Some geometric ideas for feature enhancement of diffusion tensor fields

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Abstract— **Diffusion Tensor Tomography generates a 3-dimensional 2-tensor field that encapsulates properties of probed matter. We present two complementing ideas that may be used to enhance and highlight geometric features that are present. The first is based on Ricci flow and can be understood as a *nonlinear bandpass filtering technique* that takes into account directionality of the spectral content. More specifically, we view the data as a Riemannian metric and, in manner reminiscent to reversing the heat equation, we regularize the Ricci flow so as to taper off the growth of the higher-frequency speckle-type of irregularities. The second approach, in which we again view data as defining a Riemannian structure, relies on averaging nearby values of the tensor field by weighing the summands in a manner which is inversely proportional to their corresponding distances. The effect of this particular averaging is to enhance *consensus* among neighboring cells, regarding the principle directions and the values of the corresponding eigenvalues of the tensor field. This consensus is amplified along directions where distances in the Riemannian metric are short.**

Index Terms— **Ricci flow, Riemannian geometry, nonlinear diffusion.**

I. Introduction and problem statement

Diffusion tensor imaging (DTI) is a magnetic resonance technique that allows mapping the flow and diffusion of, mainly, water molecules in tissue. Thereby, it provides valuable information about fibers and other microstructures that shape the flow, and has an even greater potential to revolutionize our ability to study abnormalities in white matter brain microstructure as well as provide models for brain connectivity. The data produced encodes the intensity and directionality of flow, e.g., in the form of an ellipsoid at every unit volume element (voxel). The data can be viewed as either a manifold of multivariable normal distributions or, alternatively, as a Riemannian metric. The latter representation allows geometric techniques and insights on how to process data and extract features. The present work is along lines of a series of earlier studies with a similar point of view, see e.g., [1], [2] and the references therein.

In the present work we consider the data as a position-dependent positive-definite quadratic form, i.e., as a Riemannian metric. The line-up of principle directions of this matrix, across voxels, may signify the presence of a fiber whereas abrupt discrepancies across a front may signify a membrane. Thus, information about tissue microstructures is encoded in the spacial correlation between the values of the metric. On the other hand, speckle fluctuations of the parameters that specify principle directions and their corresponding eigenvalues, are the inevitable consequence of measurement noise. Naturally, at places, noise masks key features and limits resolution. Thus, the problem we wish to address is on how to suppress noise while maintaining structural information in the tensorial field that is available to us. We pursue and study to complementing approaches that we outline next.

The first approach is based on the Ricci flow [3]. The Ricci flow is a geometric (non-linear) evolution on a Riemannian manifold that deforms the metric and, for manifolds with positive Ricci curvature, smooths out irregularities. Thus, formally, it is analogous to the heat equation that smooths out irregularities of, e.g., a temperature distribution. In this sense, it represents a low-pass filtering for the 3-dimensional 2-tensorial field (in continuous or a discrete space). Depending on the quality of the data, our aim may be to either smooth out irregularities, or to amplify and highlight structural features. Here, we focus on the latter. To this end, we reverse time in the Ricci flow. In a way that echoes solving backwards heat equation, the flow rapidly amplifies "high frequency" spacial irregularities. The backwards heat equation, studied for inverse heat conduction problems, requires some form of regularization [4]. Likewise, for the backwards Ricci flow we introduce a suitable regularization that tapers the fastest modes. The result appears to produce a higher quality contrast and accentuate structures that are dominant in the data.

The second approach is again based on interpreting the data as a Riemannian metric. In this, we estimate distance between neighboring voxels in the induced geometry (by the data/metric itself). We then average local values of the metric with a weighing coefficient which is inversely proportional to the distance between voxels. The process represents a geometric diffusion that allows nearby voxels to reach some form of consensus more easily than ones that are far away. As a result, dominant structures such as fibers, where lines of voxels share similar values for the metric, and where their corresponding dominant eigenvectors are lined up with the fiber, are made consistent in that that small irregularities quickly dissipate. Thus, again, the approach represents a nonlinear averaging/filtering that is designed to

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accentuate correlations between nearby voxels. The method is exceptionally fast. The result will also be exemplified on case studies on white brain matter, and appears to produce consistently the desired effect.

Either of the methods we present, can be seen as an anisotropic diffusion of the tensorial field itself. The anisotropy is intended to preserve the key features of the (already anisotropic) tensorial field. Thus, the flow (in the first method) and averaging (in the second) are intended to respect the underlying anisotropy of the field.

The paper is structured as follows. In Section II we describe notations and preliminaries. Section III gives theoretical details of techniques for feature enhancement and smoothing in the tensor fields. Section IV gives algorithmic form of the techniques discussed in Section III. While in Section V, we show examples by implementing the suggested methods to synthetically generated tensor fields as well as on MRI data.

II. Notations & Preliminaries

Our methods are based on the concept of Riemannian Geometry. We shall follow standard notations in [5], [10] and give an overview on the concepts we need.

A Riemannian metric on a smooth manifold is an inner product g on the tangent space at each point, varying smoothly from point to point. Riemannian manifolds are smooth manifolds equipped with a Riemannian metric. Locally, in a neighborhood of any point, a Riemannian manifold can be viewed as a weighted Euclidean space with weight given by the Riemannian metric g.

To define derivatives of vector fields on a Riemannian manifold, the concept of connection is introduced. Intuitively, it provides a way to "connect" nearby tangent spaces. The Levi-Civita [6] connection ∇ is a special connection that is symmetric and compatible with g. It is specified by the Christoffel symbols Γ.

Another key ingredient of Riemannian geometry is curvature. There are several different notions of curvature like Sectional curvature, Ricci curvature and Scalar curvature. The one we need in this paper is Ricci curvature [7], which we denote by R. It is a 2-tensor field on the manifold uniquely defined by the Riemannian metric g.

For our purpose, we are interested in a 3 dimensional Riemannian manifold. More specifically, we view the DTI as a 3 dimensional manifold where the diffusion tensor

$$g = [g_{ij}]_{i,j=1}^{3} \;\in\; S_{++}^{3}$$

at each point $x = (x_1, x_2, x_3)$ (or its inverse) is treated as the Riemannian metric at that point. Here S_{++}^{3} denotes the space of 3 by 3 positive definite matrices.

In DTI, one important quantity is *Fractional Anisotropy (FA)*. For a positive definite 3×3 symmetric matrix, with $\lambda_1, \lambda_2, \lambda_3$ as its eigenvalues arranged in descending order, FA is defined by

$$\frac{\sqrt{(\lambda_1-\lambda_2)^2+(\lambda_2-\lambda_3)^2+(\lambda_1-\lambda_3)^2}}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}. \tag{1}$$

FA captures the structure information of a diffusion tensor. A larger FA means that the corresponding diffusion tensor is more likely to be a useful structure.

III. Smoothing and Feature Enhancement Techniques

We view the DTI as a 3 dimensional Riemannian manifold with metric specified by the diffusion tensors. From this point of view, smoothing the diffusion tensors is equivalent to smoothing the metric of the underlying manifold. Two related smoothing techniques are presented here, first based on the idea of Ricci flow and second based on anisotropic smoothing of the tensor field.

A. Ricci Flow for Tensor Fields

Given the Riemannian manifold with metric tensor $g = [g_{ij}]$ specified by the diffusion tensors, we can compute the Ricci curvature

$$R = [R_{ij}]_{i,j=1}^{3}.$$

Both the Riemannian metric g and the Ricci curvature R are 2-tensor fields on the manifold. The idea of Ricci flow [7] is to adjust the metric g using the Ricci curvature R to smooth out irregularities in the metric. Formally, the Ricci flow is defined by the nonlinear evolution equation [7], [8]:

$$\partial_t g_{ij} = -2R_{ij}. \tag{2}$$

Now we outline the steps to compute R from g. First we compute the inverse of the metric g, this is given by

$$[g^{ij}] = [g_{ij}]^{-1}. \tag{3}$$

With $[g^{ij}]$ and $[g_{ij}]$ we can the calculate the Christoffel symbols [9] as:

$$\Gamma_{ij}^{k} = \sum_{\ell=1}^{3} \frac{1}{2} g^{kl}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) \tag{4}$$

for all $i, j, k = 1, 2, 3$. Here ∂_i denotes the directional derivative in x_i direction. In matrix form, (4) can be written as:

$$\begin{bmatrix} \Gamma_{ij}^1 \\ \Gamma_{ij}^2 \\ \Gamma_{ij}^3 \end{bmatrix} = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} \begin{bmatrix} \partial_i g_{j1} + \partial_j g_{i1} - \partial_1 g_{ij} \\ \partial_i g_{j2} + \partial_j g_{i2} - \partial_2 g_{ij} \\ \partial_i g_{j3} + \partial_j g_{i3} - \partial_3 g_{ij} \end{bmatrix} \tag{5}$$

for all $i, j = 1, 2, 3$. After obtaining the Christoffel symbols, we can readily compute the Ricci Curvature Tensor R as

$$R_{\alpha\beta} = \sum_{\rho,\lambda=1}^{3} \partial_\rho \Gamma_{\beta\alpha}^{\rho} - \partial_\beta \Gamma_{\rho\alpha}^{\rho} + \Gamma_{\rho\lambda}^{\rho}\Gamma_{\beta\alpha}^{\lambda} - \Gamma_{\beta\lambda}^{\rho}\Gamma_{\rho\alpha}^{\lambda} \tag{6}$$

for all $\alpha, \beta = 1, 2, 3$.

B. *Anisotropic Smoothing for Tensor Fields*

In this approach, unlike the previous case, we treat the DTI as a 3 dimensional Riemannian manifold with metric given by the inverse of the diffusion tensor, instead of the diffusion tensor itself. And we smooth the tensor field by averaging nearby values with weights inversely proportional to their distances. The intuition behind this approach is that the useful structures of the DTI are usually captured by the principle directions of the diffusion tensors. By taking weights inversely proportional to distances, the effect in the principle direction is emphasized. To implement this framework, we first compute the distance between any point $x = (x_1, x_2, x_3)$ and its m neighbors as:

$$d_k = \frac{1}{v_k^T D_x^{-1} v_k}, \qquad 1 \leq k \leq m, \tag{7}$$

where v_k is the coordinate vector between x and its kth neighbor, and D_x is the diffusion tensor at x. More specifically, at any point x, v_k gives the direction vector for the respective neighboring D_x^k. After obtaining these distances, we can then update the diffusion tensor at any point based on the weighted average of its neighbors as follows:

$$D_x = \alpha D_x + (1-\alpha)\Sigma_{k=1}^{m} \frac{d_k}{\Sigma_{l=1}^{m} d_l} D_x^k. \tag{8}$$

Here D_x^k is the value of diffusion tensor of the kth neighbor of x and $\alpha \in [\frac{1}{2},\ 1]$ is an adjustable parameter specifying weight of the average (weighted) of the neighboring tensors.

IV. Numerical Implementation

Techniques discussed in the previous section need to be discretized for computer implementation. Algorithm 1 and 2 show detailed implementation of the Ricci flow and the anisotropic smoothing method respectively. Both the algorithms were implemented in MATLAB to solve example problems presented in the next section.

Algorithm 1 Ricci Flow Smoothing for Tensor Fields

Require: g_{ij} (the Riemannian curvature tensor).
n (number of iterations).

for l:=1 **to** n **do**
  Compute inverse Riemannian curvature tensor g^{ij} by taking inverse of g_{ij}.
  Compute three Christoffel symbol matrices by Eq. (4).
  Calculate Ricci tensor matrix elements by Eq. (6).
  Choose step size Δ.
  Compute g_{ij} using Eq. (9) for backward Ricci flow or by Eq. (10) for forward Ricci flow.

$$g_{ij}(t+\Delta t) = g_{ij}(t) + 2\Delta R_{ij}(t) \tag{9}$$

$$g_{ij}(t+\Delta t) = g_{ij}(t) - 2\Delta R_{ij}(t) \tag{10}$$

  update $g = g_n$
end for
Noise Suppression

$$g(\frac{n}{2}) = g(\frac{n}{2}) - \lambda_{max(g(n))} g(n) \tag{11}$$

Algorithm 2 Anisotropic Smoothing for Tensor Fields

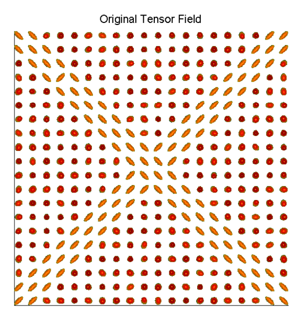
Require: D (Tensor).
n (number of iterations).
$\alpha \in [0.5,\ 1]$
v (unique distance vectors from D to each neighboring tensor).

for l:=1 **to** n **do**
  Compute inverse tensor matrix D^{-1}.
  Compute d by Eq. (7).
  Update D using Eq. (8).
end for

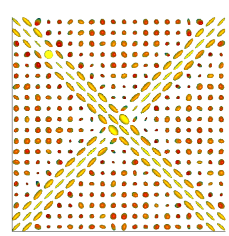
V. Examples

1) Synthetic Data Experiments: Two substrates of synthetic tensor fields were generated to experiment smoothing and feature enhancement in two dimensions. Fig. 1(a) and Fig. 2(a) show the generated synthetic tensor fields. Both the plots were generated using MATLAB function plotDTI [11].

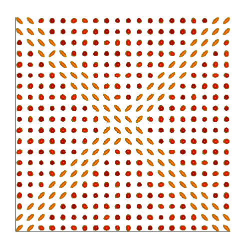
a) Backward Ricci Flow - Feature Enhancement: Performing as a non-linear inverse 'heat like' operation using Eq. (9), under-line geometric features were enhanced in the field. Fig. 1(b) and Fig. 2(b) show the enhanced features as compared to original tensor fields in Fig. 1(a) and Fig. 2(a) respectively.



(a) Original Tensor Field



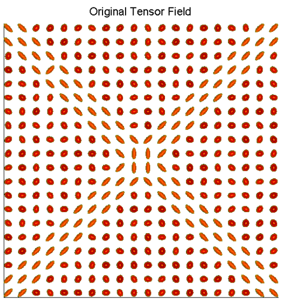
(b) Backward Ricci Flow



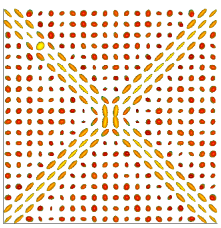
(c) Forward Ricci Flow

Fig. 1: Experiment with synthetic data set 1, (a) Original tensor field (b) Feature extraction after backward Ricci flow (c) Smoothing after forward Ricci flow.

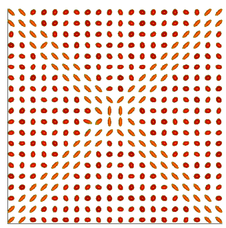
b) Forward Ricci Flow - Tensor Smoothing: Forward Ricci flow as given in Eq. 10 performs tensor field evolution resulting in overall smoothing of the field. Fig. 1(c) and Fig. 2(c) were obtained after smoothing Fig. 1(b) and Fig. 2(b) respectively. Both forward and backward Ricci flow operations are reversible in this setting under suitable constraints on the Ricci tensor.



(a) Original Tensor Field



(b) Backward Ricci Flow



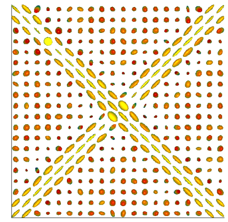
(c) Forward Ricci Flow

Fig. 2: Experiment with synthetic data set 2, (a) Original tensor field (b) Feature extraction after backward Ricci flow (c) Smoothing after forward Ricci flow.

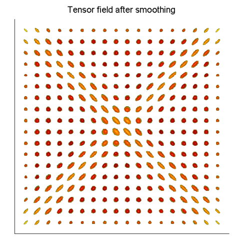
c) Anisotropic Smoothing of the Tensor Field: Anisotropic smoothing method suggested as Algorithm 2 requires very less number of iterations to smooth given tensor field. Fig. 3 shows smoothing results after four iterations.

2) MRI Data Experiments:

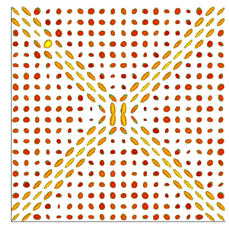
a) Ricci Flow: A healthy volunteer was scanned at Center for Magnetic Resonance Research (CMRR), University of Minnesota, to get MRI data. Fig. 4(a) shows FA map of the coronal view of the brain. Region of interest has been highlighted in red. Fig. 4(b) shows g_n which enhances features but overall image is noisy (given by Eq. 11). Fig. 4(c) shows $g(n/2)$ and Fig. 4(d) shows $g(3n/4)$. It can be seen that both Fig. 4(c) and Fig. 4(d) show higher FA in the white matter while FA remains almost unchanged in the other areas. Thus establishing noise suppression and feature enhancement, as expected. For this example, forty number of iterations (n) were used.



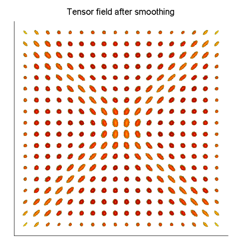
(a) Original Tensor Field



(b) After Smoothing

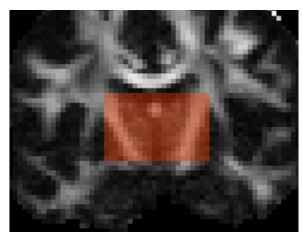


(c) Original Tensor Field

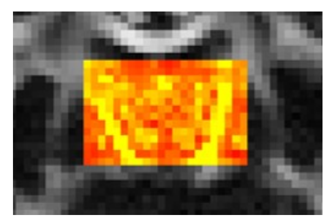


(d) After Smoothing

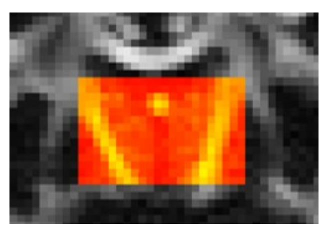
Fig. 3: Smoothing experiment with synthetic data set 1 and 2, (a) and (c) show original tensor field, while (b) and (d) show tensor fields after smoothing.



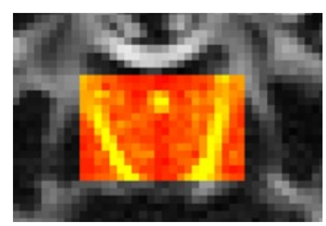
(a) Coronal plane slice showing region of interest (in red)



(b) Feature enhancement after n iterations g_n



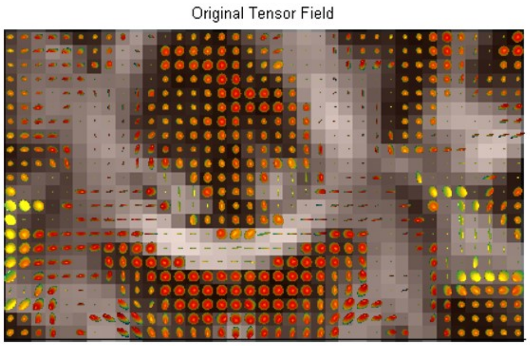
(c) $g(\frac{n}{2})$



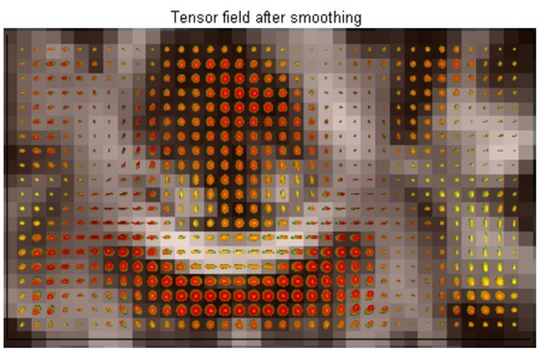
(d) $g(\frac{3n}{4})$

Fig. 4: Feature enhancement experiment (with backward Ricci flow) using MRI data. Number of iterations $n = 40$.

b) Anisotropic Smoothing: Using the same MRI data as in the previous experiment, anisotropic smoothing was performed. Fig. 4(a) shows original tensor field on a coronal slice, while Fig. 4(b) shows smooth tensor field after only four iterations of the algorithm.



(a) Original tensor field



(b) Tensor Field after smoothing

Fig. 5: Tensor field smoothing experiment (with anisotropic diffusion) using MRI data, (a) Original tensor field (b) Smooth tensor filed.

VI. CONCLUSION AND FUTURE WORK

A. Conclusion

The intend of our study has been to develop tools for de-noising as well as for feature enhancement of tensorial fields, as these arise in DTI. As noted, DTI is based on modeling diffusion of water molecules by a Gaussian process inside voxels. The covariance matrix for the anisotropic diffusion of water molecules is represented by the diffusion tensor (symmetric positive-definite 3×3 matrix). These tensor fields are often pre-processed to reduce the amount of noise arising in the acquisition process. With our presented schemes, de-noising of tensor fields can be done quite efficiently, without the need for any delicate tuning. Thus, it holds the potential of providing a useful tool in DTI imaging methodologies.

B. Future Work

In analogy to the stochastic framework of *Schrödinger bridges* [12]–[15] where a path ρ_t is constructed to interpolate two end-point in time density functions–a stochastic control problem of steering the Fokker-Planck equation, it will be interesting to control the flow of the Ricci curvature so as to interpolate, in a similar manner, tensorial distributions. That is, given two DTI tensor fields, construct a suitable homotopy/deformation linking the two. Such a problem represents a far reaching generalization of the analogous problems of controllability involving the heat equation. The value of such interpolation schemes would be to integrate DTI records across time in a way that reflects changes in the underlying tissue structure. Likewise, it will be interesting to consider the geometry of Ricci flow for the purpose of extrapolating DTI data. In this direction, it would be important to investigate possible regularization terms that may preserve and capture structural features as these evolve (e.g., in the spirit of [16]). This extension may help in learning tensor field evolution (tissue structural development) in time, which can be of significance in clinical applications.

VII. ACKNOWLEDGMENTS

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