Spectral Analysis and Analytic Interpolation

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Spectral Analysis based on covariance statistics
amounts to
an analytic interpolation problem

Applications to high resolution analysis

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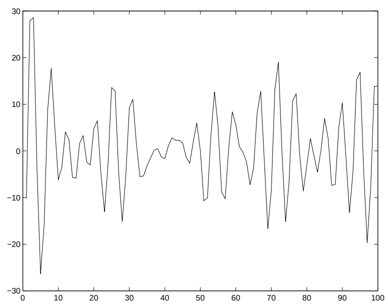
- **Signal analysis/filtering**

Problem: Given time-series data:

$$\{u_0, u_1, u_2, \ldots, u_{N-1}\}$$



determine the **power spectrum of** y,
i.e., periodicities and "color"

Earlier methodologies:

- ∘ Periodogram, FFT
- ∘ Model based (ARMA,....)
- ∘ Modern nonlinear (Maximum-entropy, maximum-likelihood,...)

  Levinson, Burg, Pisarenko, Capon, ...

Earlier "modern nonlinear methods":

- utilize (ordinary) covariance statistics
- comprise of a variety of "algorithms": Max-entropy, MUSIC, ESPRIT, etc.
- their theory is based on *orthogonal polynomials*, on the *moment problem*, and on analytic interpolation of the *Caratheodory type*

Development presented herein:

- a natural extension of modern nonlinear methods
- utilizes generalized covariance statistics (state-covariance data)
- the theory is based on generalized analytic interpolation (similar to Sarason and H_∞-matrix Nehari)
- gives rise to analogs of most prior "algorithms" (Max-entropy, MUSIC, etc.)

Earlier collaborative work with Chris Byrnes and Anders Lindquist led to a connection between analytic interpolation and spectral analysis in a special case, in the context of Nevanlinna-Pick interpolation using covariance data from a special filter bank, and showed its use for improved resolution.

$$u_k = \int_{-\pi}^{\pi} e^{jk\theta} dv(\theta)$$

$dv(\theta)$	"amplitude of complex sinusoids"
$E\{dv(\theta)^2\} \sim \rho(\theta)d\theta$	"energy density across frequencies"

Covariance statistics & spectral density

$$c_k = E\{u_t u_{t+k}\}$$

$$\Updownarrow$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-jk\theta} \underbrace{\rho(\theta)d\theta}_{d\mu(\theta)}$$

$c_0, c_1, c_2, \ldots$ autocorrelation sequence

$$\Updownarrow$$

$f(z) = \frac{1}{2}c_0 + c_1 z + c_2 z^2 + \ldots$ "positive-real" function

$$\Updownarrow$$

$$\begin{aligned} \rho(\theta) &= \mathrm{Re}(f(e^{j\theta})) \\ &= \ldots c_2 e^{-2j\theta} + c_1 e^{-j\theta} + c_0 + c_1 e^{j\theta} + c_2 e^{2j\theta} + \ldots \geq 0 \end{aligned}$$

Given finite data $c_0, \ldots, c_N$:

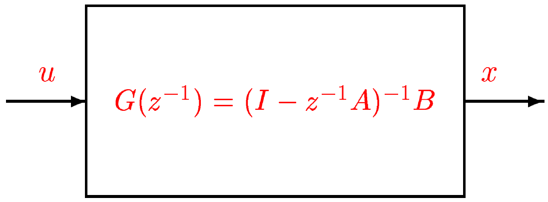
- PARAMETRIZATION OF SOLUTIONS:

$$\rho = \mathrm{Re}\left(\frac{A+BQ}{C+DQ}\right) \text{ with } Q \text{ a "free" parameter}$$

- SPECIFIC CHOICES OF Q LEAD TO DIFFERENT "METHODS":

Maximum Entropy/central solution
maximizes $\int \log(\rho(\theta))d\theta$

Pisarenko/MUSIC/ESPRIT
. . . etc.



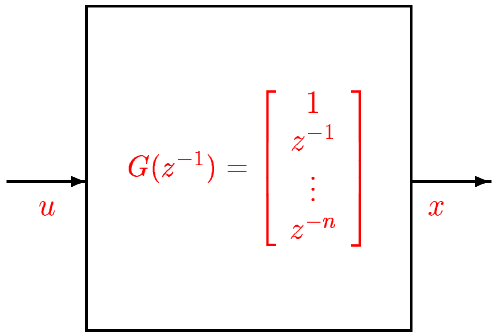
$$x_k = Ax_{k-1} + Bu_k$$

Given knowledge of A, B
what is the structure of the state-covariance matrix $\Sigma := E\{xx^*\}$*?*
i.e., given a $\Sigma \geq 0$,
how can I tell whether it qualifies a state-covariance
of the given system for a suitable stochastic input or not?

What are all spectra consistent with Σ*?*
i.e., given a legitimate-state covariance Σ for the given system,
what can I say about the spectrum of the stochastic input
which generated Σ?

$$\Sigma = \int_{-\pi}^{\pi} \left(G(e^{j\theta}) \frac{d\mu(\theta)}{2\pi} G(e^{j\theta})^* \right)$$

I.e., we want to study the correspondence $\mu(\theta) \to \Sigma$
characterize the range
and pre-images $\mu(\theta)$ for given Σ in the range



- If $G(z^{-1})$ is the "steering vector" of a uniform array, i.e.,

$$x_k = \begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ uk - n \end{bmatrix}$$

THEN

$$\Sigma = \begin{bmatrix} c_0 & c_1 & \dots & c_n \\ c_1^* & c_0 & \dots & c_{n-1} \\ \vdots & \vdots & & \vdots \\ c_n^* & c_{n-1}^* & \dots & c_0 \end{bmatrix} \text{ is (block) Toeplitz}$$

In this case: answers to earlier questions are classical

$\Sigma \geq 0$ is a state covariance of the above iff it is Toeplitz, etc.

- "Steering matrix" $G(z^{-1}) = (I - z^{-1}A)^{-1}B$
  nontrivial dynamics & many inputs

$$\begin{aligned}\Sigma &= \int_{-\pi}^{\pi} \left(G(e^{j\theta}) \frac{d\mu(\theta)}{2\pi} G(e^{j\theta})^* \right) \\ &\vdots \\ &= BH + H^*B^* + A\Sigma A^*\end{aligned}$$

Theorem: **With (A, B) controllable pair, A stable, $\Sigma \geq 0$:**

Σ is a covariance of $x_k = Ax_{k-1} + Bu_k$

$$\Leftrightarrow$$

$\Sigma = BH + H^*B^* + A\Sigma A^*$ has a solution H

$$\Leftrightarrow$$

$$\text{rank} \begin{bmatrix} \Sigma - A\Sigma A^* & B \\ B^* & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$$

"$\Rightarrow$" above is algebraic (easy),
"$\Leftarrow$" requires a construction (explicit in the next Theorem)
"$\Leftrightarrow$" is algebraic

Let Σ state-covariance and H as in previous theorem:

Theorem: All input spectra consistent with Σ are

$$d\mu(\theta) \sim \lim_{r\to 1} \mathrm{Re}\left(F(re^{j\theta})\right) d\theta$$

where

$$F(\lambda) = F_0(\lambda) + Q(\lambda)V(\lambda) \ \text{ is positive-real}$$

with data:
$F_0(\lambda) = H(I - \lambda A)^{-1}B$,
A, B **normalized so** $AA^* + BB^* = I$, **and** C, D **selected so that**
$V = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **is inner**

Theorem: There exists an LFT parametrization of all solutions F:

$$F(\lambda) = F_1(\lambda)^{-1}F_2(\lambda)$$

with

$$\left[\ F_1(\lambda),\ \ F_2(\lambda)\ \right] = \left[\ I,\ \ Q(\lambda)\ \right] K_i(\lambda)$$

where

$Q(\lambda)$ **is arbitrary positive-real and** K_i **a suitable J-inner factor**

Theorem: Define the entropy functional

$$\mathbb{I}(\rho) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\det(\rho(\theta))d\theta$$

Then the unique input spectral density consistent with Σ which maximizes $\mathbb{I}$ is

$$\rho_o(\theta) := \left(\Phi(e^{j\theta})^{-1}\Omega^{-1}\left(\Phi(e^{j\theta})^{-1}\right)^*\right)$$

where

$$\Phi(\lambda) := (B^*\Sigma^{-1}B)^{-1}B^*\Sigma^{-1}(I-\lambda A)^{-1}B,$$

and

$$\Omega := (B^*\Sigma^{-1}B)^{-1}.$$

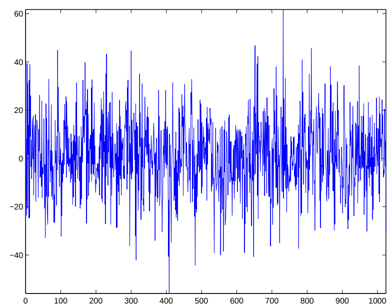
- If $\Phi(\lambda) = I - \lambda P_1 - \lambda^2 P_2 + \ldots$

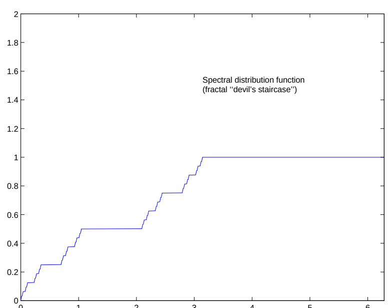
$$\hat{u}_k = P_1 u_{k-1} + P_2 u_{k-2} + \ldots,$$

has minimal variance Ω, in fact it is a **min-max optimal predictor** i.e., minimizes the prediction error uniformly over spectra consistent with Σ

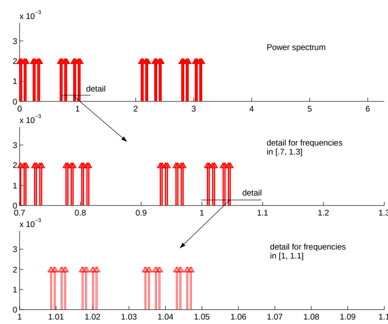
- $\Phi(\lambda)$ is the matrix analog of Szegö-Geronimus orthogonal polynomials of the 1st kind...

Fractal spectrum limits to resolution?

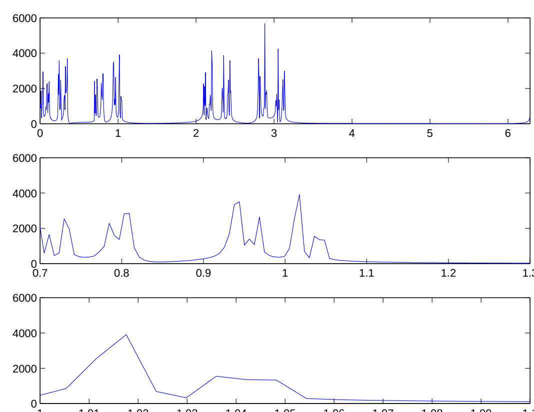




$\sigma(\theta)$

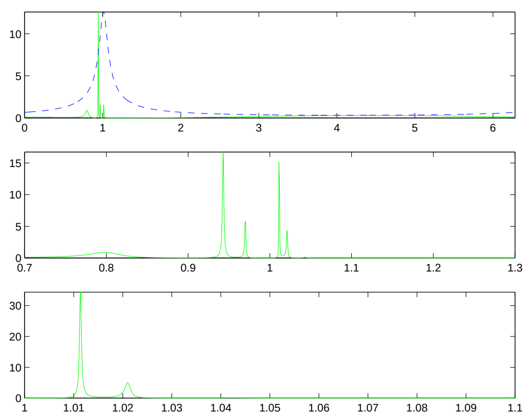


$\rho(\theta) = \dot{\sigma}(\theta)$



reconstructed via periodogram

- The spectrum is self-similar.
- Over the range [1,1.02] Fourier methods can show only one lump.



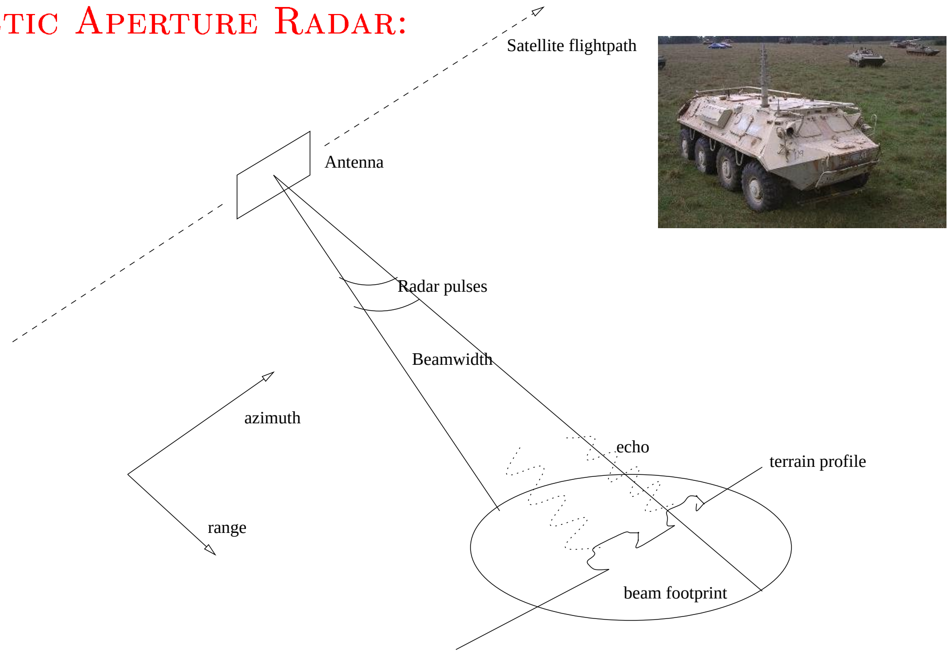
— Maximum entropy spectra
- - - Focusing filter

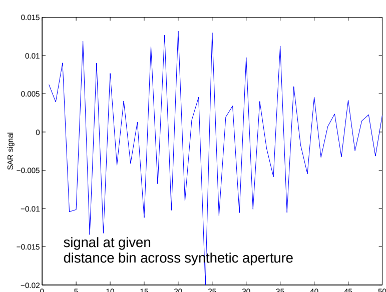
• By a proper selection of A, B, yielding statistics Σ, and using the maximum entropy interpolant we can easily resolve two lumps in the range [1,1.02]. (At the expense of resolution elsewhere.)

• The process can be repeated by using different filters to resolve different frequency bands (as in subsequent applications).

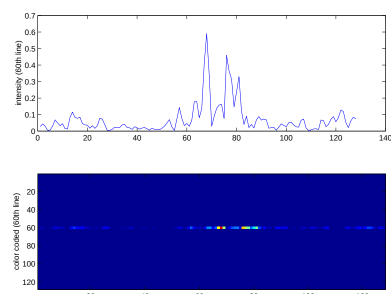
• When higher resolution is attempted the variability of the estimate increases.

Synthetic Aperture Radar:

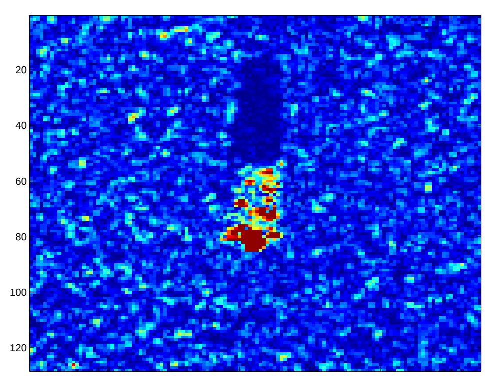




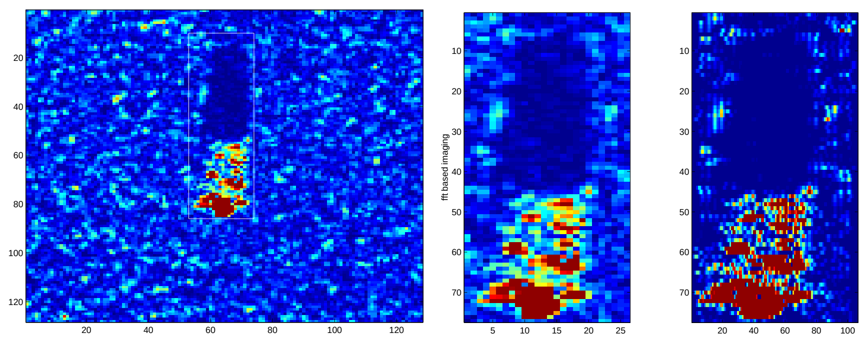
FFT
$\Rightarrow$



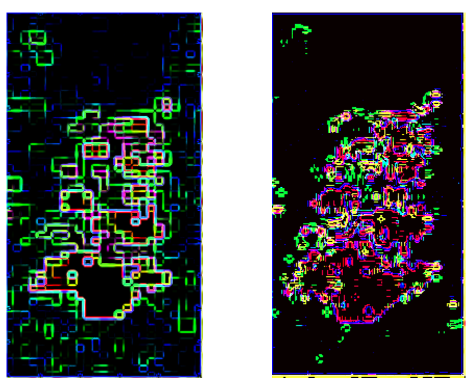
Line by line produces:



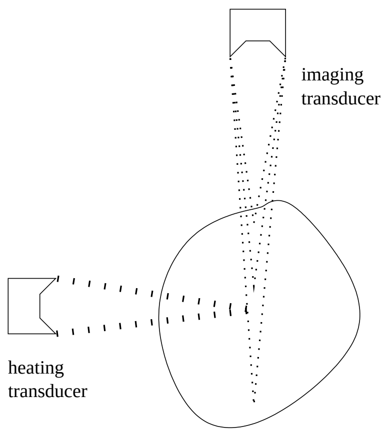
1. Armored Personel Carrier
2. MSTAR image of APC, detail, and "high resolution" reconstruction



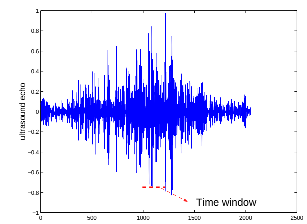
3. Using edge detection (MSTAR-image vs. high resolution)

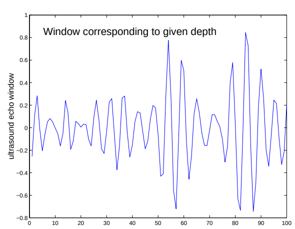


Ultrasound – Noninvasive temperature sensing

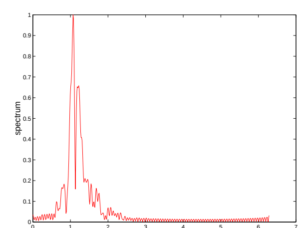


Ultrasound echo:



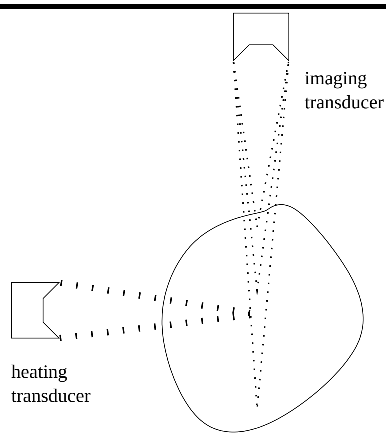


FFT
$\Rightarrow$

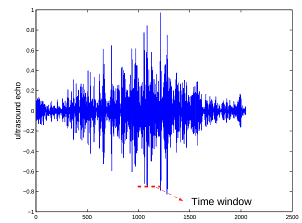


harmonic shift ⇒ temperature profile

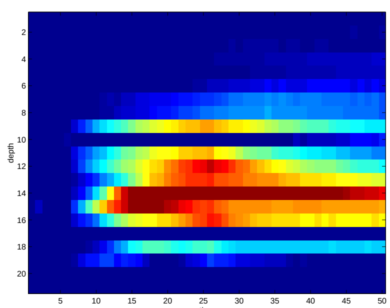
- Portions of the echo represent reflection from different depths.
- The harmonics in depth-segments of the echo shift according to local temperature
- High resolution is required to highlight accurately the temperature profile so as use such technologies for therapeutic purposes (e.g., heat-treatment of cancer)

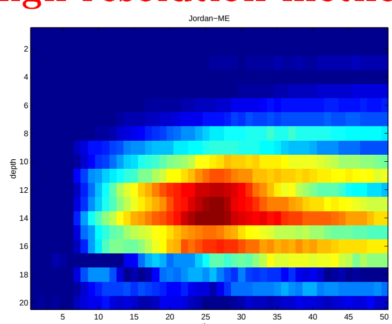


Ultrasound echo:

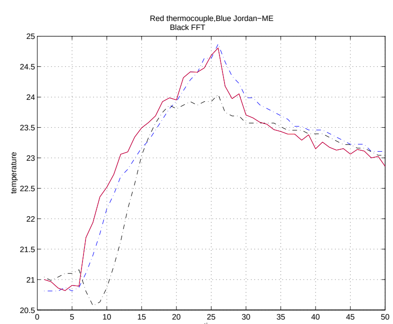


periodogram analysis vs. high resolution methods





Color-coded temperature at different depths vs. time



Comparison with thermocouple
(**thermocouple**, **periodogram**, **high resolution**)

- **Collaboration: E. Ebbini**

Summary

- state covariance statistics $\sim$ generalized analytic interpolation
- high resolution, applications

Questions and on-going research program:

- ¿how can we quantify resolution?
- tradeoffs between variance and resolution
  seeking an "H_∞-like paradigm"
- spacio-temporal dynamics and non-uniform arrays
- applications: SAR, medical imaging, polarimetry

Matlab code and references at:

`http://www.ece.umn.edu/users/georgiou`