambda \mapsto R = \int_{\mathcal{S}} G(\theta)\frac{\psi(\theta)}{<\lambda, G(\theta)>}d\theta$$

then

$$\nabla h := \frac{\partial R}{\partial \lambda} = -\int_{\mathcal{S}} G(\theta)\left(\frac{\psi(\theta)}{<\lambda, G(\theta)>}\right)^2 G(\theta)' d\theta$$

is non-singular $\forall \lambda G > 0$.

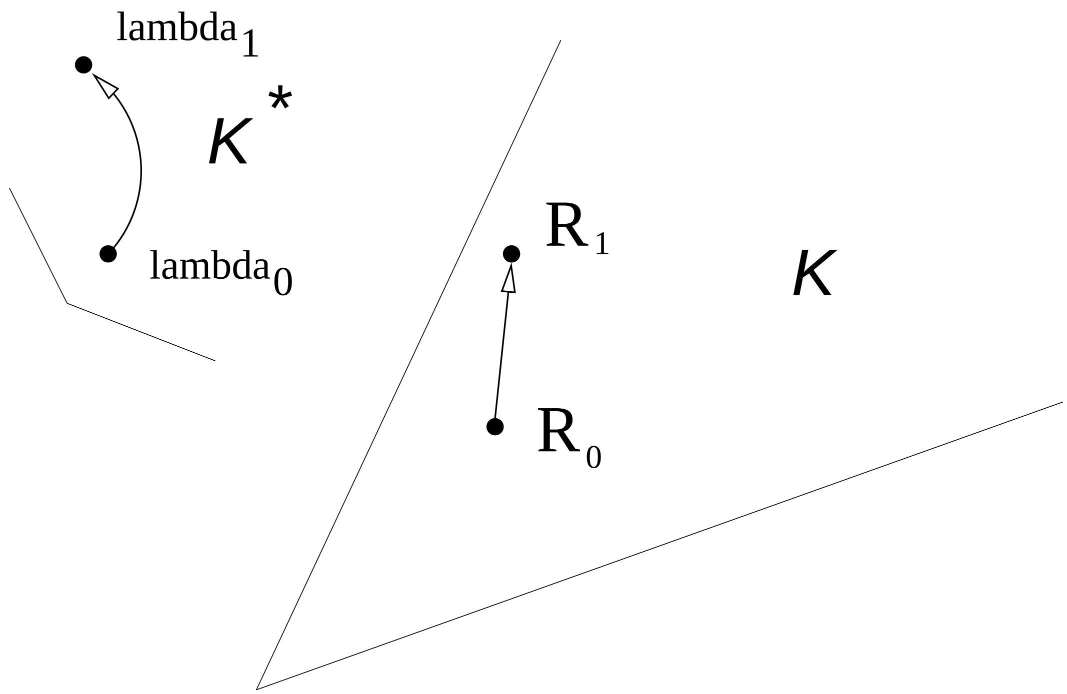
If

$$\kappa \;:\; \lambda \mapsto R = \int_{\mathcal{S}} G(\theta)\psi(\theta)e^{-<\lambda, G(\theta)>}d\theta$$

then

$$\nabla \kappa := \frac{\partial R}{\partial \lambda} = -\int_{\mathcal{S}} G(\theta)\psi(\theta)e^{-<\lambda, G(\theta)>}G(\theta)' d\theta$$

is non-singular $\forall \lambda$.



Homotopy on the R's

Construction of homotopy:

$R_\alpha := R_0 + \alpha(R_1 - R_0)$ for $\alpha \in [0, 1]$

$$\frac{dR_\alpha}{d\alpha} = R_1 - R_0, \text{ with } R_{\alpha=0} = R_0 = \int_{\mathcal{S}} G(\theta)\rho(\lambda_0, \theta)d\theta$$

$$\frac{d\lambda_\alpha}{d\alpha} = \left(\left. \frac{\partial R}{\partial \lambda} \right|_{\lambda_\alpha} \right)^{-1} (R_1 - R_0).$$

$$\frac{dR_\alpha}{d\alpha} = (1-\alpha)(R_1 - R_\alpha)$$

$$\frac{d\lambda_\alpha}{d\alpha} = (1-\alpha)\left(\left.\frac{\partial R}{\partial \lambda}\right|_{\lambda_\alpha}\right)^{-1}(R_1 - R_\alpha).$$

and, for $\alpha = 1 - e^{-t}$,

$$\boxed{\frac{d\lambda_t}{dt} = \left(\left.\frac{\partial R}{\partial \lambda}\right|_{\lambda_t}\right)^{-1}\left(R_1 - \int_{\mathcal{S}} G(\theta)\rho(\lambda_t,\theta)d\theta\right)}$$

- Minimizers of $\mathbb{S}(\psi||f)$: dim$(\mathcal{S}) = 1$

THM: Let λ_0 such that $\lambda_0 G > 0$, and

$$\frac{d\lambda_t}{dt} = \left(\frac{\partial R}{\partial \lambda}\bigg|_{\lambda_t} \right)^{-1} \left(R_1 - \int_{\mathcal{S}} G(\theta)\rho(\lambda_t, \theta)d\theta\right)$$

for $t \geq 0$ and

$$\rho(\lambda_t, \theta) = \frac{\psi(\theta)}{\langle \lambda_t, G(\theta)\rangle}.$$

If $R_1 \in \text{int}(\mathcal{K})$, then $\lambda_t \in \mathcal{K}_+^*$ for all $t \in [0, \infty)$, $\lambda_t \to \hat{\lambda} \in \mathcal{K}_+^*$, and

$$R_1 = \int_{\mathcal{S}} G(\theta)\rho(\hat{\lambda}, \theta)d\theta.$$

If $R_1 \notin \text{int}(\mathcal{K})$, then $\|\lambda_t\| \to \infty$.

- Minimizers of $\mathbb{S}(f||\psi)$: no condition on support of $\mathcal{S}$ or the dual cone

THM: For any λ_0 and

$$\boxed{\frac{d\lambda_t}{dt} = \left(\frac{\partial R}{\partial \lambda}\bigg|_{\lambda_t}\right)^{-1} \left(R_1 - \int_{\mathcal{S}} G(\theta)\rho(\lambda_t,\theta)d\theta\right)}$$

for $t \geq 0$ and

$$\rho(\lambda_t,\theta) = \psi(\theta)e^{-\langle\lambda_t,G(\theta)\rangle}.$$

If $R_1 \in \mathrm{int}(\mathcal{K})$, then $\lambda_t \to \hat{\lambda}$, remains bounded, and

$$R_1 = \int_{\mathcal{S}} G(\theta)\rho(\hat{\lambda},\theta)d\theta.$$

If $R_1 \notin \mathrm{int}(\mathcal{K})$, then $\|\lambda_t\| \to \infty$.

In both cases,

$$V(\lambda) = \|R_1 - \int_{\mathcal{S}} G(\theta)\rho(\lambda_t, \theta)d\theta\|^2$$

is a Lyapunov function , satisfying

$$\frac{dV(\lambda_t)}{dt} = -2V(\lambda_t),$$

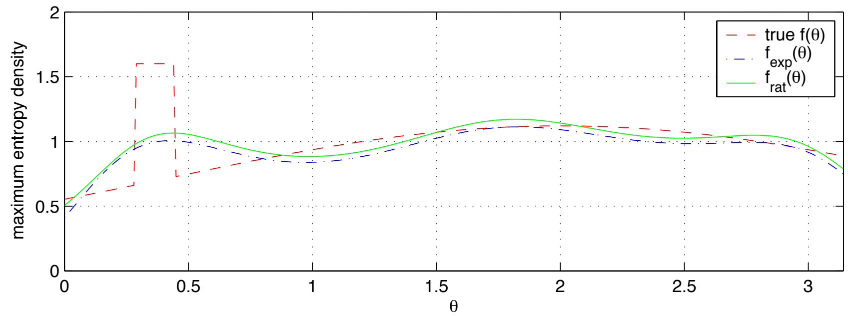
- All positive densities ρ solving $R = \int_{\mathcal{S}} G\rho$ can be obtained with a suitable ψ.

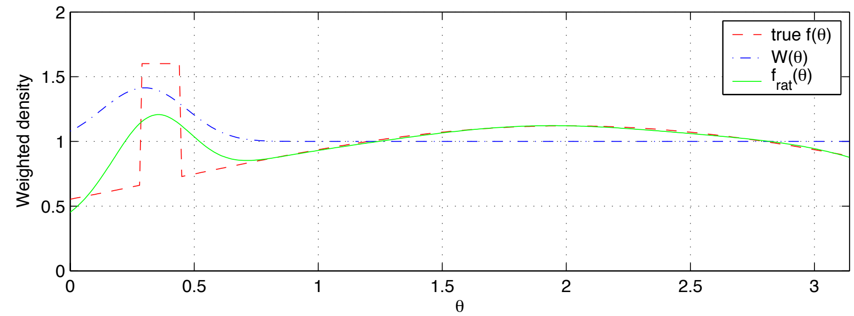
- Given R, G, ψ either solution, $\psi/\langle\lambda, G\rangle$ or, $\psi e^{-\langle\lambda, G\rangle}$, is unique.

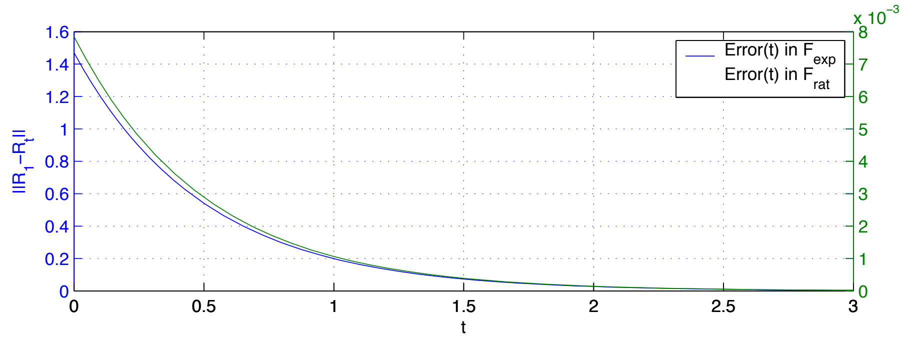
- Convergence is "fast."

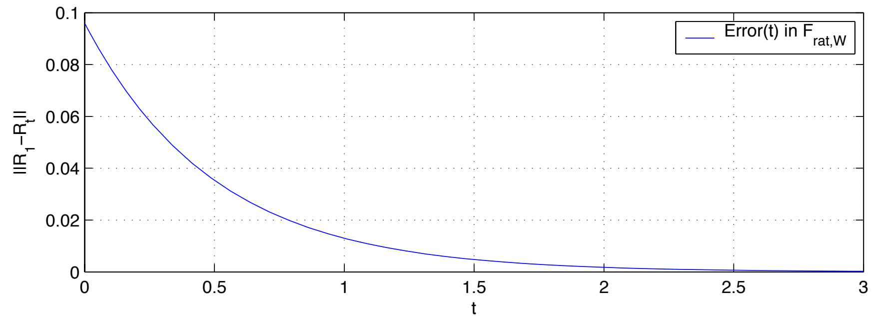
- Failure to converge $\Rightarrow$ no solution exists and $\lambda \to \infty$.

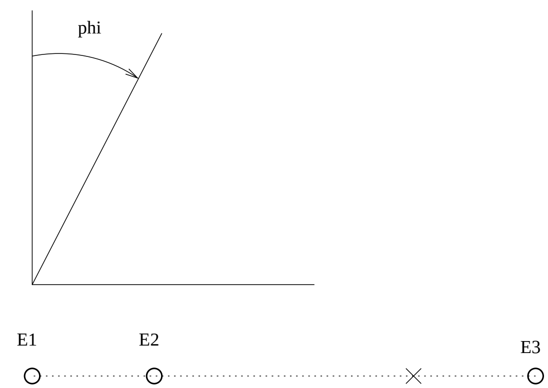
- Use of rational family requires conditions on $\mathcal{S}$ & the dual cone











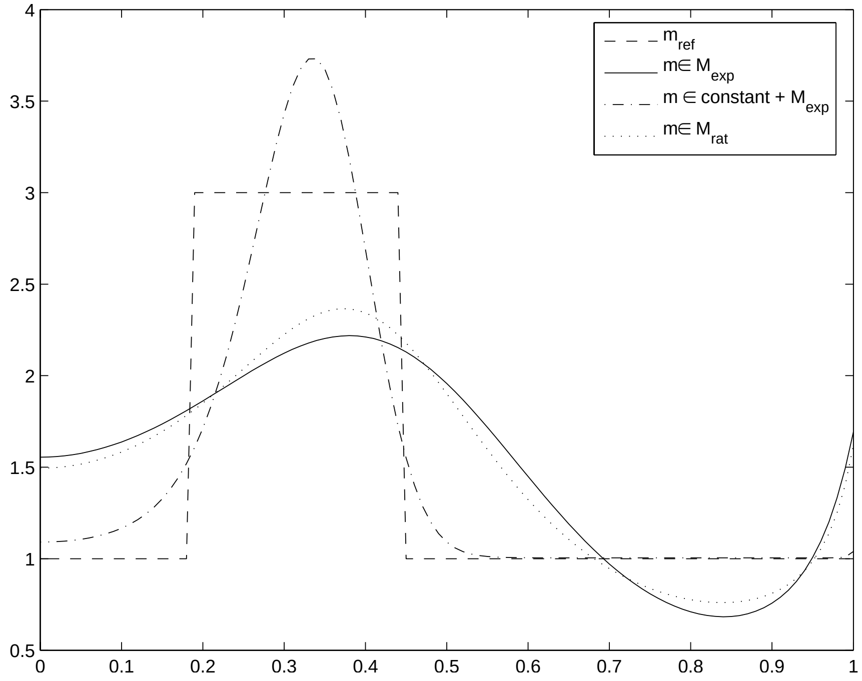
$$G(\theta) := \left[1 \;\; e^{-j\tau} \;\; e^{-j\sqrt{2}\tau} \;\; e^{-j(\sqrt{2}+1)\tau}\right]'$$

- Compute $R_{1,\text{white}} = \int G(\theta)d\theta$

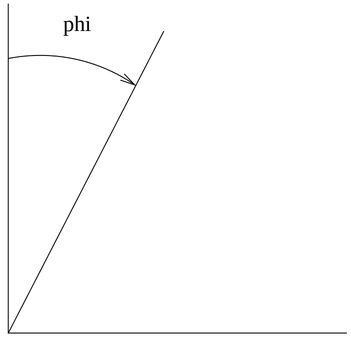
$$p_0 = \text{argmax}\{p \ : \ R_1 - pR_{1,\text{white}} \in \mathcal{K}\}.$$

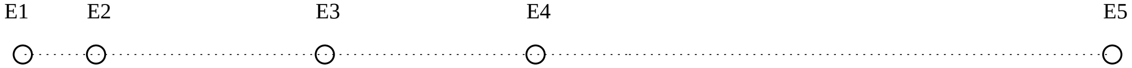
and compute $d\mu$ such that

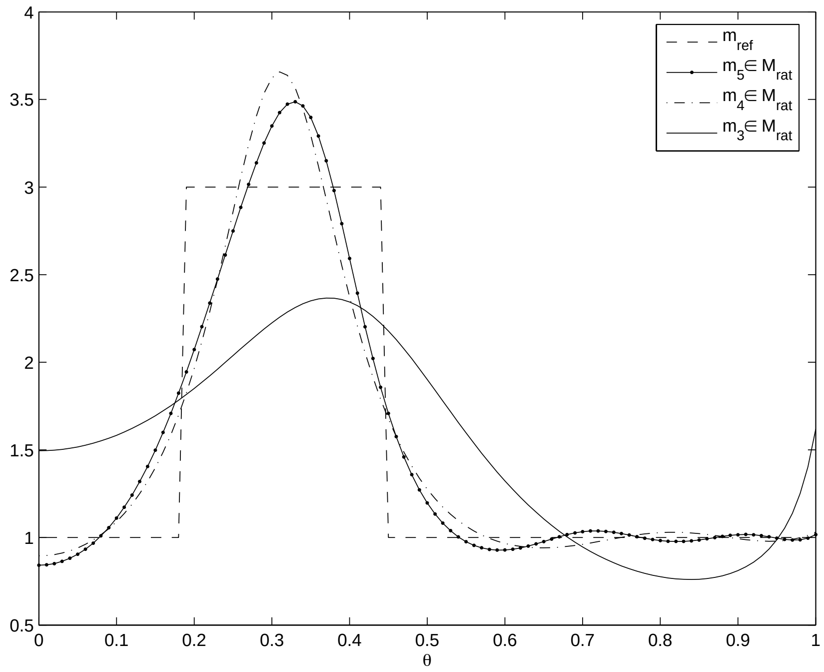
$$R_1 = \int_0^1 G(\theta)(p_0 d\theta + d\mu(\theta)).$$

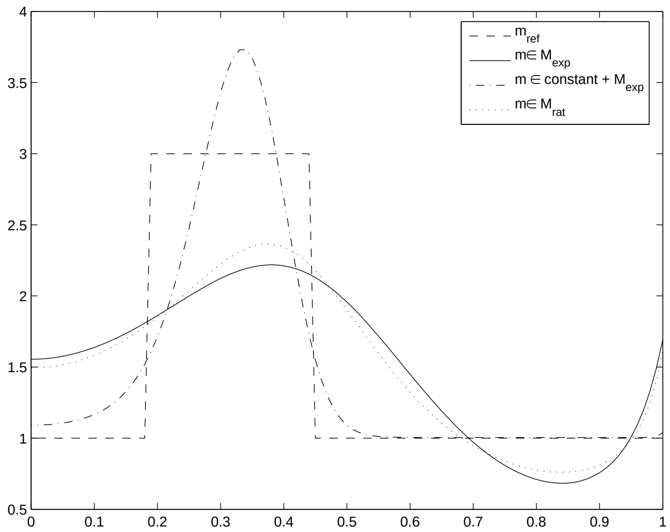


EXAMPLE: MORE "SAMPLES"



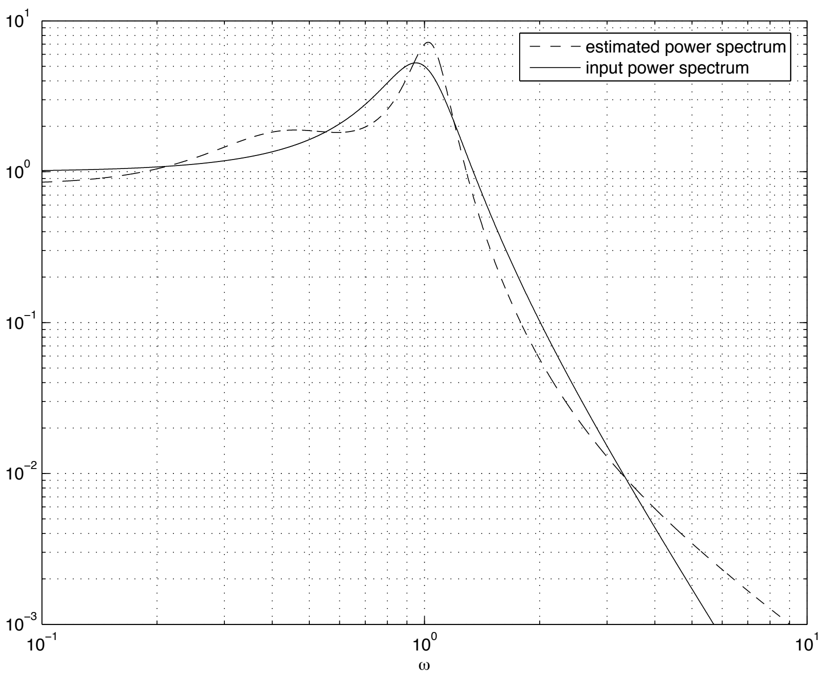


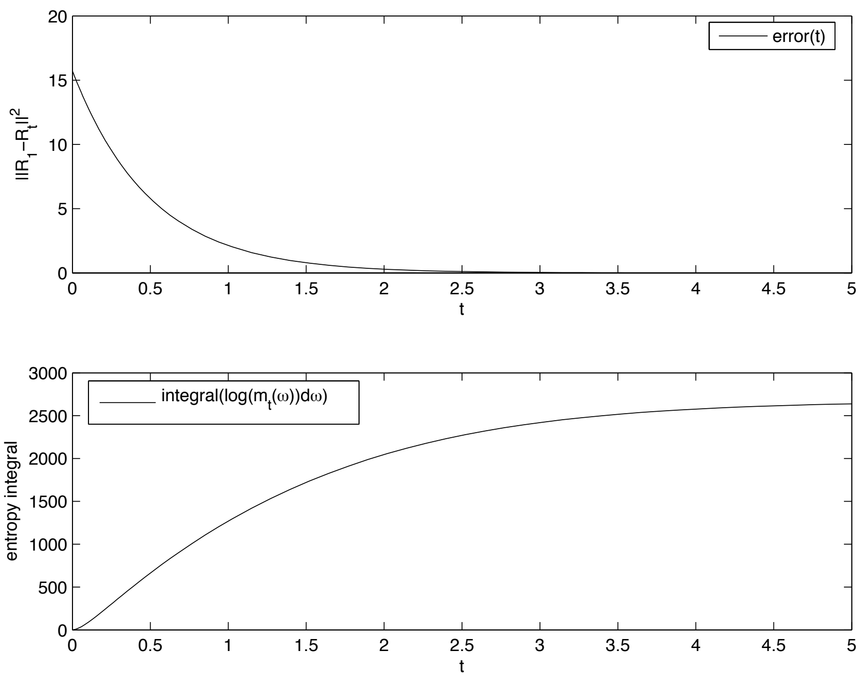




- Low-pass "sensors" $H_k(i\omega) = 1/(1 + i\omega/\tau_k)$
- stochastic input with spectral density $\rho(\omega)$
- knowledge of output covariances

$r_k = \int_{-\infty}^{\infty} g_k(\omega)\rho(\omega)d\omega$, with $g_0 = 1$, $g_k(\omega) = \frac{1}{\omega^2/\tau_k^2+1}$, $k = 0, 1, 2, 3.$

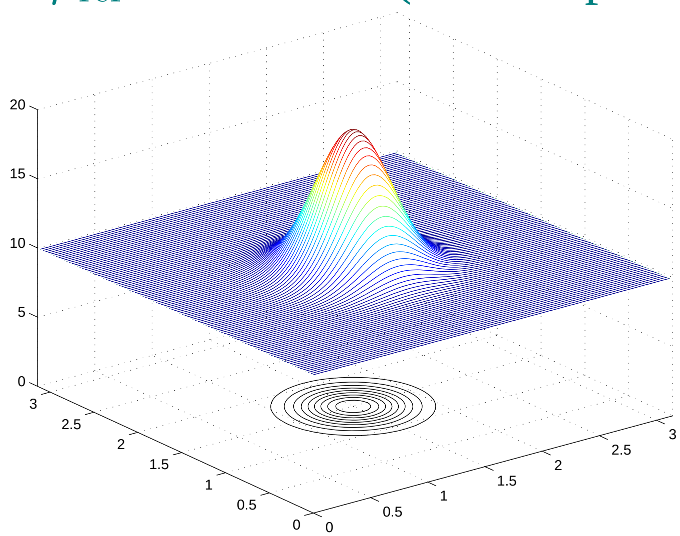


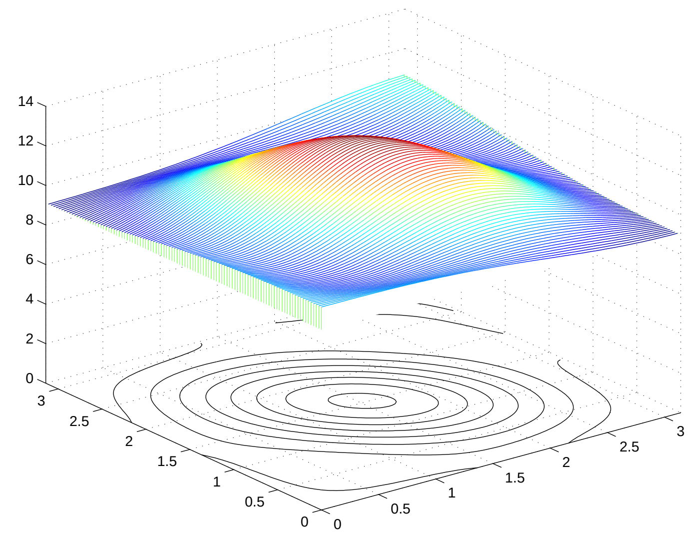


- Begin with $\rho_{\text{ref}}(\theta, \phi)$, and $g_{k,\ell}(\theta, \phi) = \cos(k\theta + \ell\phi), \quad k, \ell \in \{0, 1, 2\}$

$$R_1 = \left[\int_{\mathcal{S}} g_{k,\ell}(\theta, \phi)\rho_{\text{ref}}(\theta, \phi) d\theta d\phi \right]_{k,\ell=0}^{k,\ell=2} = \begin{bmatrix} 33.0129 & 0.3140 & -1.1417 \\ 0.3140 & -14.0469 & -0.2502 \\ -1.1417 & -0.2502 & 1.0310 \end{bmatrix}$$

- ρ_{ref} and $e^{-\langle \lambda, G \rangle}$ (for comparison).





$$R = \int_{\mathcal{S}} (G_{\text{left}} \rho G_{\text{right}}) d\theta$$

G_{left}: $\mathbb{C}^{p \times m}$-valued (C^2)
G_{right}: $\mathbb{C}^{m \times q}$-valued (C^2)
ρ: $m \times m$ Hermitian non-negative
$R \in \mathbb{C}^{p \times q}$.

(i) Given $G_{\text{left}}, G_{\text{right}}$ and R, $\exists ?\ \rho > 0$?
(ii) It $\exists$, then find a particular one.
(iii) Parametrize all ρ's.

Tangential interpolation…

- Rational matricial densities:

$$\left((G_{\text{right}}\lambda G_{\text{left}})_{\text{Hermitian}}\right)^{-1} = \text{argmin}\left\{\mathbb{S}(I||\rho) \text{ subject to } R = \int_{\mathcal{S}}(G_{\text{left}}\rho G_{\text{right}})\right\}$$

- Exponential matricial densities:

$$\frac{1}{e}e^{-\left((G_{\text{right}}\lambda G_{\text{left}})_{\text{Hermitian}}\right)} = \text{argmin}\left\{\mathbb{S}(\rho||I) \text{ subject to } R = \int_{\mathcal{S}}(G_{\text{left}}\rho G_{\text{right}})\right\}$$

Provided $\mathcal{S} \subset \mathbb{R}^1$, the homotopy construction generalizes to
$\mathfrak{F}_{\text{rat}}$: hermitian positive matrix-valued functions of the form

$$\rho = \psi^{1/2} \left((G_{\text{right}} \lambda G_{\text{left}})_{\text{Hermitian}} \right)^{-1} \psi^{1/2}$$

leading to:

$$\boxed{\frac{d\lambda}{dt} = (\nabla h(\lambda))^{-1} (R_1 - \int (G_{\text{left}} \rho(\lambda) G_{\text{right}}))}$$

$$h \; : \; \lambda \mapsto R \; = \; \int (G_{\text{left}} \psi^{1/2} \left((G_{\text{right}} \lambda G_{\text{left}})_{\text{Hermitian}} \right)^{-1} \psi^{1/2} G_{\text{right}}) dx$$

$$\nabla h \; : \; \delta\lambda \mapsto \delta R \qquad \ldots\ldots \text{ is invertible, } \ldots$$

- If $\exists\ \rho > 0:\ R = \int G_{\text{left}} \rho G_{\text{right}} d\theta$ then:

  $\lambda(t) \longrightarrow \hat{\lambda}$

  $\rho = \psi^{1/2} \left((G_{\text{right}} \hat{\lambda} G_{\text{left}})_{\text{Hermitian}} \right)^{-1} \psi^{1/2} > 0$, and $R = \int G_{\text{left}} \rho G_{\text{right}}$

  $\hat{\lambda}$ does not depend on λ_0

  $V(\lambda) = \| R_1 - \int (G_{\text{left}} \rho(\lambda) G_{\text{right}}) \|_F^2$ a Lyapunov function.

  convergence of $\dot{\lambda} = (\nabla h)^{-1} (R_1 - \int G_{\text{left}} \rho G_{\text{right}})$ exponentially fast.

  all admissible ρ's can be obtained with suitable ψ.

- If $\nexists\ d\mu > 0$, then $\|\lambda(t)\| \to \infty$

The homotopy construction generalizes to
$\mathfrak{F}_{\text{exp}}$: hermitian positive matrix-valued functions of the form

$$\rho = \psi^{1/2} e^{-\left((G_{\text{right}}\lambda G_{\text{left}})_{\text{Hermitian}}\right)} \psi^{1/2}$$

leading to:

$$\boxed{\frac{d\lambda}{dt} = (\nabla h(\lambda))^{-1}(R_1 - \int_{\mathcal{S}} (G_{\text{left}}\rho(\lambda)G_{\text{right}}))}$$

with

$\mathcal{S} \subset \mathbb{R}^k$, $k \geq 1$, and no requirements on the dual cone.

$$\boxed{R = E\{xy\} = \int_{\mathcal{S}} g_{\text{left}} \rho(\theta) g_{\text{right}} d\theta}$$

- Characterize R ✓
- Given R “find” ρ ✓
- Parametrize all ρ’s ✓
- What is the effect of the g’s $\Leftarrow$ tradeoffs resolution/variance

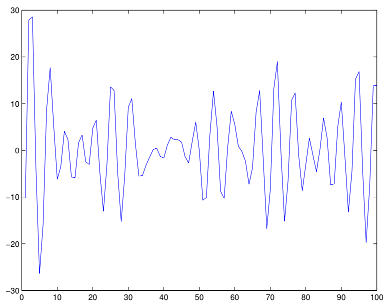
- Motivating examples & non-classical moment problems
- The classical moment problem
- General moment problems

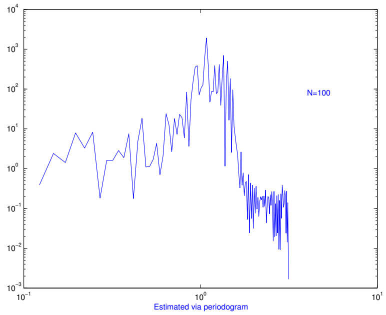
- Applications

  high resolution spectral analysis

  ultrasound non-invasive temp sensing

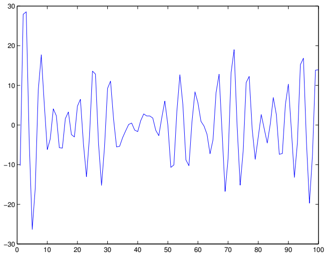
$\{u_0, u_1, u_2, \ldots, u_{N-1}\} \;\Leftrightarrow\;$ periodicities, harmonics, "color"





- ○ Periodogram, FFT
- ○ Model based (ARMA,....)
- ○ Modern nonlinear (Maximum-entropy, maximum-likelihood,...)

Time-series data



$\{u_0, u_1, u_2, \ldots, u_{N-1}\}$

select

(A, B)

$\Rightarrow$

estimate

$R \simeq \sum_k x_k x_k^*$

$\{$spectra consistent with R $\}$

where

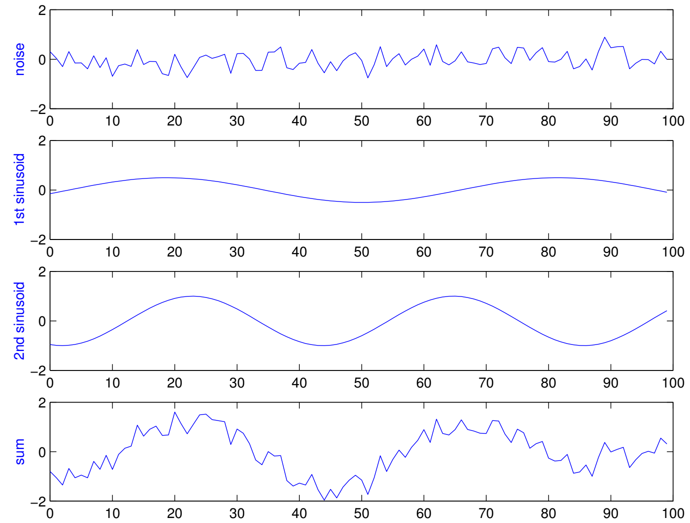
$$x_k = A x_{k-1} + B u_k$$

$$g(z) = (zI - A)^{-1} B$$

$$R = \int g(e^{j\theta})\rho(\theta) g(e^{j\theta})^* d\theta$$

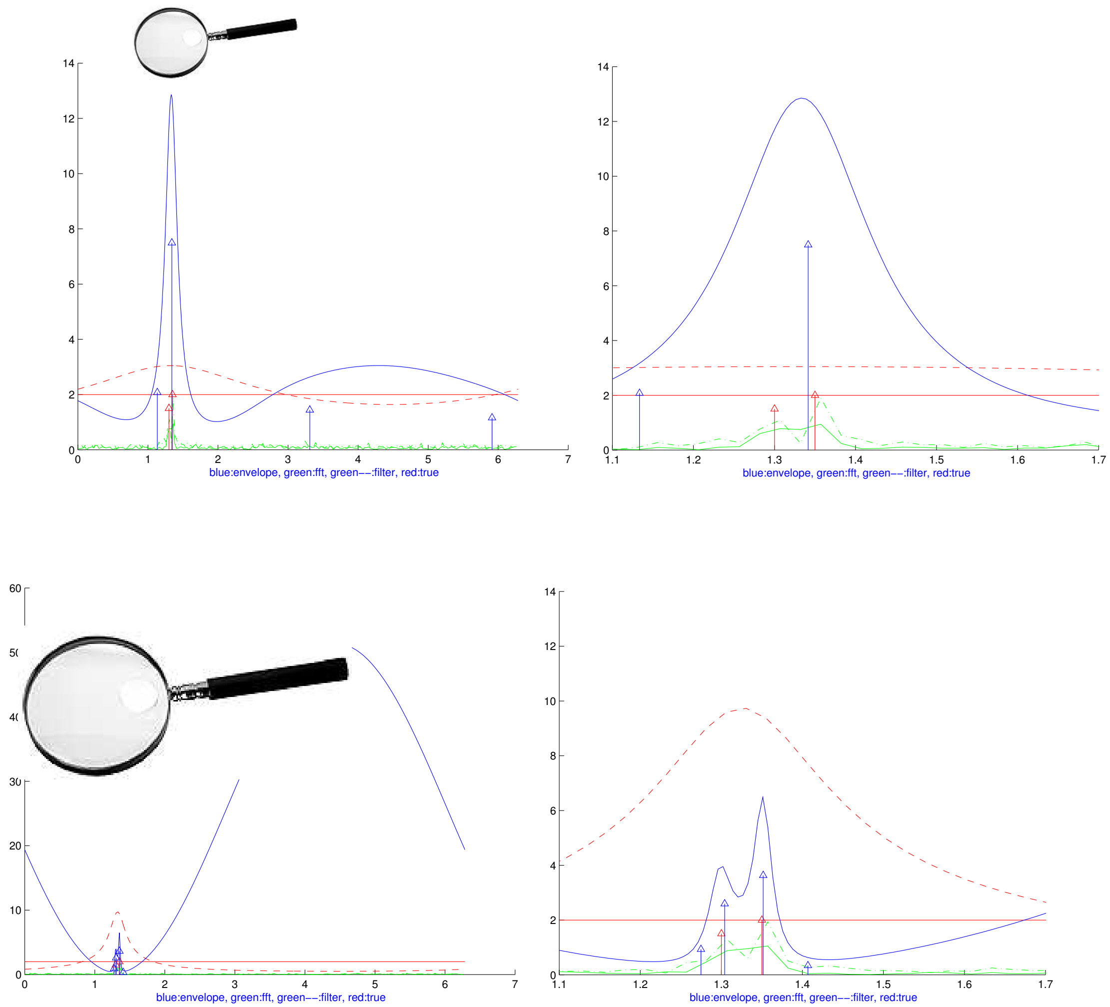
- design $g(e^{j\theta})$: tradoffs between robustness and resolution

$$\mathbf{u}_k = \nu_k + a_1 \sin(\omega_1 k + \phi_1) + a_2 \sin(\omega_2 k + \phi_2),\ k = 1, \ldots, n,$$



Noise, sinusoid 1, sinusoid 2, and their sum

$\omega_2 - \omega_1 < \frac{2\pi}{n}$ = Fourier uncertainty bound


blue:envelope, green:fft, green--:filter, red:true
blue:envelope, green:fft, green--:filter, red:true
blue:envelope, green:fft, green--:filter, red:true
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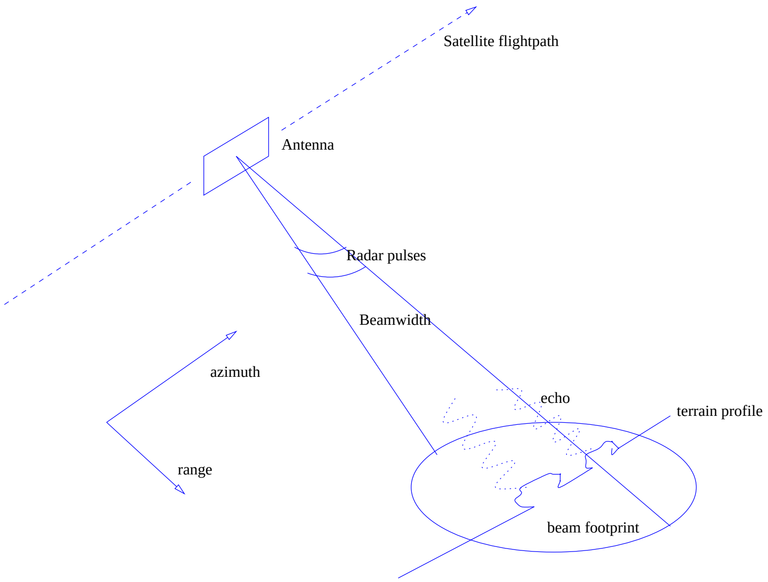
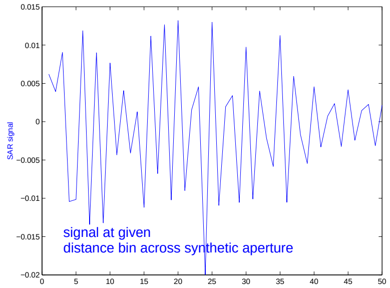
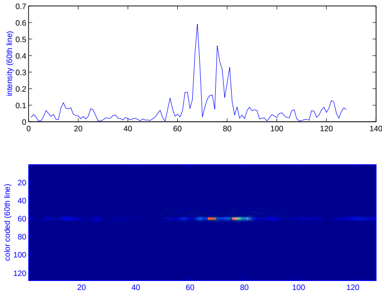
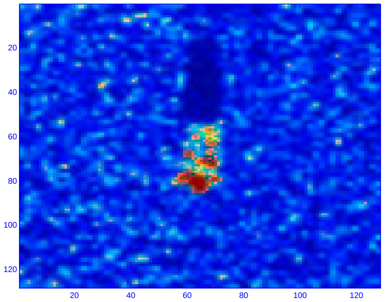

Satellite flightpath
Antenna
Radar pulses
Beamwidth
azimuth
range
echo
terrain profile
beam footprint

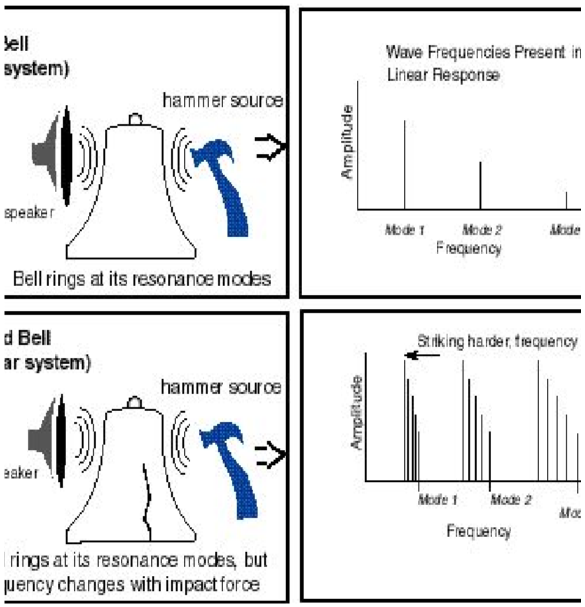

SAR signal
signal at given
distance bin across synthetic aperture

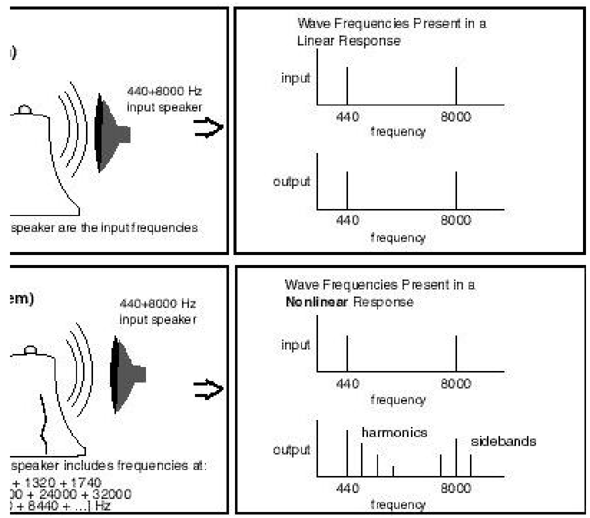
⇒

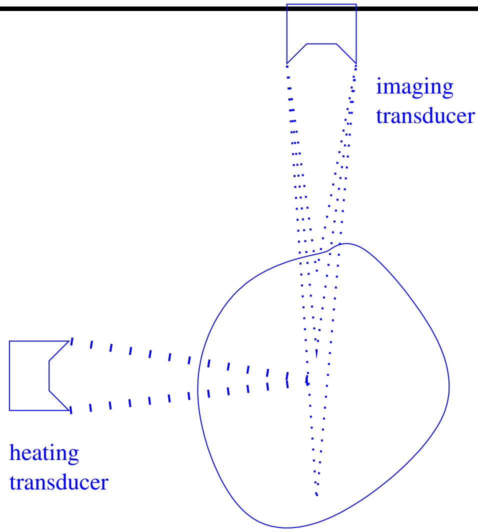

intensity (60th line)
color coded (60th line)

⇒

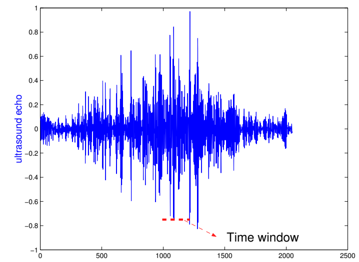


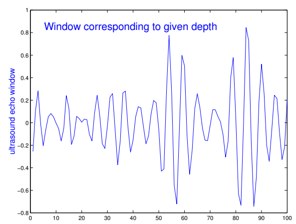

system)
hammer source
speaker
Bell rings at its resonance modes
Wave Frequencies Present in
Linear Response
Amplitude
Mode 1
Mode 2
Frequency
ar system)
hammer source
rings at its resonance modes, but
uency changes with impact force
Striking harder, frequency
Amplitude
Mode 1
Mode 2
Frequency


440+8000 Hz
input speaker
speaker are the input frequencies
Wave Frequencies Present in a
Linear Response
input
440
8000
frequency
output
440
8000
frequency
440+8000 Hz
input speaker
speaker includes frequencies at:
+ 1320 + 1740
+ 24000 + 32000
+ 8440 + ...] Hz
Wave Frequencies Present in a
Nonlinear Response
input
440
8000
frequency
harmonics
sidebands
output
440
8000
frequency

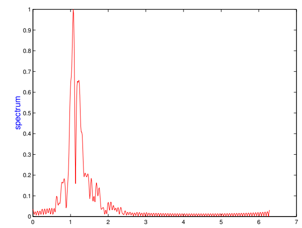


Ultrasound echo:

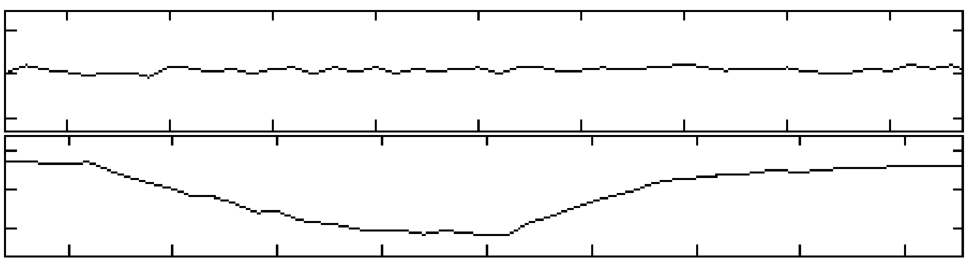




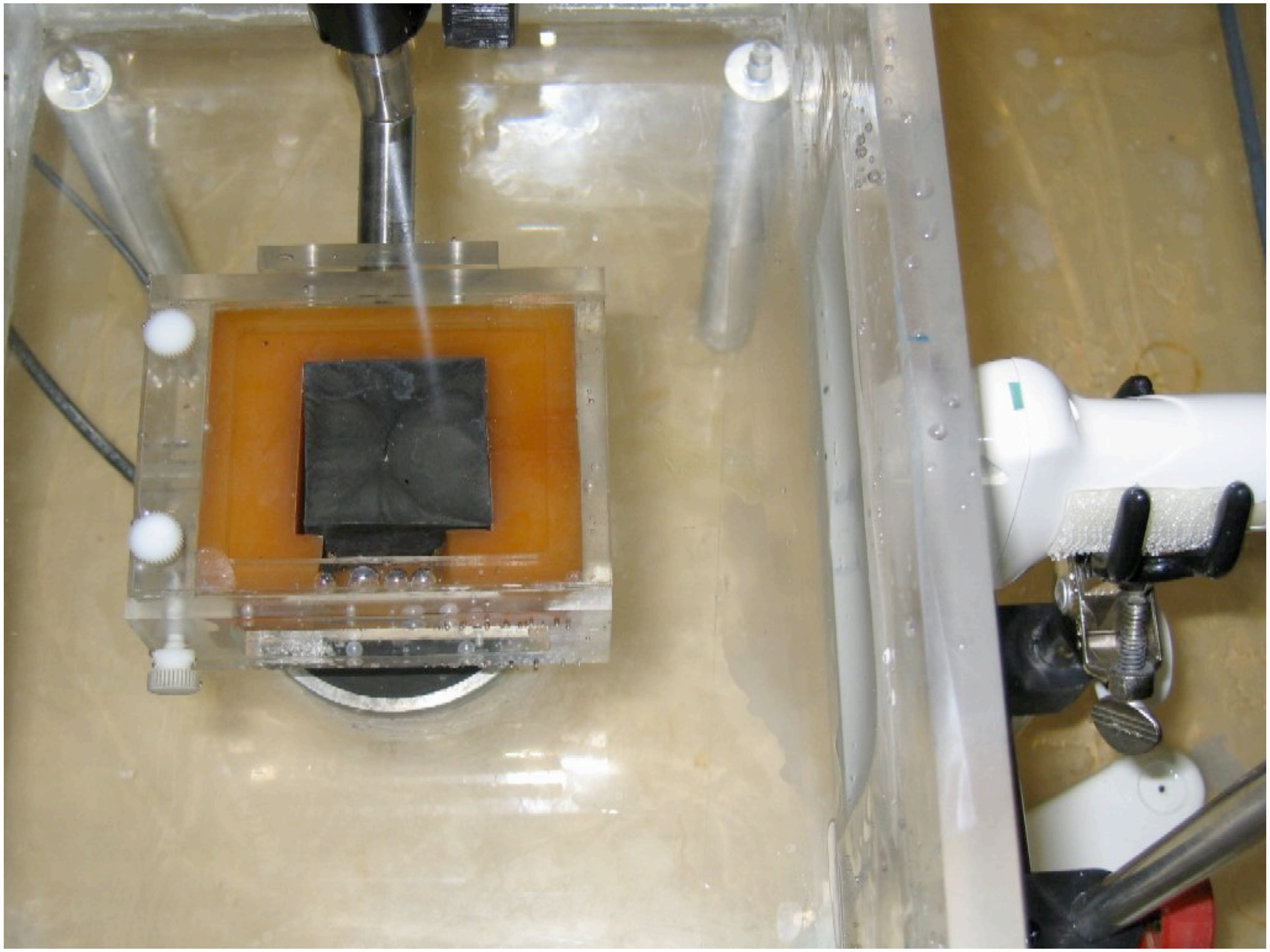
$\Rightarrow$

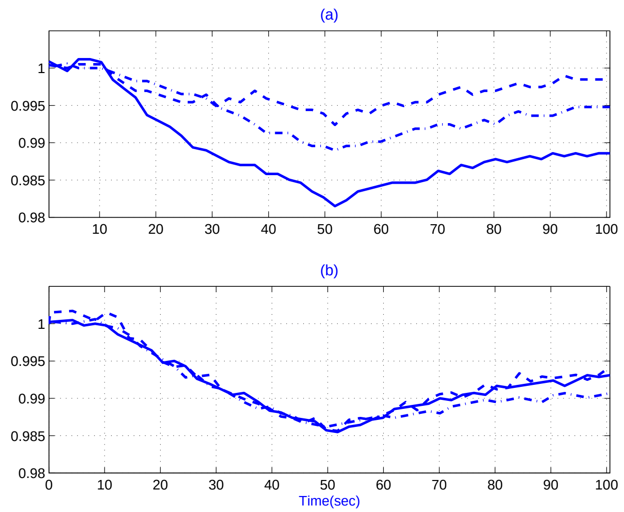


frequency shift vs. time $\Rightarrow$ temperature profile



Collaboration: E. Ebbini & A. Nasiri-Amini



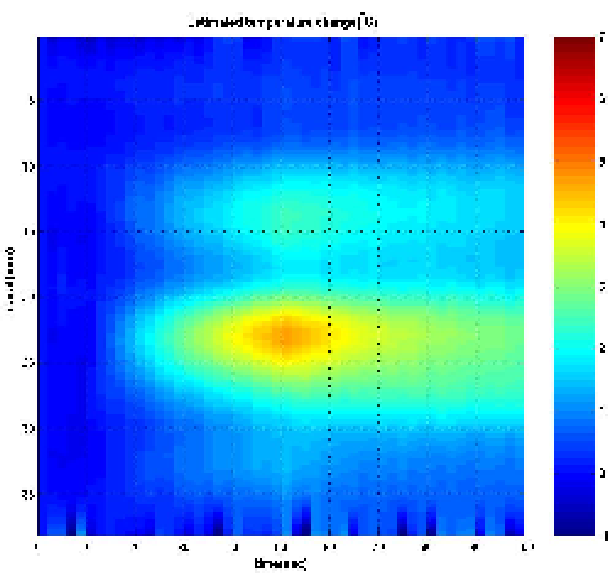


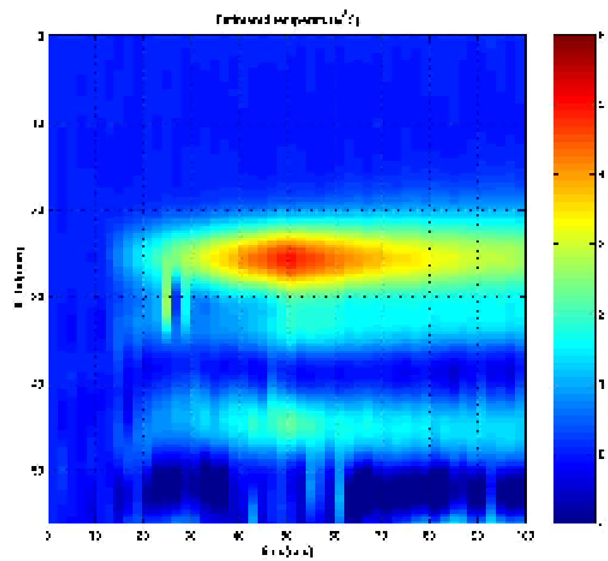
Harmonics within “passband” vs. time

(a) state-of-the art

(b) based on tuned g_k’s

Temperature field (lateral & axial) vs. time:





MOMENT PROBLEMS IN SCIENCE AND ENGINEERING

Main points: Existence and parametrization of solutions via minimizers of relative entropy

Construction via homotopy methods

Generalization matricial/bi-tangential

Family of solutions $\sim$ uncertainty set

Applications: nonuniform sampling, irregular bases (e.g., wavelets), nonuniform arrays, and spatial distribution of sensors
control design with degree constraint
linear matrix inequalities

Thank you