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Completion of partially known turbulent flow statistics via convex optimization

By A. Zare†, M. R. Jovanović† AND T. T. Georgiou†

Second-order statistics of turbulent flows can be obtained in direct numerical simulations and experiments. Even though these statistics provide invaluable insights about flow physics, they have not yet been incorporated in models to be used for control. In this report, we develop a method to account for partially available velocity correlations via stochastically forced linear dynamical models of low complexity. We formulate a convex optimization problem aimed at identifying the statistics of forcing to the linearized flow equations in order to match available velocity statistics and complete unavailable data. The solution to this covariance completion problem is used to design a spatio-temporal filter that generates the identified forcing statistics. Our modeling and optimization framework is verified using time-dependent stochastic simulations. These confirm that second-order statistics of turbulent flows can indeed be reproduced by the linearized Navier-Stokes equations with colored-in-time stochastic forcing.

1. Introduction

Nonlinear dynamical models of turbulent flows that are based on the Navier-Stokes (NS) equations typically have a large number of degrees of freedom that makes them unsuitable for control synthesis. Several techniques have been proposed for obtaining low-dimensional models that preserve the essential dynamics; these include proper orthogonal decomposition (POD) (Berkooz *et al.* 1993), balanced POD (Rowley 2005), Koopman modes (Rowley *et al.* 2009; Mezić 2013), dynamic mode decomposition (Schmid 2010; Jovanović *et al.* 2014), low-order Galerkin models (Holmes *et al.* 1998), and resolvent modes (McKeon & Sharma 2010; Moarref *et al.* 2014). However, in all of these, control actuation significantly alters the identified modes, which introduces nontrivial challenges for control design (Noack *et al.* 2011).

In contrast, linearization of the NS equations around mean-velocity gives rise to models that are well-suited for analysis and synthesis using tools of modern robust control. Furthermore, when driven by white-in-time stochastic excitation, such models have been shown to qualitatively replicate structural features of wall-bounded shear flows (Farrell & Ioannou 1993; Bamieh & Dahleh 2001; Jovanović & Bamieh 2005; Moarref & Jovanović 2012). However, it has also been recognized that white-in-time stochastic forcing is too restrictive to reproduce all statistical features of the fluctuating velocity field (Jovanović & Bamieh 2001; Jovanović & Georgiou 2010). Building on Georgiou (2002*b*,*a*); Chen *et al.* (2013), we depart from white-in-time restriction and consider low-complexity dynamical models with colored-in-time excitations that account for partially available statistics. Such statistics may come from experimental measurements or direct numerical simulations (DNS). We utilize nuclear norm minimization to develop a framework for identifying the appropriate forcing into the linearized dynamics. Based on the solution to the covariance completion optimization problem, a spatio-temporal filter is designed to

† Department of Electrical and Computer Engineering, University of Minnesota

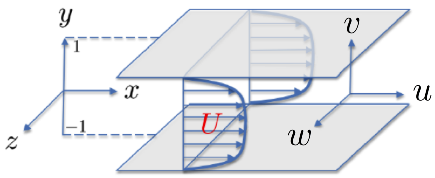

FIGURE 1. Geometry of a pressure-driven turbulent channel flow.

realize the colored-in-time forcing correlations. Our method provides a means for designing filters that not only explain the origin and directionality of input disturbances that generate the observed turbulent statistics, but also enable the realization of identified forcing structures via stochastic simulations.

Our report is organized as follows. In Section 2, we describe stochastically-forced linearization of the NS equations around turbulent mean velocity and characterize the structural constraints imposed on second-order statistics of the linearized flow equations. In Section 3, we formulate the covariance completion problem and outline procedure for designing spatio-temporal filters. In Section 4, we apply our modeling and optimization framework to a turbulent channel flow and verify our results using stochastic simulations. Finally, we summarize our developments in Section 5.

2. Problem formulation

The dynamics of incompressible Newtonian fluids are governed by the non-dimensional NS and continuity equations,

$$\begin{aligned} \mathbf{u}_t &= -(\mathbf{u}\cdot\nabla)\mathbf{u} \;-\; \nabla P \;+\; (1/\mathrm{Re}_\tau)\,\Delta\mathbf{u}, \\ 0 &= \nabla\cdot\mathbf{u}. \end{aligned}$$

We consider a pressure-driven turbulent channel, with geometry shown in Figure 1, where $\mathbf{u}$ is the velocity vector, P is the pressure, ∇ is the gradient, and $\Delta = \nabla\cdot\nabla$ is the Laplacian. The Reynolds number is defined as $\mathrm{Re}_\tau = u_\tau h/\nu$, where h is the channel half-height, $u_\tau = \sqrt{\tau_w/\rho}$ is the friction velocity, ν is the kinematic viscosity, τ_w is the wall-shear stress (averaged over horizontal directions and time), ρ is the fluid density, and t is time. In this formulation, spatial coordinates are non-dimensionalized by h, velocity by u_τ, time by h/u_τ, and pressure by ρu_τ^2.

We study the dynamics of infinitesimal fluctuations around a turbulent mean velocity, $\bar{\mathbf{u}} = (U(y), 0, 0)$, which implies translational invariance in the wall-parallel directions. Velocity fluctuations in the streamwise, x, wall-normal, y, and spanwise, z, directions are denoted by u, v, and w, and the linearized evolution model is given by

$$\begin{aligned} \dot{\boldsymbol{\psi}}(y,\mathbf{k},t) &= \mathbf{A}(\mathbf{k})\,\boldsymbol{\psi}(y,\mathbf{k},t) \;+\; \mathbf{f}(y,\mathbf{k},t) \\ \mathbf{v}(y,\mathbf{k},t) &= \mathbf{C}(\mathbf{k})\,\boldsymbol{\psi}(y,\mathbf{k},t). \end{aligned} \tag{2.1}$$

Here, $\mathbf{v} = [\,u \quad v \quad w\,]^T$ is the fluctuating velocity vector, $\mathbf{f}$ is a zero-mean stochastic forcing, and $\boldsymbol{\psi} = [\,v \;\; \eta\,]^T$ is the state vector with v and $\eta = \partial_z u - \partial_x w$ denoting the wall-normal velocity and vorticity fluctuations. The dynamical generator in Eq. (2.1) is the well-known Orr-Sommerfeld/Squire operator (around turbulent mean velocity $U(y)$) and the operator $\mathbf{C}(\mathbf{k})$ establishes a kinematic relation between $\boldsymbol{\psi}$ and $\mathbf{v}$. Both of these are parameterized by the wavenumber vector $\mathbf{k} = (k_x, k_z)$. A more detailed description of the operators in Eq. (2.1) can be found in Jovanović & Bamieh (2005).

Finite-dimensional approximations $A(\mathbf{k})$ and $C(\mathbf{k})$ of the operators $\mathbf{A}(\mathbf{k})$ and $\mathbf{C}(\mathbf{k})$ are obtained using a pseudospectral scheme with N Chebyshev collocation points in the wall-normal direction (Weideman & Reddy 2000), yielding

$$\begin{array}{rcl} \boldsymbol{\psi}_t(\mathbf{k},\, t) & = & A(\mathbf{k})\, \boldsymbol{\psi}(\mathbf{k},\, t) \;+\; \mathbf{f}(\mathbf{k},\, t), \\ \mathbf{v}(\mathbf{k},\, t) & = & C(\mathbf{k})\, \boldsymbol{\psi}(\mathbf{k},\, t), \end{array} \tag{2.2}$$

where $\boldsymbol{\psi}(\mathbf{k},\, t)$ and $\mathbf{v}(\mathbf{k},\, t)$ are complex vectors with $2N$ and $3N$ components.

2.1. *Second-order statistics of linearized NS equations*

At any $\mathbf{k}$, the steady-state covariance matrix of the velocity vector in Eq. (2.2) is given by

$$\Phi(\mathbf{k}) \;=\; \lim_{t\,\to\,\infty} \, \langle \mathbf{v}(\mathbf{k},\, t)\, \mathbf{v}^*(\mathbf{k},\, t) \rangle\,,$$

where $\mathbf{v}^*$ is the complex conjugate transpose of the vector $\mathbf{v}$, and $\langle \cdot \rangle$ is temporal ensemble averaging. The matrix $\Phi(\mathbf{k})$ contains information about second-order statistics of the fluctuating velocity field and it can be computed as

$$\Phi(\mathbf{k}) \;=\; C(\mathbf{k})\, X(\mathbf{k})\, C^*(\mathbf{k}),$$

where X is the steady-state covariance matrix of the state $\boldsymbol{\psi}$ in Eq. (2.2). When a stable system, given by Eq. (2.2), is driven by zero-mean white-in-time stochastic forcing,

$$\langle \mathbf{f}(\mathbf{k},\, t_1)\, \mathbf{f}^*(\mathbf{k},\, t_2) \rangle \;=\; Z(\mathbf{k})\, \delta(t_1 - t_2),$$

with covariance $Z(\mathbf{k}) = Z^*(\mathbf{k}) \succeq 0$, $X(\mathbf{k})$ and $Z(\mathbf{k})$ are related via the Lyapunov equation,

$$A(\mathbf{k})\, X(\mathbf{k}) \;+\; X(\mathbf{k})\, A^*(\mathbf{k}) \;=\; -Z(\mathbf{k}). \tag{2.3}$$

For homogeneous isotropic turbulence, Jovanović & Georgiou (2010) showed that the velocity covariance can be exactly matched using the linearized NS equations subject to white-in-time solenoidal forcing with appropriately selected second-order statistics. In turbulent channels, however, the matrix $Z(\mathbf{k}) = -(A(\mathbf{k})\, X(\mathbf{k}) + X(\mathbf{k})\, A^*(\mathbf{k}))$ fails to be negative semi-definite for numerically-generated covariances $X(\mathbf{k})$ of the state $\boldsymbol{\psi}$. Figure 2 shows the eigenvalues of $Z(\mathbf{k})$ in a channel flow with $\mathrm{Re}_\tau = 180$ and $\mathbf{k} = (2.5, 7)$. It can also be shown that there is no positive semi-definite completion of partially available flow statistics, which corresponds to system (2.2) driven by white-noise, and hence Z in Eq. (2.3) is sign indefinite. Thus, the second-order turbulent channel flow statistics cannot be reproduced by the linearized NS equations with white-in-time stochastic forcing. In this report, we depart from the white-in-time restriction and consider low-complexity dynamical models with colored-in-time excitations that successfully account for partially available statistics. Such statistics may come from experimental measurements or direct numerical simulations.

3. Completion of partially known statistics

Motivated by the necessity to account for turbulent flow correlations by models of low complexity, we next formulate the problem of completing partially available second-order statistics. The statistics of forcing are unknown and sought to explain the available correlations, and the complexity is quantified by the rank of the correlation structure of excitation sources. As shown by Chen *et al.* (2013), this provides a bound on the number of input channels and explains the directionality of input disturbances. While the system dynamics impose a linear constraint on admissible velocity correlations, such

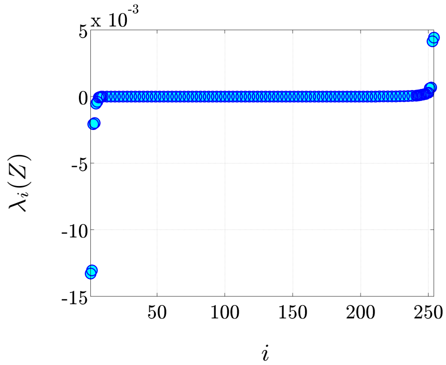

FIGURE 2. Negative eigenvalues of the forcing covariance resulting from Eq. (2.3), in a channel flow with $\mathrm{Re}_\tau = 180$ and $\mathbf{k} = (2.5, 7)$, indicate that turbulent velocity covariances cannot be reproduced by the linearized NS equations with white-in-time stochastic forcing.

an inverse problem admits many solutions for the forcing correlations. We use nuclear norm minimization to obtain correlation structures of low complexity.

3.1. *Covariance completion problem*

The covariance completion problem can be formulated as

$$\begin{array}{ll} \underset{X,\,Z}{\text{minimize}} & \|Z\|_* \\ \text{subject to} & AX + XA^* + Z = 0 \\ & (C\,X\,C^*) \circ E - G = 0 \\ & X \succeq 0. \end{array} \tag{CC}$$

Here, matrices A, C, and G denote problem data, and Hermitian matrices X and Z are optimization variables. Entries of G represent partially available second-order statistics, e.g., one-point correlations in the wall-normal direction; see Figure 3 for an illustration. These wave-number parameterized one-point velocity correlations have been obtained in direct numerical simulations (Del Alamo & Jimenez 2003; Del Alamo *et al.* 2004). The symbol $\circ$ denotes an element-wise matrix multiplication and the matrix E is the structural identity,

$$E_{ij} = \begin{cases} 1, & G_{ij} \text{ is available} \\ 0, & G_{ij} \text{ is unavailable.} \end{cases}$$

The constraint set in (CC) represents the intersection of the positive semi-definite cone and two linear subspaces: the Lyapunov-like constraint, which is imposed by the linearized dynamics, and the linear constraint, which relates X with the available statistics G.

In (CC), the nuclear norm, i.e., the sum of singular values of a matrix,

$$\|Z\|_* := \sum_i \sigma_i(Z),$$

is used as a proxy for rank minimization (Fazel 2002; Recht *et al.* 2010). The convexity of the optimization problem (CC) follows from the convexity of the nuclear norm objective and the convexity of the constraint set (Boyd & Vandenberghe 2004). This optimization problem can be formulated as a semi-definite program (SDP) whose globally optimal solution can be found efficiently using standard SDP solvers for small problems. For

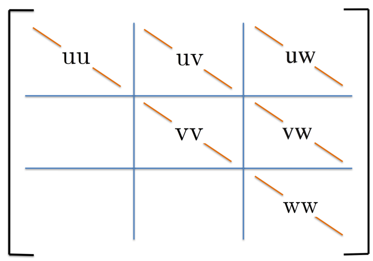

FIGURE 3. Structure of the matrix G in (CC). At each pair of horizontal wavenumbers (k_x, k_z), available second-order statistics are given by diagonal entries of the blocks in the velocity covariance matrix.

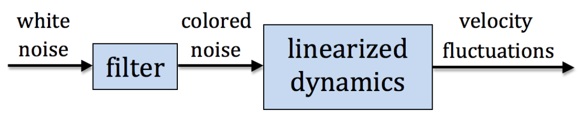

FIGURE 4. Spatio-temporal filter, given by Eq. (3.1), is designed to reproduce partially available second-order statistics of turbulent channel flow using stochastically forced linearized NS equations (2.2).

large problems, which are typical in fluid mechanics, we have been developing customized algorithms to solve (CC) and related variants of the covariance completion problem (Lin *et al.* 2013; Zare *et al.* 2014, 2015).

3.2. *Filter design*

Matrices $X(\mathbf{k})$ and $Z(\mathbf{k})$ that solve completion problem (CC) can be used to design a filter,

$$\begin{array}{rcl} \dot{\boldsymbol{\phi}}(\mathbf{k},t) & = & A_F(\mathbf{k})\,\boldsymbol{\phi}(\mathbf{k},t) \;+\; B_F(\mathbf{k})\,\mathbf{w}(\mathbf{k},t), \\ \mathbf{f}(\mathbf{k},t) & = & C_F(\mathbf{k})\,\boldsymbol{\phi}(\mathbf{k},t) \;+\; D_F(\mathbf{k})\,\mathbf{w}(\mathbf{k},t), \end{array} \tag{3.1}$$

that generates the colored-in-time forcing $\mathbf{f}(\mathbf{k},t)$ to the linearized NS equations (2.2). This filter has the same number of degrees of freedom as the linearized equations given by Eq. (2.2), and it is driven by white-in-time stochastic forcing $\mathbf{w}(\mathbf{k},t)$ with covariance $W(\mathbf{k})$. Here, $W(\mathbf{k})$ is any positive-definite matrix and the power spectrum of $\mathbf{f}(\mathbf{k},t)$ is determined by

$$\Pi_{ff}(\mathbf{k},\omega) \;=\; F(\mathbf{k},\omega)\,W(\mathbf{k})\,F^*(\mathbf{k},\omega),$$

where $F(\mathbf{k},\omega)$ is the transfer function of the filter,

$$F \;=\; C_F\,(\mathrm{i}\omega I \,-\, A_F)^{-1}\,B_F \;+\; D_F,$$

and

$$\begin{array}{rcl} A_F & = & A \,+\, B_F\,C_F^{-1}, \\ C_F & = & (-0.5\,W\,B_F^* \,+\, H^*)\,X^{-1}, \\ D_F & = & I. \end{array}$$

The input matrix B_F and the matrix H are obtained by factorizing the matrix Z into $B_F H^* + H B_F^*$; see Chen *et al.* (2013) for details.

By augmenting the linearized NS equations (2.2) with the filter dynamics, given by Eq. (3.1), we can recover partially available second-order turbulent flow statistics via a linear model. In Section 4, we use stochastic simulations of the interconnection shown in Figure 4 to demonstrate the utility of our approach.

4. Results and discussion

We next use partially available second-order statistics resulting from DNS of a turbulent channel flow with $\mathrm{Re}_\tau = 180$ (Del Alamo & Jimenez 2003; Del Alamo *et al.* 2004) to demonstrate the utility of our approach. As illustrated in Figure 3, the available statistics for our optimization problem are given by one-point velocity correlations at various wavenumber pairs. Differential operators in the wall-normal direction are approximated using $N = 51$ collocation points. In the wall-parallel directions a Fourier transform with 80×81 grid points is applied, $k_x \in [0.01, 42.5]$ and $k_z \in [0.01, 84.5]$, with the largest values of k_x and k_z being equal to those used in DNS (Del Alamo & Jimenez 2003; Del Alamo *et al.* 2004). Building on the solution to the covariance completion problem (CC) at $\mathbf{k} = (2.5, 7)$, we use stochastic simulations of the linear system (2.2)-(3.1), to show that our approach can indeed reproduce available statistical signatures of the turbulent flow. We also demonstrate that completed two-point velocity correlations compare favorably with two-point correlations resulting from DNS.

4.1. *Solution to the covariance completion problem*

Figure 5 shows that the solution to the optimization problem (CC) exactly reproduces available one-point velocity correlations of turbulent channel flow at various horizontal wavenumbers. Figures 5(a) and 5(c) display perfect matching of all one-point velocity correlations resulting from integration over horizontal wavenumbers. Because problem (CC) is not feasible for $Z \succeq 0$, perfect matching of turbulent flow statistics cannot be achieved with white-in-time stochastic forcing. Figures 5(b) and 5(d) demonstrate perfect recovery of the pre-multiplied, one dimensional energy spectrum of streamwise velocity fluctuations. Results are given in terms of streamwise (Figure 5(b)) and spanwise (Figure 5(d)) wavelengths and the wall-normal coordinate, all in inner (viscous) units. Color plots and contour lines show energy spectra resulting from DNS (Del Alamo & Jimenez 2003; Del Alamo *et al.* 2004) and from our optimization framework, respectively. These are in perfect agreement. Similar matching is observed for the wall-normal and spanwise velocity spectra as well as for the Reynolds stress co-spectrum (not shown owing to page limitations).

Figures 6(a) and 6(c) respectively display covariance matrices of the streamwise and spanwise velocity components resulting from DNS (Del Alamo & Jimenez 2003; Del Alamo *et al.* 2004). Although only diagonal elements of these matrices (denoted by black lines) were used in optimization problem (CC), Figures 6(b) and 6(d) display good recovery of unavailable two-point correlations. Thus, the stochastically driven linearized model in conjunction with the proposed optimization framework can be used to complete unavailable statistical signatures of the fluctuating velocity field.

4.2. *Verification in linear stochastic simulations*

We next conduct stochastic simulations of the linearized flow equations and compare them with DNS results. A filter that generates colored-in-time forcing $\mathbf{f}$ to the linearized NS Eqs. (2.2) is designed using the solution to (CC) at $R_\tau = 180$ and $\mathbf{k} = (2.5, 7)$. This filter is driven by the white-in-time Gaussian process $\mathbf{w}$ with zero mean and unit variance. Our simulations not only confirm that our optimization framework can recover available turbulent flow statistics by identifying forcing models for the stochastically driven linearized NS equations, but also illustrate how our results should be interpreted when

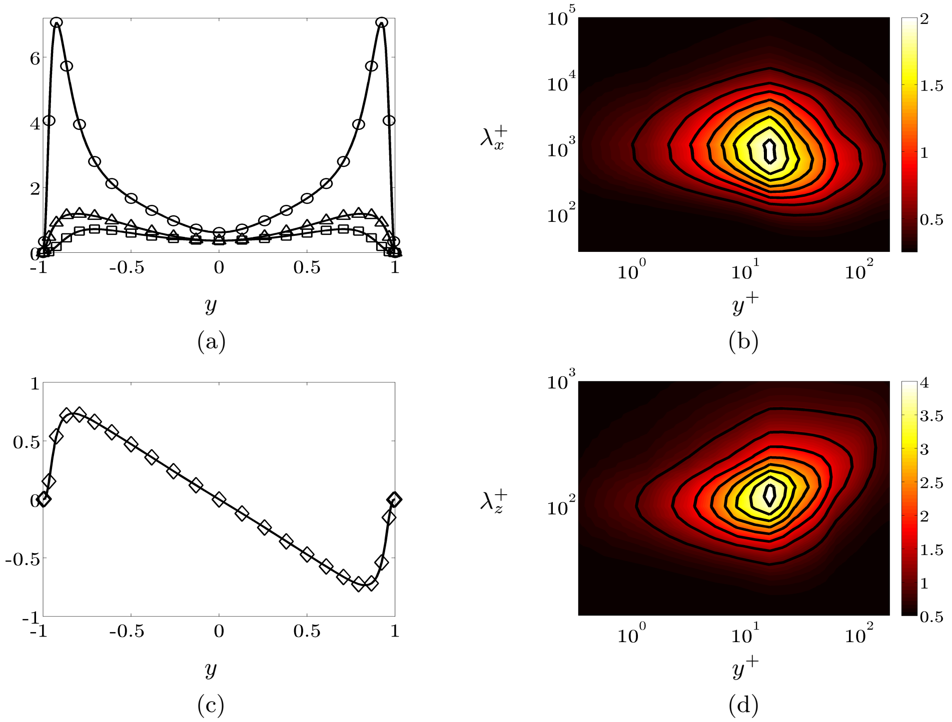

FIGURE 5. Turbulent channel flow with $\mathrm{Re}_\tau = 180$. (a) Correlation profiles of normal and (c) shear stresses resulting from DNS (Del Alamo & Jimenez 2003; Del Alamo *et al.* 2004) (–) and from the solution to (CC); uu (○), vv (□), ww (△), $-uv$ (◇). Pre-multiplied, one-dimensional energy spectrum of streamwise velocity fluctuations in terms of streamwise (b) and spanwise (d) wavelengths. Color plots: spectra resulting from DNS. Contour lines: spectra resulting from the solution to (CC).

compared with DNS or experiments. Proper comparison requires ensemble-averaging, rather than comparison at the level of individual stochastic simulations. We have thus conducted ten different stochastic simulations of the linear system Eqs. (2.2)-(3.1).

Figure 7(a) shows the time evolution of the energy (variance) of velocity fluctuations, for ten realizations of stochastic forcing to Eq. (3.1). The variance averaged over all simulations is given by the thick black line. Even though the responses of individual simulations differ from each other, the average of ten sample sets asymptotically approaches the correct value of turbulent kinetic energy in the statistical steady-state, $\mathrm{trace}\,(\Phi)$. Figures 7(b) and 7(c) respectively display the normal stress profiles in the streamwise direction and the shear stress profiles resulting from DNS and from linear stochastic simulations. We see that the averaged output of ten stochastic simulations agrees well with DNS results. This close agreement can be further improved by running additional linear simulations and by increasing the total simulation times.

5. Concluding remarks

We have developed a convex optimization framework to account for partially available second-order statistics of turbulent flows by linear stochastic models of low complexity. Available statistics originate from experiments or direct numerical simulations and consist

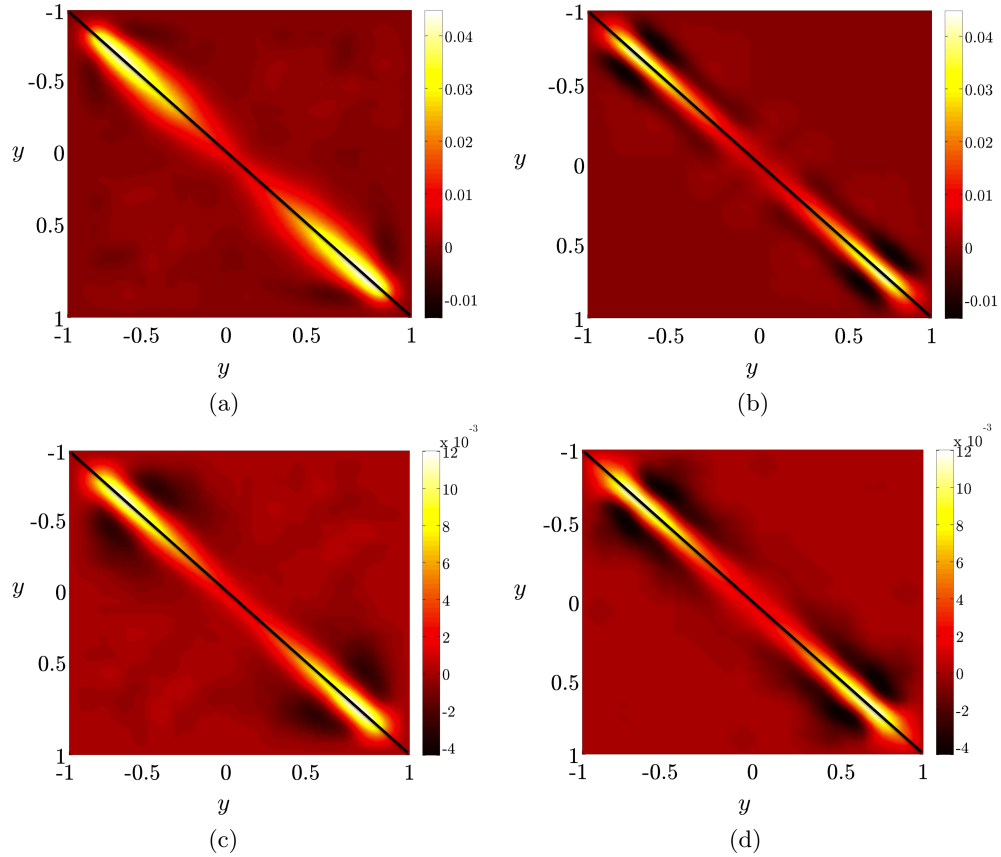

FIGURE 6. Covariance matrices resulting from (a, c) DNS (Del Alamo & Jimenez 2003; Del Alamo *et al.* 2004), and (b, d) linear stochastic simulations. (a, b) Streamwise and (c, d) spanwise velocity correlations in wall-normal coordinates at $\mathbf{k} = (2.5, 7)$. Available one-point correlation profiles represent diagonal entries of these matrices and are shown by diagonal black lines.

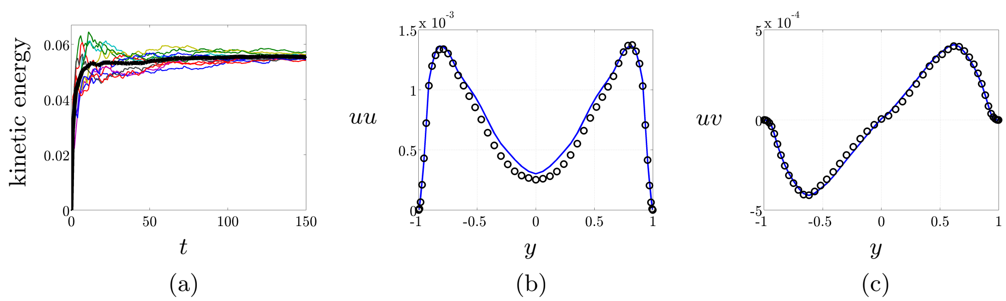

FIGURE 7. (a) Time evolution of fluctuation's kinetic energy for ten realizations of forcing to the linearized model, given by Eqs. (2.2)-(3.1), with $\mathbf{k} = (2.5, 7)$; the energy averaged over all simulations is shown by thick black line. (b) Normal stress profiles in the streamwise direction and (c) shear stress profiles resulting from DNS (–) and stochastic linear simulations (○).

of partially known velocity correlations. We utilize nuclear norm minimization as a basis for completing unavailable statistical signatures and to identify forcing models that drive the linearized flow equations. The complexity of the model is quantified by the rank of the correlation structure of excitation sources, which provides a bound on the number of input channels to the linearized NS equations.

We have solved the covariance completion problem for turbulent channel flow with $\mathrm{Re}_\tau = 180$ at all horizontal wavenumbers. Using the solution to the convex optimization problem (CC), we have designed a spatio-temporal filter that generates the colored-in-time forcing to the linearized NS equations. This filter has the same number of degrees of freedom as the finite-dimensional approximation of the linearized model given by Eq. (2.2). The power spectrum of the forcing has been obtained to match available one-point correlations and approximate unavailable two-point correlations. In addition, we have verified our modeling and optimization developments in stochastic simulations. These time-dependent simulations have confirmed that DNS-based second-order statistics can indeed be reproduced by low-complexity linear dynamical models with colored-in-time stochastic forcing.

During the summer program, we also used spanwise wall-oscillations to demonstrate the utility of our approach in model-based control design. For a turbulent channel flow with $\mathrm{Re}_\tau = 180$, we have identified the period of oscillations that yields largest drag reduction via input-output analysis of our stochastically driven linear models.

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