# PARTIAL REALIZATION
# OF COVARIANCE SEQUENCES

By

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TABLE OF CONTENTS

Abstract of Dissertation Presented to the Graduate Council
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This work is concerned with rational covariance extensions of partial sequences. Certain methods of the classical interpolation theory are exploited and a novel topological approach is developed. An associated polynomial, that we call the "dissipation polynomial", is found to be a free parameter for covariance extensions with dimension bounded by the number of data. A similar result holds for the case of matrix sequences as well.

The dissipation polynomial is found to impose an "almost recurrence" law on the SCHUR parameters of rational covariance sequences. This is done via a new approach to spectral factorization. These theoretical results, placed in the context of the applied area of spectral estimation theory, suggest some recursive procedure for pole-zero modeling.

# CHAPTER I. INTRODUCTION

The elementary notion of positivity of real numbers has found various generalizations to that of quadratic forms, operators etc. These play a key role in, not only mathematics, but many areas of applied science as well. The reason is that positivity is, in one form or another, intimately related with the manifestations of physical quantities and entities.

The motivation for this work arises from the area of stochastic processes and identification theory. The covariance function $c_s := Ey_\tau \bar{y}_{\tau+s}$, $s = 0, \pm 1, \ldots,$ of a discrete-time, zero-mean, stationary stochastic process $y_\tau$, $\tau \in \mathbb{Z}$, is characterized by the nonnegative definiteness of the Toeplitz quadratic forms

$$\sum_{s=0}^{u} \sum_{t=0}^{v} a_s c_{s-t} \bar{b}_t,$$

for $u, v = 0, 1, \ldots,$ and $a_s, b_t \in \mathbb{C}$. This is the notion of positivity which plays a central role in our work.

A stochastic process is an abstraction. The real object is a realization of the process: a time-series. The probabilistic behavior is not available. For identification and prediction, estimates of the means and covariances of the stochastic process have to be computed from some observation record. In this, a large number of issues are involved. Most of them are of a statistical nature (confidence limits, etc.). Theoretically, most of these issues are still terra incognita.

In this work we shall not be concerned about such questions but we shall assume as the given data a partial sequence of covariances $C_s := \{c_t : t = 0, 1, \ldots, s\}$. This is the standing assumption for the theoretical development of several modern spectral estimation techniques (cf. HAYKIN [1979]). These techniques seek a certain extension of $C_s$ to a covariance sequence $C = \{c_t : t = 0, 1, \ldots\}$.

The set of all covariance extensions of $C_s$ can be described by several alternative approaches that we shall recapitulate in Chapter II. However, theoretical as well as practical interest lies with a certain subclass, the rational ones.

A sequence $C = \{c_s : s = 0, 1, \dots\}$ is called rational iff there exists an integer $v$ such that the rank of the behavior (Hankel) matrices

$$B_s := [c_{t+u-1}]_{t,u=1}^{s}$$

satisfies rank $B_s \leq v$ for $s = 1, 2, \dots$. The smallest such $v$ will be called the dimension of $C$.

Rationality is directly related to existence of finite dimensional stochastic realizations of $C$. The dimension of $C$ is then precisely equal to the minimal dimension for the corresponding state-space (see for example FAURRE, CLERGET and GERMAIN [1978]). Moreover, in case $C$ is a rational sequence there exists a finite positivity test.

In the case of unconstrained rational extensions of a partial sequence, called partial realizations, a minimal dimension can be found by testing linear dependence. In case the minimal dimension partial realization is not unique, this set is parametrized by a linear space. These concepts have given rise to the elegant partial realization theory of KALMAN [1979]. In the case where the extension is also required to satisfy the covariance property the problem becomes substantially more involved (see KALMAN [1981]).

The purpose of this dissertation is, in broad terms, to elucidate the relation between rationality and positivity. The main goal is to unveil some of the issues and the nonuniqueness involved in extending the partial data $C_s$ to a rational covariance sequence $C$, paying special emphasis on the dimension of these extensions. We believe that we are reasonably successful and that certain of our results can be profitably considered in the applied area of identification.

We now give a brief description of the contents of each chapter. More detailed introductory remarks are provided at the beginning of each chapter.

Chapter II is devoted to a review of certain classical techniques and concepts that are directly related to the covariance extension problem.

In Chapter III we use the algebraic machinery developed in Chapter II, to study rational covariance extensions. It turns out that a certain polynomial is intimately related with the extension and can be chosen arbitrarily. This polynomial, that we call the dissipation polynomial, represents the zeros of the power spectrum. We found that the dissipation polynomial determines in a precise way the asymptotic behavior of some important sequence of parameters that is associated to a covariance sequence. In order to obtain this result we developed a new technique for spectral factorization.

In Chapter IV we develop a new approach that is most suited for describing the covariance extensions of dimension bounded by the number of data. In point of fact, in a large number of cases this dimension coincides with the minimal dimension. The merit of this approach lies also with the fact that it provides a novel topological proof of a certain classical result: the positive definiteness of the quadratic form associated with $C_s$ is sufficient for the existence of covariance extensions. The key result of this Chapter further shows that an essential nonuniqueness in this partial realization problem is best described in terms of the associated dissipation polynomials.

Chapter V extends some of the earlier development to the case of matrix sequences. Finally, in Chapter VI we discuss the relevance and potential of the above in modeling.

The initial motivation and part of this dissertation grew out of a joint work with KHARGONEKAR (GEORGIOU and KHARGONEKAR [1982]). The material of Chapters II and III is based upon this work.

We close this introduction with a few words on notation and standing assumptions. Throughtout this dissertation, we shall work with the field of complex numbers $\underline{\underline{C}}$. $\underline{\underline{C}}[z]$ will denote the ring of polynomials in $z$ with coefficients in $\underline{\underline{C}}$. $\underline{\underline{C}}^{n\times n}[z]$ will denote the ring of $n \times n$-matrix polynomials in $z$. " $-$ " denotes complex conjugation and if $p(z) \in \underline{\underline{C}}[z]$ then $\bar{p}(z)$ denotes complex conjugation on the coefficients of $p(z)$. We shall be dealing with both infinite sequences $C = \{c_t : t = 0, 1, \ldots, c_t \in \underline{\underline{C}}$ for $t \geq 1$ and $c_o \in \underline{\underline{R}}\}$ and finite sequences $C_s := \{c_t : t = 0, 1, \ldots, s, c_t \in \underline{\underline{C}}$ for $t = 1, 2, \ldots, s$ and $c_o \in \underline{\underline{R}}\}$. A sequence $C$ is said to be <u>positive</u> (resp. <u>nonnegative</u>) iff the Toeplitz matrices

$$T_s = [c_{t-u}]_{t,u=o}^{s}, \quad s = 0, 1, \ldots,$$

where we define $c_{-|t|} := \overline{c_{|t|}}$, are positive (resp. nonnegative) definite for all $s$. A partial sequence $C_s$ is said to be <u>positive</u> (resp. nonnegative) iff $T_s$ is positive (resp. nonnegative) definite. Thus, <u>a sequence $C$ is a sequence of covariances of a stochastic process if and only if $C$ is nonnegative</u>. Finally this notion of positivity (resp. nonnegativity) will be denoted by $> 0$ (resp. $\geq 0$).

# CHAPTER II. CERTAIN CLASSICAL APPROACHES

The purpose of this Chapter is to introduce certain mathematical concepts and techniques that are pertinent to the covariance extension problem.

We shall begin with some algebraic aspects of the theory of orthogonal polynomials relative to the unit circle. Toeplitz matrices, of the same type as the ones associated with the covariance function of a stationary stochastic process, were classically considered to induce an inner product on the space of polynomials. The special Toeplitz structure can be effectively exploited by considering a particular orthogonal basis. This gave rise to the theory of orthogonal polynomials of SZEGÖ [1939]. Since that time the theory was progressively developed by many researchers. AKHIEZER [1965], GERONIMUS [1954], [1961] and GRENANDER and SZEGÖ [1958] have given classical expositions on the subject. It was early recognized that the theory of orthogonal polynomials had strong connections with a prediction problem in the theory of stochastic processes (see GRENANDER and SZEGÖ [1958, p. 173] or the survey paper by KAILATH [1974]). This opened up areas of application of the theory, notably in stochastic problems, spectral analysis and autoregressive modeling (see the book by HAYKIN [1979] for various applied and theoretical aspects on these). Motivated by autoregressive modeling for multivariate stochastic processes, WHITTLE [1963], and WIGGINS and ROBINSON [1965] laid the first pieces of a theory of orthogonal matrix polynomials. A number of researchers have then pursued this line of research. We mention only the most recent works of YOULA and KAJANJIAN [1978], MORF, VIERA and KAILATH [1978] and especially DELSARTE, GENIN, and KAMP [1978a] that have given a rather elegant account of the theory of orthogonal matrix polynomials on the unit circle.

We restrict our attention to the scalar case and in Section 2 we give a concise exposition of those aspects of the theory that we consider to be relevant to the covariance extension problem.

The theory of orthogonal polynomials is connected to certain problems in analysis. Various aspects are discussed in AKHIEZER [1965], AKHIEZER and KREIN [1962] and KREIN and NUDEL'MAN [1977]. In particular there is a connection with a certain interpolation problem that is equivalent to the covariance extension problem. Both solvability conditions and a parametrization of all solutions can be obtained by the classical SCHUR's algorithm. The machinery of orthogonal polynomials can be used to provide a compact description of the solutions in terms of a linear fractional transformation. This is the content of Section 3. We should finally mention that a similar description of an isomorphic problem was used by DEWILDE, VIEIRA and KAILATH [1978] and also DELSARTE, GENIN and KAMP [1979].

## 1. Orthogonal Polynomials: An Algebraic Approach.

We consider an infinite sequence $C = \{c_s : s = 0, 1, \ldots,$ with $c_0$ in $\underline{\underline{R}}$ and $c_s$ in $\underline{\underline{C}}$ for $s \geq 0\}$. We define on the space $\underline{\underline{C}}[z]$ of polynomials in $z$ an inner product by

$$\left\langle \sum_{t=0}^{u} a_t z^t, \sum_{s=0}^{v} b_s z^s \right\rangle := \sum_{t=0}^{u} \sum_{s=0}^{v} a_t \bar{b}_s c_{t-s}.$$

This inner product is definite if and only if $C > 0$. Whenever $C > 0$ (resp., $\geq 0$) then the above inner product defines a norm (resp., semi-norm) on $\underline{\underline{C}}[z]$ that we shall denote by $\|\cdot\|$. We begin by discussing separately the two cases of interest: first the case $C > 0$, and second the case $C \geq 0$ but not $> 0$.

We now consider $C$ to be a positive sequence. The inner product $\langle \cdot, \cdot \rangle$ is now definite. We apply the standard orthogonalization procedure to the natural basis $\{z^s : s = 0, 1, \ldots\}$ of $\underline{\underline{C}}[z]$ to obtain an orthogonal (but not necessarily orthonormal) basis of monic polynomials $\{\Phi_s(z) : s = 0, 1, \ldots\}$. These polynomials are known as the orthogonal polynomials (of the first kind) associated to the sequence $C$, and are given by

$$\Phi_o(z) = 1,$$

(1.1)
$$\Phi_s(z) = \det \begin{bmatrix} c_o & c_{-1} & \cdots & c_{-s} \\ c_1 & c_o & \cdots & c_{-s+1} \\ \vdots & \vdots & & \vdots \\ c_{s-1} & c_{s-2} & \cdots & c_{-1} \\ 1 & z & \cdots & z^s \end{bmatrix} / \det T_{s-1}, \quad s = 1, 2, \ldots .$$

Since $\|\Phi_s(z)\|^2 = \langle z^s, \Phi_s(z)\rangle$ it follows that

(1.2)
$$\|\Phi_s(z)\|^2 = \det T_s / \det T_{s-1}, \quad s = 0, 1, \ldots,$$

where $\det T_{-1} := 1$.

The special inner product structure of $\underline{\underline{C}}[z]$ induces upon the set of orthogonal polynomials certain algebraic identities and an important parametric description. We shall now discuss these.

Let $P_s(z) \in \underline{\underline{C}}[z]$ be of degree $s$. We define the reverse polynomial

$$P_s(z)^* := z^s \bar{P}_s(z^{-1}).$$

From $\langle z^t, \Phi_s(z)\rangle = 0$ for $t = 0, 1, \ldots, s - 1$ and the Toeplitz structure of the inner product it follows that

(1.3)
$$\langle z^t, \Phi_s(z)^*\rangle = 0 \quad \text{for} \quad t = 1, 2, \ldots, s, \quad \text{and any} \quad s \geq 1.$$

Since $\Phi_s(z)$ is a monic polynomial of degree $s$ we can write

(1.4)
$$\Phi_s(z)^* = 1 - z\sum_{t=o}^{s-1} b_{s,t}\Phi_t(z),$$

for some scalars $b_{s,t}$. From (1.3) and the above we obtain

$$0 = \langle 1,\ z\Phi_t(z)\rangle - b_{s,t}\langle \Phi_t(s),\ \Phi_t(z)\rangle .$$

Hence, $b_{s,t} = \langle 1,\ z\Phi_t(z)\rangle / \|\Phi_t(z)\|^2$ is independent of $s$. This fact gives rise to the parameters

$$r_{t+1} := \langle 1,\ z\Phi_t(z)\rangle / \|\Phi_t(z)\|^2, \quad t = 0, 1, \ldots .$$

These parameters are known as the <u>SCHUR parameters</u> of the sequence C. From (1.4) we now obtain the (well known) <u>recurrence identities</u>

$$(1.5)\qquad \begin{aligned} \Phi_s(z) &= z\Phi_{s-1}(z) - \bar{r}_s \Phi_{s-1}(z)^*, \\ \Phi(z)^* &= \Phi(z)^* - r_s z\Phi_{s-1}(z), \end{aligned}$$

for $s = 1, 2, \cdots$. From the first identity we obtain

$$\|\Phi_s(z)\|^2 = \|\Phi_{s-1}(z)\|^2 - \bar{r}_s\langle \Phi_{s-1}(z)^*,\ z\Phi_{s-1}(z)\rangle$$

and also

$$0 = \langle z\Phi_{s-1}(z),\ \Phi_{s-1}(z)^*\rangle - r_s\|\Phi_{s-1}(z)^*\|^2 .$$

Combining the two we obtain

$$(1.6)\qquad \|\Phi_s(z)\|^2 = (1 - |r_s|^2)\|\Phi_{s-1}(z)\|^2, \quad s = 1, 2, \cdots .$$

This shows that the parameters $R = \{r_s : s = 1, 2, \ldots\}$ satisfy

$$(1.7)\qquad |r_s| < 1, \quad s = 1, 2, \cdots .$$

<u>These conditions (and</u> $c_0 > 0$) <u>are equivalent to</u> C <u>being positive. In fact starting from the parameter sequence</u> R <u>with</u> $|r_t| < 1$ <u>for all</u> t, <u>and</u> $c_0 > 0$, <u>we may construct a corresponding positive sequence. This correspondence is bijective.</u> Furthermore, partial

positive sequences $C_s = \{c_t : t = 0, 1, \ldots, s,$ with $T_s > 0\}$ correspond bijectively to pairs $(c_o, R_s)$ where $c_o > 0$ and $R_s = \{r_t : 1, 2, \ldots, s$ with $|r_t| < 1$ for all $t\}$. This we show below.

The sequence of parameters of a positive sequence $C$ are given by

$$r_1 = - c_1/c_o,$$

$$(1.8) \qquad r_s = - \overline{\Phi_s(0)} = (-1)^s \det \begin{bmatrix} c_1 & c_2 & \cdots & c_s \\ c_o & c_1 & \cdots & c_{s-1} \\ \vdots & \vdots & & \vdots \\ c_{-s+2} & c_{-s+3} & \cdots & c_1 \end{bmatrix} / \det T_{s-1},$$

for $s = 2, 3, \ldots,$ and they satisfy $|r_s| < 1, \quad s = 1, 2, \cdots.$

Conversely, starting from the sequence of parameters that satisfy $|r_s| < 1$, $s = 1, 2, \ldots,$ and $c_o > 0$ we obtain the corresponding sequence $C$ by solving (1.8) for $c_s$:

$$(1.9) \qquad c_s = c_o r_s \prod_{t=1}^{s-1} (1 - |r_t|^2) + (c_1 \cdots c_{s-1}) T_{s-2}^{-1} \begin{pmatrix} c_{s-1} \\ \vdots \\ c_1 \end{pmatrix},$$

for $s = 1, 2, \cdots$. The above is valid provided $\det T_{s-2} \neq 0$. But this follows from $|r_t| < 1$ for $t = 1, 2, \ldots,$ and the algebraic identity

$$\frac{\det T_t}{\det T_{t-1}} = (1 - |r_t|^2) \frac{\det T_{t-1}}{\det T_{t-2}}, \quad t = 1, 2, \ldots,$$

(which arises from (1.2) and (1.6)). Hence, (1.8) and (1.9) establish the required bijective correspondence.

We now discuss the singular case of nonnegative sequences that are <u>not</u> positive. Such a sequence is called <u>singularly nonnegative</u>.

Assume that $C \neq \{0, 0, \ldots\}$ is a singularly nonnegative sequence and let $s$ be the smallest (positive) integer for which

$$\det T_s = 0.$$

The partial sequences $\{\phi_t(z) : t = 0, 1, \ldots, s\}$ and $R_s$ are defined as earlier and (1.1) - (1.6) hold for $t = 1, \ldots, s$. However, we now have $\|\phi_s(z)\|^2 = 0$ and $R_s$ satisfies

$$(1.10) \quad |r_t| < 1, \quad t = 1, 2, \ldots, s-1, \quad \text{and} \quad |r_s| = 1.$$

The interesting feature of this singular case is that <u>any singularly nonnegative sequence</u> $C$ <u>as above, is rational and uniquely determined by</u> $C_s$ <u>(or, equivalently, by</u> $c_o$ <u>and</u> $R_s$). <u>Also any partial sequence</u> $C_s$ <u>with</u> $T_s \geq 0$ <u>and</u> $\det T_s = 0 \neq \det T_{s-1}$ <u>admits a uniquely defined singularly nonnegative extension</u> $C$. We now prove these facts.

Let $C$ be singularly nonnegative and

$$\chi(z) = z^s + a_1 z^{s-1} + \ldots + a_s$$

be a monic polynomial of least degree that satisfies $\|\chi(z)\|^2 = 0$. Evidently, $s$ is the smallest (positive) integer such that $\det T_s = 0$. Furthermore, $\chi(z) = \phi_s(z)$.

Clearly, $\|z^t\chi(z)\|^2 = 0$ for all $t \geq 0$. By the fact that $T_{s+t} \geq 0$ it follows that

$$(\underbrace{0 \ldots 0}_{t} \quad a_s \ldots a_1 \quad 1)'$$

is a zero-eigenvector of $T_{s+t}$ and therefore

$$(1.11) \quad c_{-s-t} = - a_1 c_{1-s-t+1} - \ldots - a_s c_{-t} \quad \text{for} \quad t \geq 0.$$

This shows that $C$ is a rational sequence and is in fact uniquely determined from the partial sequence $C_s$.

We now let $C_t$ be such that $T_t \geq 0$ with $\det T_t = 0$. We shall show that there exists a nonnegative extension $C$ of $C_s$ which by the above discussion is unique.

The partial sequence $C_t$ defines in the obvious way a semi-norm and an inner product on the space of polynomials of degree less than or equal to $t$. We denote these by $\|\cdot\|_t$ and $\langle \cdot , \cdot \rangle_t$ respectively. Consider now $s$ to be the smallest integer for which $\det T_s = 0$ and $\chi(z) = z^s + a_1 z^{s-1} + \dots + a_s$ be a monic polynomial of least degree that satisfies $\|\chi(z)\|_t^2 = 0$. Precisely as we did before we now obtain that

$$(1.12) \quad c_{-s-u} = - a_1 c_{-s-u+1} - \dots - a_s c_{-u} \text{ for } u = 0, 1, \dots, t - s.$$

We now extend the partial sequence $C_t$ to an infinite one $C$ using (1.12) for $u = t - s + 1, \dots$. We now show that $C \geq 0$.

For any $a(z)$ in $\underline{\underline{C}}[z]$ denote by $a \bmod \chi$ the remainder of $a(z)$ divided by $\chi(z)$. Since $\langle z^u, \chi(z) \rangle = 0$ for all $u \geq 0$ it follows that

$$\begin{aligned} \langle a(z), a(z) \rangle &= \langle a \bmod \chi, a \bmod \chi \rangle \\ &= \langle a \bmod \chi, a \bmod \chi \rangle_t \geq 0. \end{aligned}$$

Therefore $C \geq 0$.

The above considerations readily solve the covariance extension problem: <u>Given a partial sequence</u> $C_s$, <u>there exists a nonnegative extension</u> $C$ <u>of</u> $C_s$ <u>if and only if</u> $C_s \geq 0$. In particular, the following two cases are possible:

(a) <u>Nondegenerate case</u>: $C_s > 0$.
In this case the set of all nonnegative extensions of $C_s$ are in bijective correspondence with sequences of parameters $R$ that are either finite and of the form $R = \{r_t : t = s + 1, \dots, s + u$, with $|r_t| < 1$ for $t < s + u$ and $|r_{s+u}| = 1\}$ or infinite satisfying $|r_t| < 1$ for $t = s + 1, \dots$.

(b) <u>Degenerate case</u>: $C_s \geq 0$ but $C_s \not> 0$.
In this case there exists a uniquely determined nonnegative extension $C$.

We now proceed to consider certain related mathematical objects, which will be useful in the next section in showing the connection of the above with an interpolation problem.

Define the power series

$$\Gamma(z) := c_0 + 2\sum_{t=1}^{\infty} c_t z^t.$$

The function theoretic properties of $\Gamma(z)$ will be described in the next section. Here we view $\Gamma(z)$ as an algebraic object. We shall now recall the notion of partial realizations (see KALMAN [1979]) and then introduce the so-called orthogonal polynomials of the second kind of $C$ by considering certain partial realizations of

$$\bar{\Gamma}(z^{-1}) = c_0 + 2\sum_{t=1}^{\infty} c_{-t} z^{-t}.$$

Consider a formal power series $F(z^{-1})$ (in negative powers of $z$). A pair of coprime monic polynomials $(\pi(z), \chi(z))$ or, equivalently, the rational function $\pi(z)/\chi(z)$ is said to be a <u>partial realization of $F(z^{-1})$ of order</u> $s$ iff

$$[F(z^{-1})\chi(z)z^{s-\deg\chi}]_+ = \pi(z)z^{s-\deg\pi},$$

where $[\ ]_+$ denotes "<u>the polynomial part of</u>". Equivalently, the rational function $\pi(z)/\chi(z)$ is a partial realization of $F(z^{-1})$ of order $s$ if and only if the Laurent series $\pi(z)/\chi(z)$ (with the division carried out in the field of formal Laurent series in negative powers of $z$) matches $F(z^{-1})$ up to and including the coefficient of $z^{-s}$.

We now consider the power series $\bar{\Gamma}(z^{-1})$ and define a sequence of polynomials by

$$\Psi_s(z) := \frac{1}{c_0} [\bar{\Gamma}(z^{-1})\Phi_s(z)]_+,$$

where $\Phi_s(z)$ is the s-th orthogonal polynomial of the sequence C. The integer s runs over either all nonnegative integers or, a finite number of them depending on whether C is positive or nonnegative, as we discussed earlier. These polynomials are known as the <u>orthogonal polynomials of the second kind of the sequence</u> C and were introduced by GERONIMUS [1971, p. 10]. From the definition of $\Psi_s(z)$ we have that $c_0\Psi_s(z)/\Phi_s(z)$ (though not necessarily a coprime representation) is a partial realization of $\bar{\Gamma}(z^{-1})$ of order s. In the case where C is singularly nonnegative and s is the smallest integer for which $\det T_s = 0$, it follows that $\bar{\Gamma}(z^{-1})$ is actually equal to $c_0\Psi_s(z)/\Phi_s(z)$.

We now show that the orthogonal polynomials of the second kind satisfy the following recurrence identities (which, except for a sign change, are the same as (1.5)):

$$\text{(1.13)} \quad \begin{aligned} \Psi_s(z) &= z\Psi_{s-1}(z) + \bar{r}_s\Psi_{s-1}(z)^*, \\ \Psi_s(z)^* &= \Psi_{s-1}(z)^* + r_s z\Psi_{s-1}(z), \end{aligned}$$

for $s = 1, 2, \ldots$ (or a finite index sequence in case C is singularly nonnegative).

We define the transformation

$$f\colon \underline{\underline{C}}[z] \to \underline{\underline{C}}[z]\colon A(z) \mapsto \frac{1}{c_0}[A(z)(c_0 + 2\sum_{t=1}^{\infty} c_{-t}z^{-t})]_+.$$

We also define a sequence $\hat{C}$ via the relation

$$(c_0 + 2\sum_{t=1}^{\infty} c_{-t}z^{-t})(\hat{c}_0 + 2\sum_{t=1}^{\infty} \hat{c}_{-t}z^{-t}) = 1,$$

and the transformation

$$\hat{f}: \underline{\underline{C}}[z] \to \underline{\underline{C}}[z]: A(z) \mapsto \frac{1}{\hat{c}_0}[A(z)(\hat{c}_0 + 2\sum_{t=1}^{\infty} \hat{c}_{-t}z^{-t})]_+.$$

It is straightforward to check that $f\hat{f} = \hat{f}f$ is the identity transformation.

We now modify our earlier notation by adding a subscript to Toeplitz inner products (and norms) that specify the defining sequence. Thus we shall have $\langle \cdot, \cdot \rangle_C$, $\langle \cdot, \cdot \rangle_{\hat{C}}$, and if we define $I := \{1, 0, 0, \ldots\}$ then we shall also write $\langle \cdot, \cdot \rangle_I$.

We denote by $g^*$ the adjoint of the transformation $g$. We now have that

$$\langle \cdot, \cdot \rangle_C = \langle \cdot, \tfrac{1}{2}(f + f^*)\cdot \rangle_I,$$

and also

$$\langle \hat{f}\cdot, \hat{f}\cdot \rangle_C = \langle \cdot, \tfrac{1}{2}\hat{f}^*(f + f^*)\hat{f}\cdot \rangle_I = \langle \cdot, \tfrac{1}{2}(\hat{f}^* + \hat{f})\cdot \rangle_I = \langle \cdot, \cdot \rangle_{\hat{C}}.$$

Since $\|\hat{f}\cdot\|_C = \|\cdot\|_{\hat{C}}$ is certainly a semi-norm it follows that $\hat{C}$ <u>is nonnegative</u>.

Since the polynomials $\Psi_s(z)$, $s = 0, 1, \ldots,$ were defined by $\Psi_s(z) := f\Phi_s(z)$, $s = 0, 1, \ldots,$ they are the orthogonal polynomials of the first kind of the sequence $\hat{C}$. Therefore they satisfy relations of the type (1.5). In order to establish the precise form (1.13) of these relations we only need to show that

(1.14) $\quad \Phi_s(0) = -\Psi_s(0).$

From the definition of $\Psi_s(z)$ we have

$$\Psi_s(z) = \frac{1}{c_0}[(2\sum_{t=1}^{\infty} c_{-t}z^{-t})\Phi_s(z)]_+ - \Phi_s(z).$$

The constant term of

$$\frac{1}{c_0}\left[\left(2\sum_{t=1}^{\infty} c_{-t}z^{-t}\right)\Phi_s(z)\right]_+$$

equals $(2/c_o)\langle 1, \Phi_s(z)\rangle = 0$. Therefore (1.14) holds and hence (1.13) follows.

We finally want to indicate another algebraic identity that is satisfied by the two types of orthogonal polynomials that we shall refer to later on:

$$(1.15)\quad \Phi_s(z)\Psi_s(z)^* + \Phi_s(z)\Psi_s(z)^* = 2z^s h_s,$$

where $h_s$ is a scalar given by

$$(1.16)\quad h_s := \prod_{t=1}^{s}(1 - |r_t|^2) = \frac{1}{c_0}\,\|\Phi_s(z)\|^2.$$

This identity follows from (1.5) and (1.13) using induction on $s$.

## 2. Interpolation: Function Theoretic Approach.

Consider the following interpolation problem: We are given two regions $G_z$ and $G_w$ in the complex planes of the variables $z$ and $w$, and a set of pairs $(z_a, w_a)$ with $z_a$ in $G_z$ and $w_a$ in $G_w$ for all $a$ in a certain index set $I$. It is required to find a function $F(z)$ holomorphic in $G_z$ with values in $G_w$ that satisfies the interpolation constraints

$$F(z_a) = w_a \quad \text{for all } a \text{ in } I.$$

When certain of the points $z_a$ coincide, then the interpolation constraints are modified so as to assign at these points values to the successive derivatives of $F(z)$.

This problem is classical and a number of techniques have been applied to it. The books by WALSH [1956] and AKHIEZER [1965] give comprehensive expositions of the classical approaches to the problem. In recent years new functional theoretic techniques have been applied that also extend to a more general class of interpolation problems that includes interpolation with matrix-valued functions. These techniques have been developed in the work of SZ.-NAGY and FOIAS [1970] and SARASON [1967].

In this section we shall consider a particular case of the problem which is directly related to the covariance extension problem. We discuss the so-called SCHUR's algorithm that also provides a description of the solutions, and we tie up this approach with the material of the previous section.

The following classical theorem states that the notion of positivity encountered earlier is expressed in terms of a function theoretic property.

(2.1) THEOREM (see AKHIEZER [1965, p. 178]). The power series

$$\Gamma(z) := c + 2c_1 z + \dots + 2c_s z^s + \dots$$

converges in $|z| < 1$ and has $\operatorname{Re} \Gamma(z) \geq 0$ for all $z$ in $|z| < 1$ if and only if the sequence $C = \{c_s : s = 0, 1, \dots,$ with $c_0 := (c + \bar{c})/2\}$ is nonnegative.

The functions possessing the above property form the so-called class $\underset{\sim}{C}$. (This same property is known in the engineering literature as positive realness. See for example BELEVITCH [1968, p. 71].)

We shall consider the following interpolation problem: Given a partial sequence $C_s = \{c_t : t = 0, 1, \dots, s\}$ find the necessary and sufficient conditions for the existence of a function in $\underset{\sim}{C}$ whose power series expansion in $z$ begins with $c_0 + 2c_1 z + \dots + 2c_s z^s$. Also, it is required to describe the set of all solutions.

This is known as the CARATHEODORY problem. In view of Theorem (2.1) it is seen to be equivalent to the covariance extension problem.

Below we proceed to discuss the so-called SCHUR's algorithm as applied to the CARATHEODORY problem. This technique provides a parametrization of the solutions in terms of functions in class $\underset{\sim}{C}$.

The main technical result needed is the following simple

(2.2) LEMMA. The function $\Gamma_a(z)$ belongs to $\underset{\sim}{C}$ and has power series expansion in $z$ that begins with $1 + 2c_1^{(a)} z$ if and only if one of the following two conditions holds:

(a) $|c_1^{(a)}| < 1$ and

$$\Gamma_b(z) := \frac{d_a(z)\Gamma_a(z) - b_a(z)}{-c_a(z)\Gamma_a(z) + a_a(z)}$$

is in $\underset{\sim}{C}$ where $a_a(z) := (1 + z)(1 - c_1^{(a)})$,
$b_a(z) := (1 - z)(1 - c_1^{(a)})$, $c_a(z) := (1 - z)(1 + c_1^{(a)})$,
and $d_a(z) := (1 + z)(1 + c_1^{(a)})$,

(b) $|c_1^{(a)}| = 1$ and $\Gamma_a(z) = \dfrac{1 + c_1^{(a)} z}{1 - c_1^{(a)} z}$.

INDICATION OF THE PROOF. The set of functions $S(z)$ that are analytic in $|z| \leq 1$ and satisfy $|S(z_o)| \leq 1$ for all $z_o$ in $|z| < 1$ forms the so-called class $\underset{\sim}{S}$. There exists a simple relation between functions of class $\underset{\sim}{C}$ and functions of class $\underset{\sim}{S}$: $\Gamma(z)$ is in $\underset{\sim}{C}$ if and only if

(2.3) $$S(z) = \frac{1}{2}\,\frac{\Gamma(z) - \Gamma(0)}{\Gamma(z) + \overline{\Gamma(0)}}$$

is in $\underset{\sim}{S}$.

For functions of class $\underset{\sim}{S}$ it is easier to show an analogous statement (see AKHIEZER [1965, p. 101]): A function $S_a(z)$ is in $\underset{\sim}{S}$ if and only if one of the following two conditions hold:

(a') $|S_a(0)| < 1$ and

$$S_b(z) := \frac{1}{2} \frac{S_a(z) - S_a(0)}{1 - \overline{S_a(0)}S_a(z)}$$

is in $\underset{\sim}{C}$,

(b') $S_a$ is constant of modulus equal to one.

Applying now (2.3) to the above statement proves the lemma. □

In point of fact this lemma gives a description of all functions in $\underset{\sim}{C}$ in terms of certain parameters: Beginning with a function $\Gamma(z) := \Gamma_1(z) = 1 + 2c_1^{(1)}z + \ldots$, we iterate the formula

$$(2.4) \qquad \Gamma_{t+1}(z) = \frac{d_t(z)\Gamma_t(z) - b_t(z)}{-c_t(z)\Gamma_t(z) + a_t(z)},$$

for $t = 1, 2, \ldots,$ while $|c_1^{(t)}| \neq 1$. Then, $\Gamma(z)$ belongs to $\underset{\sim}{C}$ if and only if one of the following two cases holds:

(a) $|c_1^{(t)}| < 1$ for all $t$,

(b) $|c_1^{(t)}| < 1$ for $t = 1, \ldots, s - 1$ and

$$\Gamma_s(z) = \frac{1 + c_1^{(s)}z}{1 - c_1^{(s)}z} \quad \text{with} \quad |c_1^{(s)}| = 1.$$

(See also AKHIEZER [1965, p. 103].) The parameters $\rho_t := c_1^{(t)}$, $t = 1, 2, \ldots$ are called <u>SCHUR parameters of</u> $\Gamma(z)$.

The above lemma readily solves the CARATHEODORY problem: Given the partial $C_s = \{1, c_1, \ldots, c_s\}$ define $c_t^{(1)} := c_t$, $t = 1, \ldots, s$. By the lemma a function in $\underset{\sim}{C}$ exists having power series expansion that begins with $1 + 2c_1^{(1)} + \ldots + 2c_s^{(1)}z^s$ if and only if either

$$|c_1^{(1)}| = 1 \text{ and } c_t^{(1)} = (c_1^{(1)})^t \text{ for } t = 2, \ldots, s$$

or

$$|c_1^{(1)}| < 1$$

and there exists a $\underset{\sim}{C}$-function with power series that begins with

$$1 + 2c_1^{(2)} z + \ldots + 2c_{s-1}^{(2)} z^{s-1}$$

where $c_t^{(2)}$, $t = 1, \ldots, s - 1$ are obtained via the formula

$$\frac{a_1(z)(1 + 2c_1^{(1)} z + \ldots + 2c_s^{(1)} z^s) - b_1(z)}{- c_1(z)(1 + 2c_1^{(1)} z + \ldots + 2c_s^{(1)} z^s) + d_1(z)} =$$

$$= 1 + 2c_1^{(2)} z + \ldots + 2c_{s-1}^{(2)} z^{s-1} + O(z^s)$$

(where the division is carried out in the field of formal Laurent series in positive powers of $z$). In this way the problem can be transformed to an equivalent one with one interpolation constraint less. This inductive procedure is known as the SCHUR's algorithm. Iterating the above we obtain:

The CARATHEODORY problem is solvable in precisely the following two cases:

(a) Nondegenerate case: $|c_1^{(t)}| < 1$ for $t = 1, \ldots, s$.

In this case the general solution is nonunique and is obtained from

$$\Gamma_t(z) = \frac{a_t(z)\Gamma_{t+1}(z) + b_t(z)}{c_t(z)\Gamma_{t+1}(z) + d_t(z)}, \tag{2.5}$$

for $t = s, s - 1, \ldots, 1$ and $\Gamma_{s+1}$ an arbitrary function in $\underset{\sim}{C}$.

(b) *Degenerate case:* $|c_1^{(t)}| < 1$ for $t = 1, \ldots, u - 1$, with $u \leq s$, $|c_1^{(u)}| = 1$ and $c_t^{(u)} = (c_1^{(u)})^t$ for $t = 1, \ldots, s - u + 1$. In this case the solution is unique and is obtained from (2.5) iterating for $t = u - 1, \ldots, 1$ with

$$\Gamma_u(z) = \frac{1 + c_1^{(u)} z}{1 - c^{(u)} z} .$$

The property that a function $\Gamma(z)$ belongs to $\underset{\sim}{C}$ is described in terms of the parameters $\rho_t := c_1^{(t)}$, $t = 1, 2, \ldots,$ and also in terms of the parameters $r_t$, $t = 1, 2, \ldots,$ that occur in the recurrence relations of the orthogonal polynomials of a corresponding (by the Theorem (2.1)) sequence. *These two sets of parameters turn out to be equivalent.* In the rest of the section we shall show this which gives the precise connection of the SCHUR's algorithm with the material of the previous section.

We first prove the following

(2.6) LEMMA. *Let* $R^t := \{r_s : |r_s| < 1,\ s = t, t + 1, \ldots\}$ *be a sequence of parameters and* $\{\Psi_s^t(z), \Phi_s^t(z),\ s = 0, 1, \ldots\}$ *denote the associated orthogonal polynomials. Define*

$$A_s^t(z) := \Psi_s^t(z) + \Psi_s^t(z)^*, \quad B_s^t(z) := \Psi_s^t(z) - \Psi_s^t(z)^*,$$

$$C_s^t := \Phi_s^t(z) - \Phi_s^t(z)^*, \quad D_s^t(z) := \Phi_s^t(z) + \Phi_s^t(z)^*, \quad \text{and}$$

$$M_s^t(z) := \frac{1}{2}\begin{pmatrix} A_s^t(z) & B_s^t(z) \\ C_s^t(z) & D_s^t(z) \end{pmatrix} .$$

*The following algebraic identity holds*

$$(2.7) \qquad \begin{pmatrix} \Psi^t_{s+u}(z) \\ \Phi^t_{s+u}(z) \end{pmatrix} = M^t_s(z) \begin{pmatrix} \Psi^{t+s}_u(z) \\ \Phi^{t+s}_u(z) \end{pmatrix},$$

PROOF. We apply induction on $u$. For $u = 0$, (2.7) obviously holds. Assuming that it holds for $u = v$ we obtain

$$\begin{pmatrix} \Psi^t_{s+v+1}(z) \\ \Phi^t_{s+v+1}(z) \end{pmatrix} = zM^t_s(z)\begin{pmatrix} \Psi^{t+s}_v(z) \\ \Phi^{t+s}_v(z) \end{pmatrix} + r_{t+v}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M^t_s(z)*\begin{pmatrix} \Psi^{t+s}_v(z)* \\ \Phi^{t+s}_v(z)* \end{pmatrix},$$

where applying $*$ on $M^t_s(z)$ simply means to apply $*$ on the entries of $M^t_s(z)$. But

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M^t_s(z)* = M^t_s(z)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We finally obtain

$$\begin{pmatrix} \Psi^t_{s+u+1}(z) \\ \Phi^t_{s+u+1}(z) \end{pmatrix} = M^t_s(z)\begin{pmatrix} \Psi^{s+t}_{u+1}(z) \\ \Phi^{s+t}_{u+1}(z) \end{pmatrix}$$

which completes the proof. □

In the case where $R^1$ is a finite sequence $\{r_s : s + 1, \ldots, v,$ with $|r_s| < 1$ for $s < v$ and $|r_v| = 1\}$ then (2.7) still holds for $1 \leq t \leq v$, $0 \leq s$, $0 \leq u$, and $s + t + u \leq v$.

Let now $\Gamma^t(z)$ be the functions in $\underset{\sim}{C}$ with $\Gamma^t(0) = 1$ that correspond through (1.9) and Theorem (2.1) to the parameter sequences $R^t$ that we defined in the lemma. We shall show the following

(2.8) PROPOSITION. <u>Provided</u> $|r_u| < 1$ <u>for</u> $u = 1, \ldots, s + t$, <u>then the following two identities hold</u>:

$$(2.9)\qquad \Gamma^t(z) = \frac{A_s^t(z)*\Gamma^{s+t}(z) + B_s^t(z)*}{C_s^t(z)*\Gamma^{s+t}(z) + D_s^t(z)*},$$

$$(2.10)\qquad \Gamma^{s+t}(z) = \frac{D_s^t(z)*\Gamma^t(z) - B_s^t(z)*}{-C_s^t(z)*\Gamma^t(z) + A_s^t(z)*},$$

<u>for all</u> $t \geq 1$, $s \geq 0$.

PROOF. We first consider the case where $R^1 = \{r_s : s = 1, 2, \ldots,$ with $|r_s| < 1$ for all $s\}$ is an infinite sequence. We shall show that both sides of (2.9) have the same power series expansion in $z$. (Both polynomials and power series are considered as elements in the field of formal Laurent series in $z$.)

We first show that $(C_s^t(z)*\Gamma^{s+t}(z) + D_s^t(z)*)^{-1}$ exists. Indeed, the coefficient of the zero-term is

$$\Phi_s^t(0)* - \Phi_s^t(0) + \Phi_s^t(0)* + \Phi_s^t(0) = 2 \neq 0.$$

From Lemma (2.6) we have that

$$(2.11)\qquad \frac{\Psi_{s+u}^t(z)*}{\Phi_{s+u}^t(z)*} = \frac{A_s^t(z)*\Psi_u^{s+t}(z)*/\Phi_u^{s+t}(z)* + B_s^t(z)*}{C_s^t(z)*\Psi_u^{s+t}(z)*/\Phi_u^{s+t}(z)* + D_s^t(z)*}.$$

Using the above it is straightforward to show that

$$\frac{A_s^t(z)*\Gamma^{s+u}(z) + B_s^t(z)*}{C_s^t(z)*\Gamma^{s+t}(z) + D_s^t(z)*} - \frac{\Psi_{s+u}^t(z)*}{\Phi_{s+u}^t(z)*} = O(z^{s+u+1}),$$

for all $u \geq 0$. By the definition of the orthogonal polynomials of the second kind we also have that

$$\Gamma^t(z) - \frac{\Psi_{s+u}^t(z)*}{\Phi_{s+u}^t(z)*} = O(z^{s+u+1})$$

for all $u \geq 0$. This establishes (2.9).

In the case where $R^1$ is a finite sequence the above are still valid for $u \leq v$ for some maximal $v$ such that $|r_{s+t+v}| = 1$. But then

$$\Gamma^t(z) = \Psi^t_{s+v}(z)*/\Phi^t_{s+v}(z)*, \quad \text{and}$$

$$\Gamma^{s+t}(z) = \Psi^t_v(z)*/\Phi^t_v(z)*.$$

Therefore, by (2.11), it follows that (2.9) holds.

The identity (2.10) follows from (2.9) when solved for $\Gamma^{s+t}(z)$ provided the denominator in the right hand side of (2.10) is not identically zero. This we show below.

From the definition of the orthogonal polynomials of the second kind we have that

$$(2.12) \quad -\Phi^t_s(z)*\Gamma^t(z) + \Phi^t_s(z)* = O(z^{s+1}).$$

From the above and (1.15) we obtain

$$(2.13) \quad \Phi^t_s(z)\Gamma^t(z) + \Psi^t_s(z) =$$
$$= \Phi^t_s(z)\Psi^t_s(z)* + \Phi^t_s(z)*\Psi^t_s(z))/\Phi^t_s(z)* + O(z^{s+1})$$
$$= h^{(t)}_s z^s + O(z^{s+1}),$$

where $h^{(t)}_s := \prod_{u=0}^{s}(1 - |r_{t+u}|^2)$. By adding (2.12) and (2.13) we obtain

$$C^t_s(z)*\Gamma^t(z) + A^t_s(z)* = h^{(t)}_s z^s + O(z^{s+1}) \neq 0,$$

since $h^{(t)}_s \neq 0$. □

We finally show

(2.14) PROPOSITION. <u>If</u> $\Gamma(z)$ <u>denotes a function in</u> $\underset{\sim}{C}$ <u>with</u> $\Gamma(0) = 1$ <u>and</u> $\Gamma_t(z)$, $\rho_t$, $\Gamma^t(z)$, $r_t$ <u>are defined as before for</u> $t = 1, 2, \ldots$ <u>(finite or infinite), then</u> $\Gamma_t(z) = \Gamma^t(z)$ <u>and</u> $\rho_t = r_t$ <u>for all</u> $t$.

PROOF. We apply induction on $t$. By definition $\Gamma_1(z) = \Gamma(z) = \Gamma^1(z)$ and therefore $\rho_1 = r_1$. Suppose $\Gamma_t(z) = \Gamma^t(z)$ for some $t$. Then $\rho_t = r_t$ and hence, $a_t(z) = A_1^t(z)^*$, $b_t(z) = B_1^t(z)^*$, $c_t(z) = C_1^t(z)^*$, and $d_t(z) = D_1^t(z)^*$, as well. In case $|\rho_t| = |r_t| = 1$ then both sequences have terminated and we are done. If $|\rho_t| = |r_t| < 1$ then from (2.9) and (2.10) we conclude that $\Gamma_{t+1}(z) = \Gamma^{t+1}(z)$. □

(2.15) REMARK. As we mentioned earlier, interpolation ideas have a strong connection with circuit theory. For example, the celebrated DARLINGTON synthesis procedure is the analog of the SCHUR's algorithm for solving the general Nevanlinna-Pick interpolation problem. Several of these connections were pointed out and shown explicitely by DEWILDE, VIEIRA, and KAILATH [1978]. In point of fact, in that paper they derived a compact description for the solutions of a SCHUR interpolation problem that is similar to (2.9) (see DEWILDE, VIEIRA, and KAILATH [1978, p. 668] and also DELSARTE, GENIN and KAMP [1979, p. 40]). The various forms of interpolation can be interpreted in a circuit theoretic framework as synthesis with cascade connection of coupling networks. In the same framework the linear fractional transformation (2.9) is seen to correspond to a cascade connection terminated to a resistive network with impedance $\Gamma^{s+1}(z)$ (compare also with BELEVITCH [1968, p. 110, (25)]).

## CHAPTER III. RATIONAL COVARIANCE EXTENSIONS

In the first section we begin by applying the previously derived interpolation results to the study of rational covariance extensions. Certain bounds for the dimension of the various extensions are provided by this algebraic approach.

With any rational $\underset{\sim}{C}$-function or, equivalently with any rational covariance sequence there is associated a certain polynomial in $z$ and $z^{-1}$. This polynomial we call the <u>dissipation polynomial</u> of the sequence. It represents the zeros of the power spectrum or, equivalently, the zeros of an associated stochastic realization. This polynomial is completely determined up to a scalar factor by the tail of the associated parameter sequence. In point of fact, the dissipation polynomial up to a scalar factor , is an invariant of the action of "shifting and truncating" the corresponding parameter sequence.

This rather interesting result is further exploited in Section 4 in connection with asymptotic properties of rational covariance sequences. The most complete treatment up to date of the asymptotic and analytic properties of positive sequences and of the associated orthogonal polynomials has unquestionably been given by GERONIMUS [1961]. We shall apply some of his results to the case of rational sequences. A certain new aspect that emerged does not seem to have an analogue in the general case. The dissipation polynomial determines the asymptotic behavior of the parameter sequence. The sequence of parameters of a rational covariance sequence is not necessarily rational, however it is in a certain precise sense very close to being so. We shall call this property "almost rationality".

Another aspect of this development is a new algorithmic procedure for spectral factorization. This is a key problem in system theory and several approaches to it have been developed. In the discrete-time scalar case, given a rational function $\pi(z)/\chi(z)$, with $\pi(z)$, $\chi(z)$ in $\underline{\underline{C}}[z]$, that has positive real part almost everywhere on

$|z| = 1$, it is required to obtain a factorization

$$\frac{1}{2}\left\{\frac{\pi(z)}{\chi(z)} + \frac{\bar{\pi}(z^{-1})}{\bar{\chi}(z^{-1})}\right\} = \frac{\eta(z)}{\chi(x)}\,\frac{\bar{\eta}(z^{-1})}{\bar{\chi}(z^{-1})},$$

with $\eta(z) \in \underline{\underline{C}}[z]$, for the real part of $\pi(z)/\chi(z)$ on $|z| = 1$. This factorization amounts to factoring a nonnegative trigometric polynomial

$$d(z,\ z^{-1}) := \frac{1}{2}\left\{\pi(z)\bar{\chi}(z^{-1}) + \chi(z)\bar{\pi}(z^{-1})\right\} = \eta(z)\bar{\eta}(z^{-1}),$$

$z = e^{j\theta}$, as the square of the modulus of a polynomial $\eta(z)$.

The existence of such a factorization is well known. The most common approaches are a Riccati based approach (see FAURRE, CLERGE, and GERMAIN [1978]) and an algorithm due to RISSANEN and KAILATH [1972]. For different aspects of the factorization problem see ANDERSON, HITZ and DIEM [1974], DELSARTE, GENIN and KAMP [1978b], FRIEDLANDER [1982], SAEKS [1976], STRINTZIS [1972], and YOULA [1961].

In our investigations we found a new technique. This is intimately related with the above. However, it operates on both numerator and denominator of a $\underset{\sim}{C}$-function $\pi(z)/\chi(z)$ instead of simply the dissipation polynomial.

The key idea is based on the invariance of the dissipation polynomial $d(z,\ z^{-1})$ under the action of "shifting and truncating" the corresponding parameter sequence. Under this operation, certain associated rational $\underset{\sim}{C}$-functions tend to 1, uniformly on compact subsets of $|z| < 1$. Consequently, both numerator and denominator polynomials tend to the same polynomial, which turns out to be the "stable spectral factor" of $d(z,\ z^{-1})$.

## 3. Rational Covariance Extensions and the Dissipation Polynomial

A key result in partial realization theory, is that a sequence $C = \{c_t\colon t = 0,\ 1,\ \ldots\}$ is rational if and only if the power series

$$\Gamma(z) = c_o + 2\sum_{t=1}^{\infty} c_t z^t$$

defines a rational function in $z$ (see GANTMACHER [1959, Chapter V] and also KALMAN, FALB, and ARBIB [1969, Chapter 10]). Moreover, if $\Gamma(z) = \pi(z)/\chi(z)$ with $\pi(z)$, $\chi(z)$ coprime polynomials in $z$, then

$$\dim C = \max\{\deg \pi(z),\ \deg \chi(z)\}$$
$$=: \dim \Gamma(z).$$

The previously derived description for the solutions of the CARATHEODORY problem will now be applied for the study of the rational ones. Such a solution with data a partial sequence $C_s$ will be called a <u>pr</u> (<u>positive rational) - extension of</u> $C_s$.

In the degenerate case where $C_s$ is nonnegative but not positive, there exists (see page 10) a unique covariance extension which turns out to be rational. In the nondegenerate case where $C_s$ is positive the set of all pr-extensions is described in the following

(3.1) THEOREM. <u>Let</u> $C_s$ <u>be a given partial positive sequence,</u> $R_s = \{r_t\colon t = 1, \ldots, s\}$ <u>be the associated partial parameter sequence, and</u> $M_s^1(z)$ <u>be the corresponding matrix polynomial defined in Lemma</u> (2.6). <u>An irreducible rational function</u> $c_o\pi_1(z)/\chi_1(z)$, <u>with</u> $\pi_1(z), \chi_1(z) \in \underline{\underline{C}}[z]$ <u>and</u> $\pi_1(0) = \chi_1(0) = 1$, <u>is a pr-extension of</u> $C_s$ <u>if and only if there exists a unique irreducible rational</u> $\underset{\sim}{C}$<u>-function</u> $\pi_{s+1}(z)/\chi_{s+1}(z)$ <u>with</u> $\pi_{s+1}(z), \chi_{s+1}(z) \in \underline{\underline{C}}[z]$ <u>and</u> $\pi_{s+1}(0) = \chi_{s+1}(0) = 1$, <u>such that</u>

$$(3.2)\qquad \begin{pmatrix} \pi_1(z) \\ \chi_1(z) \end{pmatrix} = M_s^1(z) * \begin{pmatrix} \pi_{s+1}(z) \\ \chi_{s+1}(z) \end{pmatrix}.$$

<u>Then,</u> $R^{s+1} = \{r_{s+1}, r_{s+2}, \ldots\}$ <u>is the sequence of parameters associated with</u> $\pi_{s+1}(z)/\chi_{s+1}(z)$ <u>if and only if</u> $R^1 = \{r_1, \ldots, r_s, r_{s+1}, \ldots\}$ <u>is the sequence of parameters associated with</u> $\pi_1(z)/\chi_1(z)$. <u>Moreover, the following holds</u>

$$\dim \frac{\pi_{s+1}(z)}{\chi_{s+1}(z)} \le \dim \frac{\pi_1(z)}{\chi_1(z)} \le \dim \frac{\pi_{s+1}(z)}{\chi_{s+1}(z)} + s.$$

It is clear that any continuation $\{r_t : |r_t| < 1, t = s + 1, \dots\}$ for the partial sequence $R_s$ leads to a positive extension of $C_s$. The positivity is effectively characterized in terms of the associated sequence of parameters. This is not the case with the rationality. Due to the nonlinear transformation (2.9) between the c's and the r's the parameter sequence of a rational sequence is almost never a rational one. However, the interpolation approach shows that the rationality of a covariance sequence is completely determined by the tail of the parameter sequence.

PROOF. From (2.9) and (2.10) we readily obtain that a solution $\Gamma^1(z)$ to the interpolation problem is rational if and only if the $\underset{\sim}{C}$-function $\Gamma^{s+1}(z)$ associated with the continuation of the parameter sequence is also rational.

Let $\Gamma^{s+1}(z) = \pi_{s+1}(z)/\chi_{s+1}(z)$. The normalization conditions $\pi_1(0) = \chi_1(0) = 1$ and $\pi_{s+1}(0) = \chi_{s+1}(0) = 1$ are compatible due to the fact that

$$M_s^1(0)* = \frac{1}{2}\begin{pmatrix} 1 + r_s & 1 - r_s \\ 1 - r_s & 1 + r_s \end{pmatrix}.$$

We now derive the last inequality that provides bounds on the dimensions of various pr-extensions. We first define the matrix polynomial

$$N_u^v(z) := \frac{1}{2}\begin{pmatrix} D_u^v(z) & -B_u^v(z) \\ -C_u^v(z) & A_u^v(z) \end{pmatrix}, \tag{3.3}$$

for $v = 1, 2, \dots, n = 0, 1, \dots,$ with $A_u^v(z)$, $B_u^v(z)$, $C_u^v(z)$, $D_u^v(z)$ defined as in Lemma (2.6). Using (2.15) it can be shown by direct calculation that

$$N_u^v(z)^* \, M_u^v(z)^* = z^u h_u^v \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$h_u^v := \prod_{t=v}^{v+u} (1 - |r_t|^2).$$

Applying this to our case we have

$$h_s^1 z^s \begin{pmatrix} \pi_{s+1}(z) \\ \chi_{s+1}(z) \end{pmatrix} = N_s^1(z)^* \begin{pmatrix} \pi_1(z) \\ \chi_1(z) \end{pmatrix}.$$

Since the elements of $N_s^1(z)^*$ are polynomials of degree $s$, the above implies that

$$\deg \chi_{s+1}(z) \leq \deg \chi_{1(z)}$$

and also

$$\deg \pi_{s+1}(z) \leq \deg \pi_1(z).$$

From (3.2) we also have that

$$\deg \chi_1(z) \leq \deg \chi_{s+1}(z) + s$$

and similarly

$$\deg \pi_1(z) \leq \deg \pi_{s+1}(z) + s.$$

These prove the last inequality in (3.1). □

We now consider an irreducible rational $\underset{\sim}{C}$-function $\pi(z)/\chi(z)$. The real part of $\pi(z)/\chi(z)$ is nonnegative for all $z$ on $|z| = 1$ where it is defined. Therefore,

$$\pi(e^{j\theta})\overline{\chi(e^{j\theta})} + \overline{\pi(e^{j\theta})}\chi(e^{j\theta}) \geq 0$$

for all $\theta$ in $[-\pi, \pi]$. Following the terminology of an unpublished report by KALMAN we shall call

$$d(z, z^{-1}) := \frac{1}{2}\{\pi(z)\bar{\chi}(z^{-1}) + \bar{\pi}(z^{-1})\chi(z)\}$$

the <u>dissipation polynomial of</u> $\pi(z)/\chi(z)$. By slight abuse of terminology we shall also call a <u>dissipation polynomial</u> any polynomial $p(z, z^{-1})$ in both $z$ and $z^{-1}$, which for $z = \exp j\theta$ and all $\theta$ in $[-\pi, \pi]$ is nonnegative. Finally, let us define the <u>degree</u> of $d(z, z^{-1})$ as the largest power of $z$.

The role of the dissipation polynomial in the context of stochastic processes will be discussed in Remark (3.8). Herein, we shall see that the dissipation polynomials associated to the various pr-realizations of $C_s$ are up to a scalar factor determined completely by the choice of the $\underset{\sim}{C}$-function $\Gamma^{s+1}(z) = \pi_{s+1}(z)/\chi_{s+1}(z)$ of the previous theorem.

Let $\Gamma(z) = c_o\pi_1(z)/\chi_1(z)$ be a rational $\underset{\sim}{C}$-function and $R^1 = \{r_s : s = 1, \dots\}$ denote the associated parameter sequence. Let also $R^t = \{r_s : s = t, t+1, \dots\}$ denote the usual "shifted and truncated" parameter sequence and $\pi_t(z)/\chi_t(z)$, $d_t(z, z^{-1})$ the associated rational $\underset{\sim}{C}$-function and dissipation polynomial. We now present the following

(3.4) THEOREM. <u>In the degenerate case where</u>

$$R^1 = \{r_s : s = 1, 2, \dots, u, \ \underline{\text{with}} \ |r_s| < 1 \ \underline{\text{for}} \ 1 \leq s \leq u \ \text{and} \ |r_u| = 1\}$$

<u>we have that</u>

$$d_t(z, z^{-1}) \equiv 0 \ \underline{\text{for}} \ t = 1, 2, \dots u.$$

<u>In the nondegenerate case where</u>

$$R^1 = \{r_s : s = 1, 2, \ldots, \text{ with } |r_s| < 1 \text{ for all } s\}$$

<u>we have that</u>

$$d_1(z, z^{-1}) = h_1^t d_{t+1}(z, z^{-1}), \quad \underline{\text{for}} \quad t = 1, 2, \ldots .$$

The behavior of the dimension of $\frac{\pi_t(z)}{\chi_t(z)}$ as $t$ increases is described in

(3.5) PROPOSITION. <u>In the degenerate case where</u> $u$ <u>is as in Theorem</u> (3.4) <u>we have that</u>

$$\dim \frac{n_t(z)}{\chi_t(z)} = u - t + 1, \quad \underline{\text{for}} \quad t = 1, \ldots, u.$$

<u>In the nondegenerate case we have that the following two cases are possible</u>

$$\text{(a)} \qquad \dim \frac{\pi_{t+1}(z)}{\chi_{t+1}(z)} = \dim \frac{\pi_t(z)}{\chi_t(z)} ,$$

$$\text{(b)} \qquad \dim \frac{\pi_{t+1}(z)}{\chi_{t+1}(z)} = \dim \frac{\pi_t(z)}{\chi_t(z)} - 1.$$

<u>Furthermore, case</u> (a) <u>is equivalent to each of the following two conditions:</u>

$$\text{(a')} \quad \dim \frac{\pi_t(z)}{\chi_t(z)} = d_t(z, z^{-1}),$$

$$\text{(a'')} \quad \dim \frac{\pi_t(z)}{\chi_t(z)} = \deg (\pi_t(z) + \chi_t(z)).$$

We will now prove (3.4) and (3.5).

PROOF OF (3.4). In the degenerate case we have

$$\pi_t(z) = \Psi^t_{u-t+1}(z)^*,$$

and

$$\chi_t(z) = \Phi^t_{u-t+1}(z)^*.$$

From (1.15) we obtain $d_t(z, z^{-1}) = h^t_{u-t+1}$. Since $|r_u| = 1$ it follows that

$$h^t_{u-t+1} = \prod_{v=t}^{u}(1 - |r_v|^2) = 0$$

In the nondegenerate case we obtain from (1.15) that

$$d_1(z, z^{-1}) = \frac{1}{2}\{\pi_1(z)\bar{\chi}_1(z^{-1}) + \bar{\pi}_1(z^{-1})\chi_1(z)\}$$

$$= \frac{1}{2}\{\Psi^1_t(z)\bar{\Phi}^1_t(z^{-1}) + \bar{\Psi}^1(z^{-1})\Phi^1_t(z)\}\{\pi_{t+1}(z)\bar{\chi}_{t+1}(z^{-1}) + $$

$$+ \bar{\pi}_{t+1}(z^{-1})\chi_{t+1}(z)\}$$

$$= h^t_1 d_{t+1}(z, z^{-1}). \quad \square$$

PROOF OF (3.5). In the degenerate case,

$$\chi_t(z) = \Phi^t_{u-t+1}(z)^*,$$

and

$$\pi_t(z) = \Psi^t_{u-t+1}(z)^*,$$

for $t = 1, 2, \ldots, u$. Also, $|\Phi^t_{u-t+1}(0)| = |\Psi^t_{u-t+1}(0)| = |r_u| = 1$

is different from zero. Therefore,

$$\deg \chi_t(z) = \deg \pi_t(z) = u - t + 1$$

In the nondegenerate case, using (2.4) we obtain

$$z\pi_{t+1}(z) = \frac{1}{2}\{\pi_t(z) - \chi_t(z)) + z(\pi_t(z) + \chi_t(z))\}/(1 + r_t),$$

and

$$z\chi_{t+1}(z) = \frac{1}{2}\{\chi_t(z) - \pi_t(z)) + z(\chi_t(z) + \pi_t(z))\}/(1 - r_t).$$

This shows that both $\deg \pi_{t+1}(z)$ and $\deg \chi_{t+1}(z)$ are less than or equal to $\max\{\deg \pi_t(z), \deg \chi_t(z)\}$. Moreover, this difference can be at most one. Therefore (a) and (b) are the only two possibilities.

In the case where $\max\{\deg \chi_t(z), \deg \pi_t(z)\} = \deg(\pi_t(z) + \chi_t(z))$ then clearly $\deg \pi_{t+1}(z) = \deg \chi_{t+1}(z) = \deg(\pi_t(z) + \chi_t(z)) = \dim \pi_t(z)/\chi_t(z)$. In the case where the above does not hold then both $\deg \pi_{t+1}(z)$ and $\deg \chi_{t+1}(z)$ are less than $\dim \pi_t(z)/\chi_t(z)$. This establishes the equivalence of (a) and (a").

Consider now the identity

$$\frac{1}{2}\{\pi_t(z)\,\bar{\chi}_t(z^{-1}) + \bar{\pi}_t(z^{-1})\chi_t(z)\} = d_t(z, z^{-1})$$

$$= d_t^{(s)} z^s + \dots + d_t^{(o)} + \dots + \bar{d}_t^{(s)} z^{-s},$$

where we assume that $d_t^{(s)} \neq 0$. It is easy to see that $\deg(\pi_t(z) + \chi_t(z)) = \deg d_t(z, z^{-1})$. This establishes the equivalence of (a') and (a"). □

In view of the above, the dissipation polynomial of the various pr-extensions of $C_s$ can be arbitrarily chosen by appropriate choice of the C-function $\pi_{s+1}(z)/\chi_{s+1}(z)$. The set of all rational $\underset{\sim}{C}$-functions which have dissipation polynomial fixed up to a scalar factor, is described in the following

(3.6) PROPOSITION. *Let* $d(z, z^{-1})$ *be an arbitrary dissipation polynomial of degree* $s$. *Then for any polynomial* $a(z)$ *such that* $a(z)$

<u>and</u> $z^{s}d(z, z^{-1})$ <u>are coprime polynomials there exists a unique rational</u> $\underset{\sim}{C}$<u>-function</u> $\Gamma(z) = \pi(z)/\chi(z)$ <u>with</u> $\Gamma(0) = 1$ <u>and such that</u>

$$(3.7) \quad d(z, z^{-1}) = \frac{1}{2}\{\pi(z)\bar{\chi}(z^{-1}) + \bar{\pi}(z^{-1})\chi(z)\}$$

and

$$\pi(z) - \chi(z) = za(z).$$

<u>Conversely, for any rational</u> $\underset{\sim}{C}$<u>-function</u> $\Gamma(z)$ <u>having</u> $\Gamma(0) = 1$ <u>and dissipation polynomial</u> $d(z, z^{-1})$ <u>there exists a corresponding polynomial</u> $a(z)$ <u>as above.</u>

PROOF. Let $a(z)$ be any polynomial such that $z^{s}d(z, z^{-1})$ and $a(z)$ are coprime. Then

$$\frac{1}{4}d(z, z^{-1}) + a(z)\bar{a}(z^{-1})$$

is a dissipation polynomial which is zero on $|z| = 1$. Therefore, this polynomial factors into a product $b(z)\bar{b}(z^{-1})$ where $b(z)$ is coprime with $a(z)$ and has no root in $|z| \leq 1$.

Consider the function $S(z) := a(z)/b(z)$. Clearly we have that

$$1 - |S(z)|^{2} = \frac{1}{4}\,\frac{d(z, z^{-1})}{b(z)\bar{b}(z^{-1})} \geq 0,$$

for $z = \exp j\theta$ and $\theta$ in $[-\pi, \pi]$. Since $b(z)$ has no root in $|z| \leq 1$ it follows from the maximum modulus principle that $S(z)$ is in $\underset{\sim}{S}$. Therefore by (2.3)

$$\Gamma(z) = \frac{b(z) - za(z)}{b(z) + za(z)}$$

is in S. It can be readily checked that $\Gamma(z)$ has dissipation polynomial $d(z, z^{-1})$.

Conversely, for any $\Gamma(z) = \pi(z)/\chi(z)$ in $\underset{\sim}{C}$ where $\Gamma(0) = 1$ and (3.7) hold $a(z) = (\pi(z) - \chi(z))/z$ is the required polynomial. This is shown by reversing the previous argument. □

The algebraic approach we followed in this section gives only rough bounds on the dimension of the various pr-extensions of $C_s$. For example, given any dissipation polynomial of a certain degree u we can always find a corresponding $\underset{\sim}{C}$-function of the same degree. Then, by Theorem (3.1) we can obtain a pr-extension with dimensions between u and s + u.

In the next chapter, a different approach will be followed. It will be shown that, with an appropriate choice of the extension, we can always achieve dimension equal to s. Furthermore, it will be shown that for a large number of cases, s is the smallest possible dimension of any pr-extension.

(3.8) REMARK. The role of the dissipation polynomial in the context of stochastic realization will be now discussed. At the same time, certain quantities that will be used in the next section will now be introduced.

Consider a covariance sequence $C = \{c_s : s = 0, 1, \dots\}$. The nonnegativity of C or, equivalently, the covariance property is a necessary and sufficient condition for the existence of a nondecreasing function $\sigma(\theta)$, with $\theta$ in $[-\pi, \pi]$ such that

$$c_t = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-jt\theta} d\sigma(\theta), \; t = 0, \pm 1, \pm 2, \dots$$

(see AKHIEZER [1965, p. 180]). This function is called <u>spectral distribution</u> (<u>of</u> C, <u>or of a corresponding stochastic process</u>). The derivative $\sigma'(\theta)$ of $\sigma(\theta)$ exists almost everywhere in $[-\pi, \pi]$ and is called <u>spectral density</u>.

The $\underset{\sim}{C}$-function $\Gamma(z)$ associated to C as in Theorem (2.1) admits the following integral representation (see AKHIEZER [1965, p. 179]):

$$(3.9) \quad \Gamma(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{j\theta} + z}{e^{j\theta} - z}\, d\sigma(\theta)$$

Our interest rests in the case where $C$ is also a rational sequence. In this case $\sigma(\theta)$ consists of two parts:

$$(3.10)\quad \sigma(\theta) = \sigma_a(\theta) + \sigma_j(\theta),$$

where $\sigma_a(\theta)$ is an absolutely continuous nondecreasing function and $\sigma_j(\theta)$ is a nondecreasing function with finitely many points where the function increases. Furthermore, the derivative $\sigma_a'(\theta)$ is a rational function in $e^{j\theta}$. For more details see DOOB [1953, p. 542] and GRENANDER and SZEGÖ [1958, p. 5].

The decomposition (3.10) induces via (3.9) the representation

$$(3.11)\quad \Gamma(z) = \frac{\pi(z)}{\chi(z)} = \frac{\pi_a(z)}{\chi_a(z)} + \frac{\pi_j(z)}{\chi_j(z)}$$

where $\pi_a(z)/\chi_a(z)$ is a $\underset{\sim}{C}$-function with the property that $\chi_a(z_o) \neq 0$ for all $z_o$ in $|z| \leq 1$ and $\pi_j(z)/\chi_j(z)$ is of the form

$$(3.12)\quad \frac{\pi_j(z)}{\chi_j(z)} = \sum_{u=1}^{\deg \chi_j} \rho_u \frac{\exp(j\theta_u) + z}{\exp(j\theta_u) - z}$$

where $\rho_u$ are positive scalars.

Let $d(z, z^{-1})$, $d_a(z, z^{-1})$ and $d_j(z, z^{-1})$ denote the dissipation polynomials of the above three functions (in the obvious notation). From (3.12) it immediately follows that $d_j(z, z^{-1}) \equiv 0$. Relation (3.11) now implies that

$$(3.13)\quad d(z, z^{-1}) = d_a(z, z^{-1})\chi_j(z)\bar{\chi}_j(z^{-1}).$$

A stochastic realization of $C$ is a dynamical system $\Sigma$ that under certain stochastic input and initial states generates an output process $y_\tau$ that realizes $C$ via the covariance function. Any stochastic process $y_\tau$ can be decomposed into a superposition of two uncorrelated stochastic processes

$$y_\tau = y_{a,\tau} + y_{j,\tau}$$

where $y_{a,\tau}$ is the so-called purely nondeterministic part and $y_{j,\tau}$ the deterministic part. (This is called the Wold decomposition. For more information see GRENANDER and SZEGÖ [1958, Ch. 10] or functional analysis literature where it has been widely used, e.g., HELSON [1964, p. 10]))

In the case where C is a rational sequence then the above decomposition is in correspondence with (3.10). The part $y_{j,\tau}$ can be realized by superposition of sinusoidal signals with frequencies determined from the roots of $\chi_j(z)$ whereas $y_{a,\tau}$ can be realized by a single input system having white noise input and transfer function $\eta_a(z)/\chi_a(z)$ where

$$(3.14)\quad d_a(z, z^{-1}) = \eta_a(z)\bar{\eta}_a(z^{-1}).$$

The relations (3.13) and (3.14) indicate the role of the dissipation polynomial in this context.

Returning to the covariance extension problem it is natural to consider factorizations of the form (3.13) for the dissipation polynomials of both $\pi_1(z)/\chi_1(z)$ and $\pi_{s+1}(z)/\chi_{s+1}(z)$ in Theorem (3.1). Whenever both parts in (3.10) are present it is not necessarily true that (in the obvious notation) $\chi_{1,j}(z) = \chi_{s+1,j}(z)$. Due to this fact it appears that formula (3.2) in Theorem (3.1) is simply a computational tool and does not seem to have a stochastic interpretation.

## 4. Asymptotic Properties of the Spectral Zeros

We begin by discussing a procedure for spectral factorization. This result will be subsequently used to elucidate the role of the dissipation polynomial on the asymptotic behavior of the sequence of parameters.

Let $\Gamma^1(z) = \pi_1(z)/\chi_1(z)$ be a rational $\underset{\sim}{C}$-function with $\Gamma^1(0) = 1$ and $R^1 = \{r_s : s = 1, 2, \ldots\}$ the associated parameter sequence. We denote by $R^t$ the usual truncated sequences, $\pi_t(z)/\chi_t(z)$ the associated $\underset{\sim}{C}$-functions and $d_t(z, z^{-1})$ the dissipation polynomials, where $t = 1, 2, \ldots.$ We shall consider only the nondegenerate case where $d_1(z, z^{-1}) \not\equiv 0$. In this case by a well known factorization theorem (see GRENANDER and SZEGO [1958, p. 20]) there exists a polynomial $\eta_1(z)$, with $\eta_1(0) = 1$, and a positive scalar $\gamma_1$ such that

$$(4.1) \quad d_1(z, z^{-1}) = \gamma_1 \eta_1(z) \bar{\eta}_1(z^{-1}).$$

If we require that $\eta_1(z_o) \neq 0$ for all $z_o$ in $|z| < 1$ and also $\eta_1(0) = 1$, then both $\gamma_1$ and $\eta_1(z)$ are uniquely determined. This polynomial $\eta_1(z)$ we call the <u>stable spectral factor of</u> $d_1(z, z^{-1})$.

We similarly define $\gamma_t$, $\eta_t(z)$. From Theorem (3.4) we clearly have that

$$\eta_1(z) = \eta_t(z), \quad \text{for} \quad t = 1, 2, \ldots,$$

and also

$$\gamma_1 = \prod_{s=1}^{t} (1 - |r_s|^2) \gamma_t.$$

We now set

$$\pi_t(z) = 1 + a_1^{(t)} z + \ldots + a_s^{(t)} z^s,$$

and

$$\chi_t(z) = 1 + b_1^{(t)} z + \ldots + b_s^{(t)} z^s.$$

Clearly,

$$(4.2) \quad r_t = a_1^{(t)} - b_1^{(t)},$$

and by (2.4)

$$(4.3) \quad \pi_{t+1}(z) = 1/2[(\pi_t(z) - \chi_t(z))/z + (\pi_t(z) + \chi_t(z))]/(1 + r_t),$$

(4.4) $\chi_{t+1}(z) = 1/2[(\chi_t(z) - \pi_t(z))/z + (\pi_t(z) + \chi_t(z))]/(1 - r_t)$,

for $t = 1, 2, \ldots$. Iterating the above we obtain the sequence of pairs $(\pi_t(z), \chi_t(z))$, $t = 1, 2, \ldots$.

The following theorem states that (4.2), (4.3), *and* (4.4) *provide an algorithmic procedure to obtain* $\eta(z)$.

(4.5) THEOREM. *Let* $R^1$, $\pi_t(z)$, $\chi_t(z)$, *and* $\eta(z)$ *be as above. Then*

$$\lim_{t\to\infty} \pi_t(z) = \lim_{t\to\infty} \chi_t(z) = \eta(z).$$

PROOF. First we need to recall certain function theoretic results:

Let $\Gamma(z)$ be a $\underset{\sim}{C}$-function, and for simplicity assume $\Gamma(0) = 1$. Let $\sigma_a'(\theta)$ be the associated spectral density function that is given by

$$\sigma_a'(\theta) = \operatorname{Re} \Gamma(e^{j\theta})$$

a.e. on $[-\pi, \pi]$.

(4.6) STATEMENT. *The following are equivalent*:

(a) $\ell n\, \sigma_a'(\theta)$ *is integrable in* $[-\pi, \pi]$,

(b) *there exists a function* $m(z)$ *in* $H^2$ (*the usual Hardy space; see* RUDIN [1966, p. 328]) *such that*

$$\sigma_a'(\theta) = |m(e^{j\theta})|^2$$

(c) *a.e. on* $[-\pi, \pi]$,

(d) $\prod_{t=1}^{\infty}(1 - |r_t|^2) > 0$,

(e) $\sum_{t=1}^{\infty}|r_t|^2 < +\infty$.

For a proof of the above statement see GERONIMUS [1961, pages 20 and 159].

In case the above equivalent conditions hold we may find a function $m(z)$ that also has inverse that is analytic in $|z| < 1$.

Then $m(z)$ is

$$m(z) = \exp \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{j\theta} + z}{e^{j\theta} - z} \ln \sigma_a'(\theta)\, d\theta \quad , \ |z| < 1$$

and also

$$\gamma := |m(0)|^2$$

(4.7) $$= \exp - \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln \sigma_a'(\theta)\, d\theta$$

$$= \prod_{t=1}^{\infty} (1 - |r_t|^2)^{-1}$$

(See GERONIMUS [1961, pages 20, 21, and 158].) Furthermore, if $\alpha_s := \det T_{s-1}/\det T_s$, $s = 1, 2, \ldots,$ and $\alpha_o := 1$, then

$$\varphi_s(z)^* := \alpha_s \Phi_s(z)^*, \ s = 0, 1, \ldots$$

converges to $m(z)^{-1}$ as $s \to \infty$. This convergence is satisfying the following inequality (GERONIMUS [1961, Theorem 4.10]):

(4.8) $$|\varphi_s(z)^* m(z) - 1| \leq \gamma \delta_s \left\{ \frac{\gamma \delta_s}{1 + \gamma} + \frac{1}{\sqrt{1 - |z|}} \right\},$$

for $|z| < 1$ and where

$$\delta_s \leq \left( \sum_{t=s}^{\infty} |r_t|^2 \right)^{1/2}.$$

(Note that $\delta_s < +\infty$ because of (4.6).)

In case $\Gamma(z) = \pi(z)/\chi(z)$ is a rational $\underset{\sim}{C}$-function with dissipation polynomial $d(z, z^{-1}) \not\equiv 0$, the $\ln \sigma_a'(\theta)$ is integrable and in fact

$$m(z) = \gamma^{1/2} \eta(z)/\chi(z),$$

where $\eta(z)$, $\gamma$ satisfy (4.1) and $\eta(z)$ is the stable spectral factor of $d(z, z^{-1})$.

We now apply (4.8) to the $\underset{\sim}{C}$-functions $\pi_t(z)/\chi_t(z)$ for $t = 1, 2, \ldots,$ and for $s = 0$ :

$$|m_t(z) - 1| = |\gamma^{1/2} \frac{\eta(z)}{\chi_t(z)} - 1|$$

$$\leq \gamma_t \delta_{o,t} \left\{ \frac{\gamma_t \delta_{s,t}}{1 + \gamma_t} + \frac{1}{\sqrt{\phantom{1}} - |z|} \right\},$$

where $\delta_{s,t} \leq \left( \sum_{u=s+t-1}^{\infty} |r_u|^2 \right)^{1/2}$ . Since $\gamma_1 > 0$ and $\delta_{1,1} < +\infty$, it follows that $\lim \gamma_t = 1$ and $\lim_{t=\infty} \delta_{o,t} = 0$. Consequently,

$$|\gamma_t^{1/2} \eta(z)/\chi_t(z) - 1| \to 0$$

as $t \to \infty$, uniformly on compact subsets of $|z| < 1$. Hence,

$$\lim_{t\to\infty} \chi(z) = \eta(z)$$

Similarly we can show that $\lim_{t\to\infty} \pi_t(z) = \eta(z)$. □

Using the above we now want to study the asymptotic behavior of the sequence of parameters of rational covariance sequences. We begin by a motivating

(4.10) EXAMPLE. Consider the rational function

$$\frac{\pi(z)}{\chi(z)} = \frac{1 + (\alpha + r)z}{1 + (\alpha - r)z}$$

of degree one. Necessary and sufficient conditions for $\pi(z)/\chi(z)$ to be in $\underset{\sim}{C}$ is that

$$|r| \leq 1,$$

and

$$|\alpha| \leq 1 - |r|.$$

The sequence of parameters (provided $|r| \neq 1$) is determined by

$$(4.11) \quad \begin{aligned} r_1 &= r, \\ r_2 &= -\frac{\alpha r}{1 - |r|^2}, \end{aligned}$$

and the nonlinear recurrence law

$$(4.12) \quad r_{s+2} = \frac{r_{s+1}^2}{r_s(1 - |r_{s+1}|^2)}, \quad \text{for} \quad s = 1, 2, \ldots.$$

Rewriting the above in the following form

$$\frac{r_{s+2}}{r_{s+1}} = \frac{r_{s+1}}{r_s} \cdot \frac{1}{1 - |r_{s+1}|^2}$$

we obtain

$$(4.13) \quad \frac{r_{s+2}}{r_{s+1}} = \frac{r_2}{r_1} \cdot \frac{1 - |r_1|^2}{\prod_{t=1}^{s+1}(1 - |r_t|^2}.$$

By considering the factorization of the dissipation polynomial of $\pi(z)/\chi(z)$ we have

$$\begin{aligned} d(z, z^{-1}) &= \alpha z + (1 - |r|^2 + |\alpha|^2) + \bar{\alpha} z^{-1} \\ &= \gamma\eta(z)\bar{\eta}(z^{-1}) \\ &= \gamma(\beta z + (1 + |\beta|^2) + \bar{\beta} z^{-1} \end{aligned}$$

where $\eta(z) = 1 + \beta z$ is the stable spectral factor of $d(z, z^{-1})$.

Therefore, $\alpha = \gamma\beta$. Also by the result of GERONIMUS [1961, Theorem 8.2]

$$\gamma = \prod_{t=1}^{\infty}(1 - |r_t|^2) .$$

Combining the above two facts with (4.10) and using (4.13) we obtain

$$\lim_{s\to\infty} \frac{r_{s+2}}{r_{s+1}} = \frac{r_2(1 - |r_1|^2)}{r_1\gamma} = \frac{\alpha}{\gamma} = \beta$$

Equivalently,

$$(4.14) \quad \lim_{s\to\infty} \frac{r_{s+2} - \beta r_{s+1}}{r_{s+1}} = 0.$$

This shows that as $s \to \infty$ the sequence of parameters satisfies more accurately a linear recurrence law. Since in general $r_s$, $s = 1, 2, \ldots$, might take also zero values we consider the equivalent statement: for all $\epsilon > 0$ there exists an $s_o$ such that for all $s \geq s_o$

$$|r_{s+2} - \beta r_{s+1}| \leq \epsilon \max \{|r_{s+1}|, |r_{s+2}|\}.$$

This motivates the following

(4.15) DEFINITION. A sequence $R = \{r_s : s = 1, 2, \ldots\}$ is said to be almost rational iff there exists a polynomial $p(z) = 1 + \beta_1 z + \ldots + \beta_u z^u$ such that for all $\epsilon > 0$ there exists an integer $s_o$ with the property that for all $s \geq s_o$

$$|r_{s+u} + \beta_1 r_{s+u-1} + \ldots + \beta_u r_s| \leq \epsilon \max_{s \leq t \leq s+u} \{|r_t|\}.$$

for all $s \geq s_o$.

A polynomial $p(z)$ with the above properties is said to be an almost recurrence polynomial for $R$.

(4.16) THEOREM. Let $C$ be a rational positive sequence, $R$ be the associated parameter sequence and $\eta(z)$ be the stable spectral factor

<u>of the associated dissipation polynomial. Then</u> R <u>is almost rational and</u> $\eta(z)$ <u>is an almost recurrence polynomial for</u> R.

PROOF. Denote by $\pi_t(z)/\chi_t(z)$ the rational $\underset{\sim}{C}$-function associated with the usual truncated parameter sequences $\tilde{R}^t$, and let these functions have power series expansions

$$\frac{\pi_t(z)}{\chi_t(z)} = 1 + 2 \sum_{s=1}^{\infty} c_s^{(t)} z^s, \quad t = 1, 2, \ldots.$$

The relation between $\{c_s^{(t)}, s = 1, 2, \ldots\}$ and $R^t$ is given by the following formulas

$$c_1^{(t)} = r_t.$$

$$c_2^{(t)} = r_{t+1}(1 - |r_t|^2) + c_1^{(t)} c_1^{(t)},$$

$$\vdots$$

$$c_{s+1}^{(t)} = r_{t+s} \prod_{u=o}^{s-1}(1 - |r_{t+u}|^2) + $$

$$+ (c_1^{(t)} \ldots c_s^{(t)})(T_{s-1}^{(t)})^{-1} \begin{pmatrix} c_s^{(t)} \\ \vdots \\ c_1^{(t)} \end{pmatrix}$$

Let

$$\eta(z) = 1 + \beta_1 z + \ldots + \beta_u z^u,$$

and

$$\chi_t^{(z)} = 1 + b_1^{(t)} z + \ldots + b_u^{(t)} z^u.$$

By the previous theorem $\lim_{t\to\infty} \chi_t(z) = \eta(z)$. Therefore

$$(4.18) \quad |r_{t+u} + \beta_1 r_{t+u-1} + \ldots + \beta_u r_t| \leq |r_{t+u} + b_1^{(t)} r_{t+u-1} + \ldots + b_u^{(t)} r_t| + \epsilon \max_{t \leq s \leq t+u} |r_s|,$$

for all $s \geq s_o(\epsilon)$ and $\epsilon > 0$. The polynomial $X_t(z)$ satisfies

$$c_{u+1}^{(t)} + b_1^{(t)} c_u^{(t)} + \ldots + b_u^{(t)} c_1^{(t)} = 0.$$

From the above and (4.17) we now have

$$(4.19) \quad |r_{t+u} + b_1^{(t)} r_{t+u-1} + \ldots + b_u^{(t)} r_t| \leq \max_{t \leq s \leq t+u} |r_s| \cdot f_t$$

where $f_t$ is a polynomial function in $r_{t+s-1}$, $b_s^{(t)}$, $(\det T_{s-1}^{(t)})^{-1}$, for $s = 1, \ldots, u$, that when viewed as a function of the $r_s$'s has has zero constant term. We shall now show that

$$(4.20) \quad \lim_{t\to\infty} f_t = 0$$

In the case of rational positive sequences $\ell n \sigma_a'(\theta)$ is integrable and therefore by the result of GERONIMUS [1961, Theorem 8.2] the parameter sequence R is squarely summable. Hence

$$(4.21) \quad \lim_{t\to\infty} r_t = 0$$

In view of (4.16) it follows that

$$\lim_{\substack{t\to\infty \\ s \text{ fixed}}} c_s^{(t)} = 0.$$

(Note that $\lim_{s\to\infty, t:\text{fixed}} c_s^{(t)}$ is not necessarily zero.) Then

$$(4.22) \quad \lim_{t\to\infty} \det T_{s-1}^{(t)} = 1$$

for all s. Also by Theorem (4.15)

$$(4.23) \quad \lim_{t\to\infty} b_s^{(t)} = \beta_s,$$

for $s = 1, \ldots, u$. From (4.21), (4.22), and (4.23) we conclude (4.20). Finally (4.18), (4.19) and (4.20) imply that for all $\epsilon > 0$ there exists an $s_o(\epsilon)$ so that

$$|r_{t+u} + \beta_1 r_{t+u-1} + \cdots + \beta_u r_t| \leq \epsilon \max_{t \leq s \leq t+u} |r_s|,$$

for all $t \geq s_o(\epsilon)$. □

(4.21) REMARK. In the applied literature on time-series analysis it has been noted (see for example BOX and JENKINS [1970, p. 179]) that the asymptotic behavior of the partial autocorrelation coefficients of rational power spectra, that are precisely the SCHUR parameters of our setting, is "dominated by damped exponentials". However, no precise statement of this seems to have been proven. Moreover, in case the almost recurrence polynomial has roots on $|z| = 1$, the above statement is not absolutely correct. For example take $\Gamma(z) = 1 - z$. Then the sequence of parameters is given by

$$r_t = \frac{(-1)^t}{t+1}$$

The asymptotic behavior of this sequence is not dominated by exponentials.

# CHAPTER IV. A TOPOLOGICAL APPROACH

In this chapter we develop an alternative approach to the study of pr-extensions of $C_s$. We focus our attention to pr-extensions of dimension less than or equal to s. Our key result will be an implicit description of this set.

In Section 5 we show that for a nonempty open subset of the data-set of partial positive sequences $C_s$ there exist no pr-extensions of dimension strictly less than s. This result justifies our interest in pr-extensions of dimension s.

After a brief exposition in Section 6 of some basic facts about the topological degree, we derive in Section 7 our key result: <u>For almost any dissipation polynomial of degree less than or equal to</u> s <u>there exists a corresponding pr-extension of</u> $C_s$ <u>of dimension at most</u> s. We should note that according to the results of the previous chapter this dimension could be as large as 2s.

This result further provides a novel proof of the classically known fact that the positivity of $C_s$ is a sufficient condition for the existence of solutions to the CARATHEODORY problem.

Also, most important, this topological approach provides an implicit description of a nonuniqueness inherent in this partial realization problem.

## 5. Covariance Extensions of Dimension s.

The following well known proposition gives conditions for a rational function to belong to $\underset{\sim}{C}$.

(5.1) PROPOSITION. <u>An irreducible rational function</u> $c_o\pi(z)/\chi(z)$, <u>with</u> $\pi(z)$, $\chi(z)$ <u>in</u> $\underline{\underline{C}}[z]$ <u>and</u> $c_o\pi(0)/\chi(0) = c_o \in \underline{\underline{R}}$, <u>is in</u> $\underset{\sim}{C}$ <u>if and only if</u>

(a) $\quad d(z, z^{-1}) := \frac{c_o}{2} \{\pi(z)\bar{\chi}(z^{-1}) + \bar{\pi}(z^{-1})\chi(z)\}$

<u>is a dissipation polynomial, and</u>

(b) $\frac{1}{2}\{\pi(z) + \chi(z)\}$ <u>has no root in</u> $|z| \leq 1$.

We let $C_s = \{c_t : t = 0, 1, \ldots, s\}$ be a positive sequence and we consider a rational function $c_o\pi(z)/\chi(z)$ with power series expansion in $z$ that begins with

$$c_o + 2\sum_{t=1}^{s} c_t z^t.$$

A rational function with this property will be called a <u>partial realization of</u> $C_s$. Thus, a partial realization $c_o\pi(z)/\chi(z)$ of $C_s$ is a pr-extension of $C_s$ if and only if (a) and (b) of Proposition (5.1) hold.

If $c_o\pi(z)/\chi(z)$ is a partial realization of $C_s$ and $\pi(z)$, $\chi(z)$ have degree less than or equal to $s$ then $\pi(z)$, $\chi(z)$ and $b(z) := \{\pi(z) + \chi(z)\}/2$ are related via the following nonsingular linear transformations

$$(5.2) \qquad c_o\pi(z) = [(c_o + 2c_1 z + \ldots + 2c_s z^s)\chi(z)]_o^s,$$

and

$$(5.3) \qquad c_o b(z) = [(c_o + c_1 z + \ldots + c_s z^s)\chi(z)]_o^s,$$

where $[\;]_o^s$ denotes truncating the powers of $z$ outside $[0, s]$. We want to consider when $C_s$ admits pr-extensions of dimension strictly less than $s$. This is given in the following

(5.4) LEMMA. <u>There exists a pr-extension of</u> $C_s$ <u>of dimension strictly less than</u> $s$ <u>if and only if there exists a polynomial</u> $b(z)$ <u>of degree less than or equal to</u> $s$ <u>such that for the polynomials</u> $\chi(z)$ <u>and</u> $\pi(z)$ <u>obtained through (5.2) and (5.3) the following hold</u>

(a') $d(z, z^{-1}) = \frac{c_0}{2}\{\pi(z)\bar{\chi}(z^{-1}) + \bar{\pi}(z^{-1})\chi(z)\}$

<u>is a dissipation polynomial,</u>

(b') $b(z) = \frac{1}{2}\{\pi(z) + \chi(z)\}$ <u>has no root in</u> $|z| < 1$, <u>and</u>

(c') $\pi(z)$, $\chi(z)$ <u>have a nontrivial common factor</u>.

The essential point is that (b') is a closed condition as compared with (b) in Proposition (5.1).

PROOF. Suppose $b(z)$, $\pi(z)$ and $\chi(z)$ satisfy the conditions of the lemma and let $\pi_0(z)$, $\chi_0(z)$ be coprime polynomials such that

$$\frac{\pi(z)}{\chi(z)} = \frac{\pi_0(z)}{\chi_0(z)} \quad .$$

Then $\deg \pi_0(z)$ and $\deg \chi_0(z)$ are less than $s$. Also $d_0(z, z^{-1}) = \{\pi_0(z)\bar{\chi}_0(z^{-1}) + \bar{\pi}_0(z^{-1})\chi_0(z)\}/2$ is a dissipation polynomial and $\pi_0(z) + \chi_0(z)$ has no zero in $|z| < 1$. In order for $\pi_0(z)/\chi_0(z)$ to be a pr-extension of $C_s$ we only need to show that $\pi_0(z) + \chi_0(z)$ has no root on $|z| = 1$.

Suppose $\pi_0(z_0) + \chi_0(z_0) = 0$ for some $z_0$ with $|z_0| = 1$. Then

$$|\pi_0(z_0) + \chi_0(z_0)|^2 = d_0(z_0, z_0^{-1}) + |\pi_0(z_0)|^2 + |\chi_0(z_0)|^2 = 0.$$

Since $d_0(z_0, z_0^{-1}) \geq 0$ it follows that $\pi_0(z_0) = \chi_0(z_0) = 0$, which contradicts the hypothesis that $\pi_0(z)$ and $\chi_0(z)$ were coprime. Therefore $\pi_0(z) + \chi_0(z)$ has no root in $|z| \leq 1$ and $\pi_0(z)/\chi_0(z)$ is a pr-extension of $C_s$ with dimension strictly less than $s$.

The converse is trivial. □

Consider now $Y$ to be the set of nonnegative sequences $C_s = \{c_t : t = 0, 1, \ldots, s\}$ that for simplicity we assume $c_0 = 1$.

The interior $Y^o$ of $Y$ is the set of positive sequences $C_s$. We shall show that

(5.5) PROPOSITION. The set of partial nonnegative sequences $C_s$ that admit no pr-extension of dimension strictly less than $s$ is an open subset of $Y$.

Clearly, this would also imply

(5.6) PROPOSITION. The set of partial positive sequences $C_s$ that admit no pr-extension of dimension strictly less than $s$ is an open subset of $Y^o$.

PROOF of Proposition (5.5). Denote by $X$ the space of polynomials $b(z)$ of degree less than or equal to $s$ with $b(0) = 1$. The subset of $X$ where $(b')$ of Lemma (5.4) holds can be shown to be compact. Since $Y$ is also a compact space it follows that the subset of pairs

$$(b(z),\ C_s) \in X \times Y$$

where $(a')$ to $(c')$ of Lemma (5.4) hold is also compact. The projection onto $Y$ being a continuous map, implies that the subset of nonnegative sequences $C_s$ (which by Lemma (5.4) admits a pr-extension of dimension strictly less than $s$) is compact. The complement of this set is therefore open.

The fact that this set is nonempty follows by considering the partial sequence $C_s = \{1, 0, \ldots, 0, 1/2\}$. Clearly, $C_s$ is positive and moreover there is not even partial realization of $C_s$ of dimension less than $s$. Hence, there is no pr-extension of $C_s$ with dimension less than $s$ either. □

Given $C_s$, whether there exists a pr-extension of dimension strictly less than $s$ is a decidable question. It can be answered by applying the decision methods developed by TARSKI [1951] and SEIDENBERG [1954] (see also JACOBSON [1974, Chapter V ]) to the conditions of Lemma (5.4). However, these are very involved and a simpler criterion is lacking. In fact, in the Appendix we shall indicate the set of conditions that needs to be tested for the first nontrivial case.

But, the set of pr-extensions of dimension less than or equal to $s$ is known to be always nonempty. We focus our study on this set. We shall use concepts of homotopy and degree theory for this. So we now make a brief digression and introduce the essentials of degree theory.

## 6. Basic Degree Theory.

The "degree" of this section refers to a notion of topological degree soon to be defined. The object of study of degree theory is the solution set of an equation $d = f(b)$ where $f$ is a mapping between two topological spaces. The main question concerns the existence and the number of solutions for a given $d$.

Let $S$ be an open subset of some topological space $X$, $f$ a continuous map from $S$ into a topological space $D$, and $d$ be a point in $D$. The aim of degree theory is to define an integer valued function $\deg(f, S, d)$, called the <u>degree of</u> $f$ <u>at</u> $d$ <u>relative to</u> $S$, with the properties that

(a) $\deg(f, S, d)$ is an estimate of the number of solutions of $d = f(b)$ in $S$,

(b) $\deg(.,.,.)$ be continuous in the arguments, and

(c) $\deg(.,.,.)$ be additive in the domain $S$, i.e., whenever $S_1 \cap S_2 = \emptyset$ then $\deg(f, S_1 \cup S_2, d) = \deg(f, S_1, d) + \deg(f, S_2, d)$.

As usual, <u>when</u> $S$ <u>is a subset of topological space</u> $X$ <u>we denote by</u> $\bar{S}$, $\partial S$, <u>and</u> $S^o$ <u>the closure, the boundary, and the interior of</u> $S$ respectively. The exposition below is following NAGUMO [1951], SCHWARZ [1965, Ch. III], and LLOYD [1978, Ch. I] where we refer for additional information and detailed proofs.

Let X and W both denote the Euclidean space $\underline{\underline{R}}^N$ with the usual topology. (The reason for this redundant notation will become clear below.) The set S is assumed to be open and bounded subset of X. The maps that we consider are continuously differentiable in an open subset containing $\bar{S}$. The set of such mappings is denoted by $C^1(\bar{S})$ and topologized by the norm

$$\|f\|_1 := \sup_{\substack{b\in S \\ 1\le t\le N}} |f_t(b)| + \sup_{\substack{b\in S \\ 1\le s,t\le N}} \left|\frac{\partial f_t}{\partial b_s}(b)\right| .$$

Given $f \in C^1(\bar{S})$, $Z_f(\bar{S})$ denotes the set of points d in W such that there exists a point b in $f^{-1}(d)$ where the Jacobian $J_f(b)$ is zero.

Suppose now that $f \in C^1(\bar{S})$ and $d \in W$ but $d \notin f(\partial S) \cup Z_f(\bar{S})$. The <u>degree of</u> f <u>at</u> d relative to S is defined by

$$\deg (f, S, d) := \sum_{\{b\in f^{-1}(d)\cap S\}} \operatorname{sign} J_f(b).$$

The definition is extended to points $d_o$ that belong to $Z_f(\bar{S})$, but do not belong to $f(\partial S)$, by letting

$$\deg (f, S, d_o) = \deg (f, S, d)$$

for any $d \notin f(\partial S) \cup Z_f(\bar{S})$ and d "sufficiently close" to $d_o$.

The fact that this is well-defined and the precise meaning of the term "sufficiently close" are described by the following

(6.1) THEOREM. <u>Let</u> f <u>be as above and</u> $d_1$, $d_2$ <u>belong to the same component of</u> $W \setminus f(\partial S)$. <u>Suppose also that neither of them belongs to</u> $Z_f(\bar{S})$. <u>Then</u>

$$\deg (f, S, d_1) = \deg (f, S, d_2).$$

A simple consequence of the definition of degree is

(6.2) PROPOSITION. Let $d \notin f(\partial S)$. Then, $\deg (f, S, d) \neq 0$ implies that $d \in f(S)$.

A notion that is crucial for the development of the next section is that of homotopy: A $C^1$-homotopy between two elements $f_o$ and $f_1$ in $C^1(\bar{S})$ is a function

$$H : \bar{S} \times [0, 1] \to \underline{\underline{R}}^N$$

such that if $H_x$ denotes the map $b \mapsto H(b, x)$, then $H_o = f_o$, $H_1 = f_1$, $H_x \in C^1(\bar{S})$ for all $x$ in $[0, 1]$ and also $\|H_x - H_y\|_1 \to 0$ as $x \to y$. This last condition says that $H$ is a continuous function in the parameter $x$.

The following is a very powerful result that we shall use in the next section.

(6.3) THEOREM. Let $f_o$, $f_1$ be in $C^1(\bar{S})$, and $H$ be a $C^1$-homotopy between $f_o$ and $f_1$. If $d \notin H(\partial S, x)$ for all $x$ in $[0, 1]$ then

$$\deg (f_o, S, d) = \deg (f_1, S, d).$$

## 7. Dissipation Polynomials and Covariance Extensions of Dimension $s$.

In this section we prove the following key result.

(7.1) THEOREM. Let $C_s = \{c_t : t = 0, 1, \ldots, s\}$ be a partial positive sequence and

$$d(z, z^{-1}) := d_s z^s + \ldots + d_1 z + 1 + \bar{d}_1 z^{-1} + \ldots + \bar{d}_s z^{-s}$$

be a dissipation polynomial (of degree $\leq s$). Then there exists a pair of polynomials $(\pi(z), \chi(z))$ with $\deg \pi(z)$, $\deg \chi(z)$ less than or equal to $s$, and a positive scalar $k$ such that the following two conditions hold

(a) $c_o\pi(z)/\chi(z)$ <u>is a pr-extension of</u> $C_s$,

(b) $kd(z, z^{-1}) = \frac{1}{2}\{\pi(z)\bar{\chi}(z^{-1}) + \bar{\pi}(z^{-1})\chi(z)\}$.

We now elaborate on the implications of the above theorem with two immediate corollaries.

(7.2) COROLLARY. <u>Consider a partial positive sequence</u> $C_s$. <u>There exists always a pr-extension of</u> $C_s$ <u>with dimension at most equal to</u> s.

In this way we have circumvented the need for the algebraic machinery of orthogonal polynomials or of interpolation theory in order to establish that the positivity of $C_s$ is a sufficient condition for the existence of solutions to the CARATHEODORY problem. This is essentially a problem in analysis and an approach like ours seems to be absent. Furthermore, with this new approach we obtain some additional information about the set of pr-extensions of dimension s.

(7.3) COROLLARY. <u>Consider the partial positive sequence</u> $C_s = \{c_t : t = 0, 1, \ldots, s\}$ <u>where not all of</u> $c_t$, $t = 1, \ldots, s$ <u>are zero. Then, for almost any dissipation polynomial</u> $d(z, z^{-1})$ <u>of degree less than or equal to</u> s <u>there exists an associated pr-extension of</u> $C_s$ <u>with precisely dimension</u> s.

We now proceed to the

PROOF of Theorem (7.1). We again denote by X the space of polynomials with constant term 1 and with degree less than or equal to s.

Any $b(z) \in X$ defines through (5.2) and (5.3) a unique pair of polynomials $(\pi(z), \chi(z)) \in X^2$ such that $c_o\pi(z)/\chi(z)$ is a partial realization of $C_s$, i.e., has power series expansion that begins with

$$c_o + 2c_1z + \ldots + 2c_sz^s.$$

The correspondence $b(z) \to (\pi(z), \chi(z))$ is certainly bijective, whereas the correspondence $b(z) \to \pi(z)/\chi(z)$ is clearly not.

To any pair $(\pi(z), \chi(z))$ as above we associate the polynomial in $z$ and $z^{-1}$:

$$d(z, z^{-1}) := d_s z^s + \dots + d_1 z^{-1} + d_o + \bar{d}_1 z^{-1} + \dots + \bar{d}_s z^{-s}$$

$$= \frac{c_o}{2}\{\pi(z)\bar{\chi}(z^{-1}) + \bar{\pi}(z^{-1})\chi(z)\}.$$

The constant term $d_o$ equals

$$d_o = \tfrac{1}{2}[[\chi(z)\Gamma(z)]_o^s \bar{\chi}(z^{-1})]_o^o + \tfrac{1}{2}[[\bar{\chi}(z^{-1})\bar{\Gamma}(z^{-1})]_{-s}^o \chi(z)]_o^o$$

$$= [\bar{\chi}(z^{-1})(c_{-s}z^{-s} + \dots + c_s z^s)\chi(z)]_o^o$$

$$= \|\chi(z)\|_s^2,$$

where $\| \ \|_s$ denotes the norm that $C_s$ induces on the space of polynomial of degree less than or equal to $s$ (see page 15). Since $\chi(0) = 1$ and $C_s > 0$ then

$$d_o = \|\chi(z)\|_s^2 \neq 0$$

Let $W$ be the space of "symmetric" polynomials

$$\underset{\sim}{d}(z, z^{-1}) = \underset{\sim}{d}_s z^s + \dots + \underset{\sim}{d}_1 z + 1 + \bar{\underset{\sim}{d}}_1 z^{-1} + \dots + \bar{\underset{\sim}{d}}_1 z^{-s},$$

with constant term equal to 1. With $d(z, z^{-1})$ as above we define the map

$$\varphi_{C_s} : X \to W: (1 + b_1 z + \dots + b_s z^s) \mapsto$$

$$\mapsto \left(\frac{d_s}{d_o} z^s + \dots + \frac{d_1}{d_o} z + 1 + \frac{\bar{d}_1}{d_o} z^{-1} + \dots + \frac{\bar{d}_s}{d_o} z^{-s}\right).$$

Both X and W are Euclidean spaces of the same dimension, and $\varphi_{C_s}$ is continuously differentiable in X.

We now consider two open subsets $S \subseteq X$ and $P \subseteq W$, where S is the subset of polynomials $b(z)$ that satisfy

$$b(z_o) = 0 \text{ implies } |z_o| > 1,$$

and P is the subset that consists of all $\underset{\sim}{d}(z, z^{-1}) \in W$ that satisfy

$$\underset{\sim}{d}(z_o, z_o^{-1}) \geq 0 \text{ for all } z_o \text{ on } |z_o| = 1.$$

Therefore $\bar{S}$ is the set of ("stable") polynomials $b(z)$ such that

$$b(z_o) = 0 \text{ implies } |z_o| \geq 1,$$

and $\bar{P}$ is the set of dissipation polynomials with constant term equal to the identity.

The statement of the Theorem can be easily seen to be equivalent to: <u>for any</u> $d \in \bar{P}$ <u>there exists a</u> $b \in \bar{S}$ <u>such that</u>

$$d = \varphi_{C_s}(b).$$

Therefore we need to show that $\varphi_{C_s}(\bar{S}) \supseteq \bar{P}$.

Since the roots of a polynomial depend continuously on the coefficients (see MARDEN [1966]) it follows that S is open. Also because the roots of every $b(z)$ in S lie in $|z| > 1$ it follows that S is bounded. We shall first show that

$$(7.4) \quad \varphi_{C_s}(S) \supseteq P.$$

For the particular sequence $C_s^o = \{1, 0, \ldots, 0\}$ the map $\varphi$ takes the simple form

$$\varphi_{C_s^o} : b(z) = 1 + b_1 z + \ldots + b_s z^s \mapsto b(z)\bar{b}(z^{-1}) / \sum_{t=o}^{s} |b_t|^2.$$

It is straightforward to show that

(7.5) $\deg(\varphi_{C_s^o}, S, d) = 1$ for any $d$ in $P$.

In fact the computations can easily be done for $d(z, z^{-1}) = 1$, and since $\varphi_{C_s^o}(\partial S) = \partial P$ we use Theorem (6.1) to establish (7.5).

The set of positive partial sequences is connected (this is obvious especially when we consider the SCHUR parametrization; see page 11). Therefore we can follow a path within the set of positive sequences from $C_s^o$ to any other positive sequence $C_s$. In this way we construct a continuous homotopy $H(b, x)$ between $\varphi_{C_s^o}$ and $\varphi_{C_s}$.

We now show that

(7.6) $\varphi_{C_s}(\partial S) \cap P = \emptyset$ <u>for any</u> $C_s > 0$.

Suppose $b(z) \in \partial S$. Then $b(z_o) = 0$ for some $z_o$ with $|z_o| = 1$. Therefore, if $d(z, z^{-1}) = \varphi_{C_s} b(z) \in \bar{P}$, then $|b(z_o)|^2 = 0$ implies

$$d(z_o, z_o^{-1}) = 0.$$

Consequently, $d(z, z^{-1}) \in \partial P$ and (7.6) is proven.

If $H(b, x)$ is a homotopy as above, then

$$H(\partial S, x) \cap P \neq \emptyset,$$

for all $x$ in $[0, 1]$. By Theorem (6.3) we conclude that

$$\deg(\varphi_{C_s}, S, d) = 1,$$

for any $C_s > 0$ and any $d$ in $P$. Hence, by Proposition (6.2) it follows that

$$\varphi_{C_s}(S) \supseteq P$$

for all $C_s > 0$. By the compactness of $\bar{S}$ we also have that

$$\varphi_{C_s}(\bar{S}) \supseteq \bar{P}. \quad \square$$

We want to close this chapter with the following

(7.7) CONJECTURE. The correspondence between dissipation polynomials $d(z, z^{-1})$ and pairs of polynomials $(\pi(z), \chi(z))$ with $\pi(0) = \chi(0) = 1$, in Theorem (7.1) is bijective.

The conjecture is certainly true for the trivial sequence $C_s = \{1, 0, \ldots, 0\}$. We were also able by direct computation of the Jacobian to show that it holds in a neighborhood of $d(z, z^{-1}) = 1$. But a proof is still lacking. We should mention that the map $\varphi_{C_s}$ is not analytic, therefore

$$\deg(\varphi_{C_s}, S, d) = 1$$

for all $d$ in $P$ does not imply that the cardinality of $\varphi_{C_s}^{-1}(d) \cap S$ is one.

# CHAPTER V. THE MATRIX CASE

## 8. The Matrix Covariance Extension Problem

Given an n-variate, zero-mean, stationary stochastic process $y_\tau$, $\tau \in \underline{\underline{Z}}$ we denote by

$$c_s := Ey_\tau \tilde{y}_{\tau+s}, \quad s = 0, 1, \ldots,$$

the covariance $n \times n$-matrix-function of $y_\tau$. In this chapter we shall use "$\sim$" to denote the "complex conjugate transpose of ".

The covariance sequence $C = \{c_s : s = 0, 1, \ldots\}$ is characterized by the nonnegative definiteness of the block Toeplitz matrices

$$T_s = [c_{t-u}]_{t,u=o}^{s}, \quad s = 0, 1, \ldots,$$

where now $c_{-|t|} := \tilde{c}_{|t|}$. (See for example GIHMANN and SKOROHOD [1974, p. 196].)

Thus, we define a matrix-sequence

$$C = \{c_s : s = 0, 1, \ldots, \text{ with } c_o \text{ Hermitian}\}$$

to be <u>positive</u> (resp. <u>nonnegative</u>) iff the associated block Toeplitz matrices $T_s$ are positive (resp. nonnegative) definite for all $s$. We similarly define the partial matrix sequence $C_s = \{c_t : t = 0, 1, \ldots, s\}$ to be <u>positive</u> (resp. <u>nonnegative</u>) iff $T_s$ is a positive (resp. nonnegative) definite matrix.

This notion of matricial positivity is again related to an analytic property of the matrix-valued power series

$$\Gamma(z) := c + 2\sum_{s=1} c_s z^s.$$

(8.1) THEOREM (see KOVALISHINA and POTAPOV [1982]). <u>The power series</u> $\Gamma(z)$ <u>converges in</u> $|z| < 1$, <u>and</u>

$$\Gamma(z) + \widetilde{\Gamma(z)}$$

<u>is a nonnegative definite matrix for all</u> $z$ <u>in</u> $|z| < 1$ <u>if and only if the sequence</u>

$$C = \{c_s : s = 0, 1, \ldots, \text{ with } c_o = (c + \tilde{c})/2\}$$

<u>is nonnegative.</u>

Matrix-valued functions that satisfy the above conditions will again be said to belong to class $\underset{\sim}{C}$.

The following is now the <u>matrix CARATHEODORY problem:</u> Given a partial sequence $C_s$, find necessary and sufficient conditions for the existence of a matrix-valued $\underset{\sim}{C}$-function with power series that begin with

$$c_o + 2\sum_{t=1}^{s} c_t z^t.$$

The matrix CARATHEODORY problem seems to have been considered only recently by IL'MUSKIN [1974], and KOVALISHINA [1974]. See also AROV and KREIN [1981], DELSARTE, GENIN and KAMP [1979] and KOVALISHINA and POTAPOV [1982]. As an interpolation problem it can also be approached through the functional analytic techniques of SZ.-NAGY and FOIAS [1970]. See e.g., HELTON [1980].

In the next two sections we will consider the subclass of rational solutions and carry out some of the program followed in the scalar case.

The matrix sequence $C$ is said to be <u>rational</u> iff there exists an integer $v$ such that for all $s \geq v$ the block-behavior (Hankel) matrices

$$B_s = [c_{t+u-1}]_{t,u=1}^{s}$$

have the same rank. This integer $v$ will again be called the <u>dimension of</u> C.

The rationality of $C$ is equivalent to $\Gamma(z)$ defining a rational function in $z$. In this case $\Gamma(z)$ can be represented as a matrix fraction $P(z)Q(z)^{-1}$ (right) or $Q(z)^{-1}P(z)$ (left).

Suppose that $C$ is rational and that $P(z)Q(z)^{-1}$ is a right matrix fractional representation of $\Gamma(z)$ where $P(z)$ and $Q(z)$ are right coprime polynomial matrices, i.e. there exist $A(z)$, $B(z) \in \underline{\underline{C}}^{n\times n}[z]$ such that $A(z)P(z) + B(z)Q(z) = I$. Then it can be shown that the dimension of $C$ is equal to the maximum of the degrees of $P(z)$ and $Q(z)$. (In this chapter, "I" *will denote the* $n \times n$ *identity matrix*.)

In Section 9 we present a generalization of our topological approach for the matrix CARATHEODORY problem. We shall draw similar conclusions as in the scalar case: (a) the positivity of the partial sequence $C_s$ is sufficient for the existence of solutions, and (b) for almost any matrix-dissipation polynomial of degree less than or equal to $s$ there exists a corresponding rational solution of dimension less than or equal to $s$.

In Section 10 we shall give a brief account of the basic results that come out of the algebraic approach and SCHUR's algorithm when applied to this matrix-interpolation problem.

## 9. The Topological Approach.

We begin by establishing the matrix version of Proposition (5.1).

(9.1) PROPOSITION. *A rational function* $P(z)Q(z)^{-1}$ *with* $Q(z)$, $P(z)$ *in* $\underline{\underline{C}}^{n\times n}[z]$ *and* $P(0)Q(0)^{-1}$ *Hermitian positive definite, is in* $\underset{\sim}{C}$ *if and only if*

$$\det [Q(z_o) + P(z_o)] = 0 \ \text{implies} \ |z_o| > 1,$$

*and*

$$D(z, z^{-1}) := \frac{1}{2} \{\tilde{Q}(z^{-1})P(z) + \tilde{P}(z^{-1})Q(z)\}$$

*has nonnegative definite values for all* $z$ *such that* $|z| = 1$.

As in the scalar case we call $D(z, z^{-1})$ the *dissipation (matrix-polynomial of* $P(z)Q(z)^{-1}$. By a slight abuse of our terminology we shall also call any matrix polynomial in $z$ and $z^{-1}$ satisfying the above property a dissipation polynomial.

PROOF. An $n \times n$ matrix-function $S(z)$ is said to be in class $\underset{\sim}{S}$ iff it is analytic in $|z| < 1$ and

$$I - S(z)\widetilde{S(z)} \geq 0$$

for all $z$ in $|z| < 1$. These functions are considered in operator theory (see SZ.-NAGY and FOIAS [1970]) where they are called "contractive" and in circuit theory (see BELEVITCH [1968]) where they are called "bounded". The relation between $\underset{\sim}{C}$-functions $\Gamma(z)$ with $\Gamma(0)$ Hermitian positive definite, and $\underset{\sim}{S}$-functions $S(z)$ is given by (see for example [DELSARTE, GENIN, and KAMP [1979, p. 39])

$$(9.2) \quad S(z) = \frac{1}{2}(\Gamma(0) - \Gamma(z))(\Gamma(0) + \overline{\Gamma(z)})^{-1},$$

and

$$(9.3) \quad \Gamma(z) = (I - zS(z))(I + zS(z))^{-1}\Gamma(0).$$

So we let $\Gamma(z) = P(z)Q(z)^{-1}$. Without loss of generality we can assume that $\Gamma(0) = I$. Then we obtain

$$S(z) = \frac{1}{2}(Q(z) - P(z))(Q(z) + P(z))^{-1}.$$

$S(z)$ is in $\underset{\sim}{S}$ if and only if

$$\det (Q(z_o) + P(z_o)) = 0 \ \text{ implies } \ |z_o| > 1$$

and

$$I - S(z)\widetilde{S(z)} \geq 0,$$

for all $z$ in $|z| \leq 1$. By the maximum modulus principle it is sufficient to test this on the boundary of the region of analyticity:

$$I - S(z)\tilde{S}(z^{-1}) = 2(\tilde{Q}(z^{-1}) + \tilde{P}(z^{-1}))^{-1}D(z, z^{-1})(Q(z)+P(z))^{-1} \geq 0$$

for all $z$ on $|z| = 1$. Clearly this holds if and only if

$$D(z, z^{-1}) \geq 0. \quad \square$$

(9.4) THEOREM. Let $C_s = \{c_t : t = 0, 1, \ldots, s\}$ be any partial positive $n \times n$-matrix sequence and

$$(9.5) \quad D(z, z^{-1}) := d_s z^s + \ldots + d_1 z + I + \tilde{d}_1 z^{-1} + \ldots + \tilde{d}_s z^{-s},$$

be an $n \times n$-matrix dissipation polynomial. Then there exists a pair $(P(z), Q(z)) \in (\underline{C}^{n\times n}[z])^2$ with $\deg P(z)$, $\deg Q(z)$ less than or equal to $s$, and a positive definite matrix $K$ such that the following two conditions hold

(a) $P(z)Q(z)^{-1}$ is a pr-extension of $C_s$,

(b) $K^{1/2}D(z, z^{-1})K^{1/2} = \frac{1}{2}\{\tilde{Q}(z^{-1})P(z) + \tilde{P}(z^{-1})Q(z)\}$.

(With $(\ )^{1/2}$ we denote the "Hermitean square root of".)

This theorem establishes that when the partial sequence $C_s$ is positive, the matrix CARATHEODORY problem is solvable. Our technique does not seem to be possible to extend to the singular case when $C_s$ is only nonnegative. However, it provides information about the solutions of dimension $s$, precisely as it did in the scalar case: For a generic set of dissipation polynomials of degree less than or equal to $s$, we can associate pr-extensions of $C_s$ of precisely dimension $s$. For the complement of this set we can associate pr-extensions of dimension less than $s$.

The idea of our proof is similar to the one we gave for the scalar case. However, certain new features require the use of a more sophisticated technique. The main new aspect is that, in contrast to the scalar case, matrix polynomials with no determinental zeros in

$|z| < 1$ and with constant term the identity matrix, do not form a bounded subset of the space of the coefficients. For example, the polynomial

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + z \begin{pmatrix} 1/2 & a \\ 0 & 1/2 \end{pmatrix}$$

has nonvanishing determinant for all $|z| < 1$ and all values of $a$ as well. We circumvent this by considering our stability set on a certain compact manifold.

PROOF. Let $P(z)Q(z)^{-1}$, with $P(z)Q(z)$ in $\underline{\underline{C}}^{n\times n}[z]$, be a partial realization of $C_s$ of dimension less than or equal to $s$. Then $P(z)$ and $Q(z)$ are related by

$$(9.6) \qquad P(z) = [(c_o + 2c_1 z + \dots + 2c_s z^s)Q(z)]_o^s.$$

Clearly, $Q(0)$ is nonsingular. We now define the polynomial $B(z)$ by

$$(9.7) \qquad B(z) = [(c_o + c_1 z + \dots + c_s z^s)Q(z)]_o^s,$$

and the polynomial

$$(9.8) \qquad \underset{\sim}{D}(z, z^{-1}) = \underset{\sim}{D}_s z^s + \dots + \underset{\sim}{D}_1 z + \underset{\sim}{D}_o + \tilde{\underset{\sim}{D}}_1 z^{-1} + \dots + \tilde{\underset{\sim}{D}}_s z^{-s}$$

$$:= \frac{1}{2}\{\tilde{Q}(z^{-1})P(z) + \tilde{P}(z^{-1})Q(z)\}$$

in $z$ and $z^{-1}$. By Proposition (9.1), $P(z)Q(z)^{-1}$ is in $\underset{\sim}{C}$ if and only if $\det B(z)$ has no roots in $|z| \leq 1$ and $\underset{\sim}{D}(z, z^{-1})$ is nonnegative definite for all $z$ on $|z| = 1$. We notice that the pair $(Q(z), P(z))$ is defined up to a right unimodular factor. Therefore, so is $B(z)$. Moreover, $\det B(0) \neq 0$.

Thus, we consider the space $X$ of polynomials

$$\{B(z) = B_o + B_1 z + \ldots + B_s z^s,\ B(z) \in \underline{\underline{C}}^{nxn}[z]\}$$

of degree less than or equal to $s$. In this space we consider the subset $M$ defined by

$$(9.10)\qquad \begin{aligned} &\sum_{t=o}^{s} \tilde{B}_t B_t = I, \\ &B_o > 0, \text{ and } B_o \text{ upper triangular.} \end{aligned}$$

$M$ is a smooth compact manifold of real dimension $2sn^2$. That $M$ is a smooth manifold follows from the open condition $\det B_o \neq 0$. Compactness follows from the fact that by the first condition any entry of $B_t$, $t = 0, \ldots, s$ has modulus less than or equal to 1. (It is also easy to show that $M$ is orientable, but we will not need this fact here.)

The correspondence between $B(z)$ in $M$ and partial realizations $P(z)Q(z)^{-1}$ of $C_s$ with dimension less than or equal to $s$ is clearly surfective. If

$$Q(z) = Q_o + Q_1 z + \ldots + Q_s z^s,$$

then the polynomial $\underset{\sim}{D}(z, z^{-1})$ obtained by (9.8) satisfies

$$D_o = (Q_o \quad Q_1 \quad \ldots \quad Q_s) \begin{pmatrix} c_o & \tilde{c}_1 & \ldots & \tilde{c}_s \\ c_1 & c_o & \ldots & \tilde{c}_{s-1} \\ \vdots & & & \vdots \\ c_s & c_{s-1} & \ldots & c_o \end{pmatrix} \begin{pmatrix} Q_o \\ Q_1 \\ \vdots \\ Q_s \end{pmatrix} .$$

Since $\det Q_o \neq 0$ and $C_s > 0$, it follows that $\underset{\sim}{D}_o > 0$. Therefore the following map is well-defined

$$\begin{aligned} \varphi_{C_s} :\ & M \rightarrow W \\ & B(z) \mapsto \underset{\sim}{D}_o^{-1/2} \underset{\sim}{D}(z, z^{-1}) \underset{\sim}{D}_o^{-1/2}, \end{aligned}$$

where $W$ is the space of the polynomials $D(z, z^{-1})$ as in (9.5), i.e., such that $D(z, z^{-1}) = \tilde{D}(z^{-1}, z)$ and with constant term equal to the identity matrix.

Consider the submanifold $S$ of $M$ of polynomials with determinant nonvanishing in $|z| \leq 1$, and the submanifold $P$ of $W$ of polynomials $D(z, z^{-1})$ that have nonnegative definite values for all $z$ on $|z| = 1$. To complete the proof of the theorem we need to show that for all $C_s > 0$ we have that

$$\bar{P} \subseteq \varphi_{C_s}(\bar{S}).$$

Precisely as in the scalar case it can be shown that

$$\varphi_{C_s}(\partial s) \cap P = \emptyset.$$

Also the set of positive partial matrix sequences $C_s$ is pathwise connected. By using the homotopy invariance property of the degree the proof that was given for the scalar case works in this case also. More precisely $\varphi_{C_s}$ is certainly a continuous map between manifolds. ($\varphi_{C_s}$ is only continuous because we require taking the Hermitian square root.) Now MILNOR [1965] defines the degree for $C^1$-mappings between manifolds. However, as remarked by LLOYD [1978, p. 32] the definition immediately extends to the continuous case simply by taking $C^1$ approximations. (An explicit argument can be found in SCHWARTZ [1965, Chapter V] and LLOYD [1978, Chapter I].) Finally, we note that as before the degree $\deg(\varphi_{C_s}, S, D)$ can be seen to be $1$ by considering the point

$$D(z, z^{-1}) = I$$

and the trivial sequence

$$C_s = \{I, 0, \ldots, 0\}.$$

The proof now proceeds precisely as in the scalar case. □

## 10. The Algebraic Approach

The description of all solutions to the matrix CARATHEODORY problem can be found in AROV and KREIN [1981] and KOVALISHINA and POTAPOV [1982]. It is given in the so-called "completely nondegenerate case", when $C_s$ is a positive sequence. In the general case when $C_s$ is nonnegative but not positive there exists no closed form expression for the solutions. However, some standard techniques in operator theory can be used to deal with this case (cf. SZ.-NAGY and FOIAS [1970, p. 188]).

In this section, having presented our topological approach, we wish to give a brief account of the basic results and ideas of the algebraic approach, which essentially relies again on Schur's algorithm. We shall apply this to the case of rational $\underset{\sim}{C}$-functions.

We begin by describing the SCHUR's algorithm for the case of matrix $\underset{\sim}{C}$-functions. The main technical fact is given in the following

(10.1) LEMMA. *Let $\Gamma_t(z)$ be a matrix-valued function which has a power series expansion around the origin that begins with $I + 2r_t z$, where $I - r_t\tilde{r}_t > 0$. Then $\Gamma_t(z)$ is in $\underset{\sim}{C}$ if and only if there exists a $\underset{\sim}{C}$-function $\Gamma_{t+1}(z)$ such that*

$$(10.2)\quad \Gamma_t(z) = [a_t(z)\Gamma_{t+1}(z) + b_t(z)][c_t(z)\Gamma_{t+1}(z) + d_t(z)]^{-1},$$

*where*

$$a_t(z) = (I - r_t\tilde{r}_t)^{-1/2}(I - r_t) + z(I - \tilde{r}_t r_t)^{-1/2}(I - \tilde{r}_t),$$

$$b_t(z) = (I - r_t\tilde{r}_t)^{-1/2}(I + r_t) - z(I - \tilde{r}_t r_t)^{-1/2}(I + \tilde{r}_t),$$

$$c_t(z) = (I - r_t\tilde{r}_t)^{-1/2}(I - r_t) - z(I - \tilde{r}_t r_t)^{-1/2}(I - \tilde{r}_t),$$

$$d_t(z) = (I - r_t\tilde{r}_t)^{-1/2}(I + r_t) + z(I - \tilde{r}_t r_t)^{-1/2}(I + \tilde{r}_t).$$

PROOF. The SCHUR's recurrence relation for the matrix case is given by (see for example DELSARTE, GENIN, and KAMP [1979, (36)])

$$S_t(z) = (I - \tilde{r}_t r_t)^{-1/2} (\tilde{r}_t + zS_{t+1}(z))(I + zr_t S_{t+1}(z))^{-1}(I - r_t\tilde{r}_t)^{-1/2}.$$

Assuming that $I - r_t\tilde{r}_t > 0$, then $S_t(z)$ is in $\underset{\sim}{S}$ if and only if $S_{t+1}(z)$ is in $\underset{\sim}{S}$. Applying the bilinear transformation (9.2) we obtain the corresponding recurrence relations for the class $\underset{\sim}{C}$-functions. □

Formula (10.2) can be solved for $\Gamma_{t+1}(z)$ in terms of $\Gamma_t(z)$ and provides an inductive procedure for solving the CARATHEODORY problem in the completely nondegenerate case.

In the completely nondegenerate case a matrix-version of (2.9) can also be obtained (see for example AROV and KREIN [1981]).

Here we shall apply the lemma to rational $\underset{\sim}{C}$-functions $P_t(z)Q_t(z)^{-1}$ and consider the behavior of the dimension and the dissipation polynomial under the action of "truncating" the sequence of SCHUR parameters $r_t$ or, equivalently, as $t$ increases. We now have the following

(10.3) THEOREM. Let $\Gamma_t(z)$ and $\Gamma_{t+1}(z)$ be in $\underset{\sim}{C}$ and related as in Lemma (10.1). Then $\Gamma_t(z)$ is rational if and only if $\Gamma_{t+1}(z)$ is rational. In this case, there exist right coprime representations $\Gamma_t(z) = P_t(z)Q_t(z)^{-1}$ and $\Gamma_{t+1}(z) = P_{t+1}(z)Q_{t+1}(z)^{-1}$ with $P_t(z)$, $Q_t(z)$, $P_{t+1}(z)$, $Q_{t+1}(z)$ in $\underset{=}{C}^{n\times n}[z]$ such that

$$(10.4)\quad \begin{pmatrix} P_t(z) \\ Q_t(z) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a_t(z) & b_t(z) \\ c_t(z) & d_t(z) \end{pmatrix} \begin{pmatrix} P_{t+1}(z) \\ Q_{t+1}(z) \end{pmatrix}.$$

Also, if $D_t(z, z^{-1})$ (resp. $D_{t+1}(z, z^{-1})$) denotes the associated dissipation polynomial

(10.5) $D_t(z, z^{-1}) = D_{t+1}(z, z^{-1})$.

<u>Moreover, the following are equivalent</u>

(a) $\dim \Gamma_t(z) = \dim \Gamma_{t+1}(z)$,

(a') $\dim \Gamma_t(z) = \deg D_t(z, z^{-1})$,

(a") $\dim \Gamma_t(z) = \deg (P_t(z) + Q_t(z))$.

For the proof we need the following

(10.6) LEMMA. <u>The following identity holds</u>

$$\frac{1}{4}\begin{pmatrix} \tilde{d}_t(z^{-1}) & \tilde{b}_t(z^{-1}) \\ \tilde{c}_t(z^{-1}) & \tilde{a}_t(z^{-1}) \end{pmatrix}\begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} = I.$$

PROOF. By direct computation. □

We now proceed to the

PROOF of Theorem (10.3). Suppose $\Gamma_{t+1}(z)$ is a rational $\underset{\sim}{C}$-function and is equal to $P_{t+1}(z)Q_{t+1}(z)^{-1}$, where $P_{t+1}(z)$, $Q_{t+1}(z)$ are right coprime matrix polynomials. Define $P_t(z)$, $Q_t(z)$ via (10.4). It can be checked that $c_t(z)P_{t+1}(z) + d_t(z)Q_{t+1}(z)$ is invertible as a power series in $z$. By Lemma (10.1) it follows that $\Gamma_t(z) = P_t(z)Q_t(z)^{-1}$. Furthermore, it holds that $P_t(z)$, $Q_t(z)$ are right coprime.

Indeed, since $P_{t+1}(z)$, $Q_{t+1}(z)$ are right coprime there exist polynomial matrices $A(z)$ and $B(z)$ such that $A(z)P_{t+1}(z) + B(z)Q_{t+1}(z) = I$. Hence, from Lemma (10.6) and (10.4) we obtain that there exist polynomial matrices $A_1(z)$ and $B_1(z)$ such that

$$A_1(z)P_t(z) + B_1(z)Q_t(z) = z^{2n}I.$$

But both $\det P_t(0)$ and $\det Q_t(0)$ can be checked to be different from zero. Therefore $P_t(z)$ and $Q_t(z)$ are in fact right coprime.

The converse follows similarly by considering the identity

$$(10.7)\quad z^{2n}\begin{pmatrix} P_{t+1}(z) \\ Q_{t+1}(z) \end{pmatrix} = \begin{pmatrix} \tilde{d}_t(z) & \tilde{b}_t(z) \\ \tilde{c}_t(z) & \tilde{a}_t(z) \end{pmatrix} \begin{pmatrix} P_t(z) \\ Q_t(z) \end{pmatrix}$$

that follows from Lemma (10.6).

Relation (10.5) follows by considering

$$D_t(z,\ z^{-1}) = \tfrac{1}{2}(\tilde{Q}_t(z^{-1})\tilde{P}_t(z^{-1}))\begin{pmatrix} P_t(z) \\ Q_t(z) \end{pmatrix},$$

and applying Lemma (10.6).

Finally, the equivalence of (a), (a'), and (a") can be shown as in the scalar case (see Proposition (3.5)). $\square$

Thus, the precise analogues of certain facts that were seen to hold in the scalar case, apply to the matrix case as well. We expect that the results of Section 5 extend to the matrix case also, and that the matrix dissipation polynomial determines the asymptotic behavior of the matrical SCHUR parameters.

## CHAPTER VI. APPLIED ASPECTS OF THE COVARIANCE EXTENSION PROBLEM

In this final chapter we want to discuss the relevance of the covariance extension problem to the applied area of time-series modeling. This area involves a large number of issues that we shall not touch upon (e.g., issues of statistical nature, see BOX and JENKINS [1970], or of the essential difference between prediction for time-series and prediction for stochastic processes, see FURSTENBERG [1960]). Instead we shall consider as our point of departure, the knowledge of a partial (sampled) covariance sequence $C_s$.

Based on these data, certain schemes have been proposed that yield a unique rational covariance extension for $C_s$. These schemes form the base of modern nonlinear methods for spectral estimation (cf., HAYKIN [1979, pages 36 and 103]). We begin Section 11 by considering the so-called "maximum entropy" (ME) method in the context of our previous development.

The ME method proposes the use of a particular pr-extension of $C_s$ that has constant dissipation polynomial. The constancy of the dissipation polynomial makes the construction of a corresponding stochastic realization trivial (since the problem of spectral factorization is avoided altogether). Moreover, this construction turns out to be recursively updated as the data set increases. This latest property is precisely the recurrence relation satisfied by the orthogonal polynomials and was established in this context by LEVINSON [1947]. Recursiveness is very important in practical applications as it provides an efficient approximation procedure. In point of fact, this is the underlying philosophy in the ladder structure constructions in modern digital filter design.

However, the absence of zeros in the power spectrum obtained by the ME method in certain cases gives rise to undesirable phenomena (see HERRING [1980] and the references therein). Motivated by the need for more general pole-zero approximating techniques, for the covariance function of stochastic processes DEWILDE, VIERA, and KAILATH [1978], and DEWILDE and DYM [1981a and 1981b] (see also RUCKEBUSH [1978] and ROSENCHER and CLERGET [1979]) have placed the problem in a more general context of Nevanlinna-Pick interpolation theory. However, these investigations do not seem to illuminate the basic partial realization problem where the data is simply $C_s$. Pole-zero modeling in the context of partial realization setting remains "a nonlinear and implicit problem, and there is no possibility for recursively updated realizations of increasing order" (see BENVENISTE and CHAURRE [1981]).

In Section 12 we shall indicate that this might not be precisely so. Certainly, as it appears from our results of Chapter III, an essential part in obtaining pr-extensions of $C_s$ with nontrivial dissipation polynomial is in obtaining information about the dissipation polynomial or, equivalently, the zeros of the corresponding power spectrum. (Our results of Chapter III, in particular Theorem (4.15) suggests that the parameter sequence might be used for that. This is a point that requires further investigation.) We should note that this information is already assumed in the approximation theories of DEWILDE an DYM [1981a and 1981b]. Now, provided such information is available we shall indicate a way that this can be incorporated in the modeling process in an efficient way. Theorem (12.4) will describe a recursive construction for pr-extensions that have approximately fixed zero-structure.

## 11. The ME-Method and Some General Discussion.

Let $C_s$ be a partial positive (scalar) covariance sequence. The simplest possible choice for an admissible extension of the partial parameter sequence $R_s$ is certainly the trivial extension $\{r_{s+t} : r_{s+t} = 0 \text{ for } t = 1, \ldots\}$. This extension amounts to choosing $\Gamma_{s+1}(z) = 1$ in Theorem (3.1). The associated pr-extension of $C_s$ is simply

$$\frac{\pi(z)}{\chi(z)} = \frac{\Psi_s(z)^*}{\Phi_s(z)^*} .$$

This particular extension has a certain uniqueness property, namely, it maximizes

$$e_\infty := \lim_{t\to\infty} \|\Phi_t(z)^*\|^2,$$

i.e., the distance in the $\| \ \|$-norm of 1 from the closure of the manifold spanned by positive powers of $z$. It is immediate from (1.6) that in this case

$$\|\Phi_t(z)^*\|^2 = \|\Phi_s(z)^*\|^2,$$

for all $t \geq s$, and hence

$$e_\infty = \lim_{t\to\infty} \|\Phi_t(z)^*\|^2 = \|\Phi_s(z)^*\|^2.$$

We now would like to explain the importance of this quantity in the prediction theory of stochastic processes. The inner product $\langle\cdot,\cdot\rangle$ that was defined in Section 1 relative to a nonnegative sequence $C$, can be extended from the space of polynomials in $z$ to a more general space of functions on $|z| = 1$. This is done by realizing this inner product via a Stieljes integral

$$\langle a(z),\ b(z)\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} a(z)\overline{b(z)}d\sigma(\theta),\ z = \exp j\theta,$$

with $a(e^{j\theta}),\ b(e^{j\theta}) \in L_2[d\sigma(\theta)]$ the Hilbert space of squarely integrable functions on $|z| = 1$ with respect to the measure $d\sigma(\theta)$, ($\sigma(\theta)$ being the spectral distribution function of $C$, cf., Remark (3.8)). Let now $y_\tau$, $\tau \in \underline{\underline{Z}}$ be a (zero-mean, stationary) stochastic process having $C$ as covariance sequence. Let also $L_2(y)$ denote the Hilbert space generated by $y_\tau$ with the inner product defined by

$$\langle f, g \rangle := E\, f\bar{g},$$

$f, g \in L_2(y)$, and $E$ denoting the expectation operator. Then, the mapping

$$y_\tau \mapsto e^{j\tau\theta}$$

extends to an isometry between $L_2[d\sigma(\theta)]$ and $L_2(y)$ (cf., GRENANDER and SZEGÖ [1958, p. 175]). Via this mapping any linear problem in $L_2(y)$ can be translated into one in $L_2[d\sigma(\theta)]$ and conversely. Let now $g_s(z)$ be any polynomial in $z$ with $g_s(0) = 1$ and degree less than or equal to $s$. Clearly, due to the orthogonality properties (1.3),

$$\|\Phi_s(z)*\|^2 = \inf_{g_s(z)} \|g_s(z)\|^2.$$

Therefore,

$$e_\infty = \lim_{t\to\infty} \|\Phi_t(z)*\|^2 = \lim_{t\to\infty} \inf_{\substack{a_u \\ 1\le u\le t}} E\, |y_0 - \sum_{u=1}^{t} a_u y_{-u}|^2$$

is the square of the variance of the prediction error of $y_\tau$ at an instant $\tau = 0$ based on observations in the past $\tau < 0$. And this is maximized by the choice $\Gamma_{s+1}(z) = 1$ over all pr-extensions of the partial data $C_s$.

The quantity $e_\infty$ is an essential characteristic of a stochastic process and describes the "predictability" of the process. One can show that (see GERONIMUS [1961, p. 158])

$$e_\infty = \prod_{t=1}^{\infty} (1 - |r_t|^2)$$

$$= \exp \{\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln \sigma_a'(\theta)\, d\theta\}.$$

The stochastic process is called deterministic iff $e_\infty = 0$, and nondeterministic otherwise. $e_\infty$ is also directly related to a notion of entropy rate of a stochastic process in the sense of Shannon (see HAYKIN [1979, p. 80]) and this gives the name to the method.

On the basis of the above it has been argued that the so-obtained pr-extension of $C_s$ is maximally noncommital to the unavailable data (see JAYNES [1968]). This is indeed so as far as the prediction problem is concerned. However, the prediction should be more of an "excuse" than a "reason" (see FURSTENBERG [1960, p. 7]). In point of fact, KALMAN [1981] argues that the partial sequence of parameters contains certainly more information than merely the fact that these parameters are all of modulus less than 1. In KALMAN [1981] it is also suggested that some minimal dimension pr-extension is perhaps the right object to consider. Unfortunately, the description of the minimal dimension pr-extensions seems to face intractable difficulties. In point of fact, an equivalent question was considered by YOULA and SAITO [1967] in a circuit theoretic context. Currently, this problem is unsolved. In Appendix A we shall indicate some computational difficulties that arise in the simplest nontrivial case.

## 12. On Pole-Zero Modeling.

We begin by assuming knowledge of a number of "influential zeros" in the power spectrum of a stochastic process. This rather loose term appears to have a rather definite meaning in the more application-oriented literature. See for example MAKHOUL [1976, p. 115]. It is also stated that pole-zero modeling is not simple and not well-understood. We shall present a simple recursive way to incorporate the "zero" information in the modeling process.

Let $C_s$ be a partial positive sequence and $d(z, z^{-1})$ be a given dissipation polynomial of degree less than or equal to $s$. From the results of Section 7 we know that there exists an associated pr-extension of $C_s$ of degree less than or equal to $s$.

In principal this pr-extension can be found as a preimage of $d(z, z^{-1})$ under $\varphi_C$. Certain techniques have been recently developed to provide constructive algorithmic procedures for obtaining a solution of homotopy-based existence results (see KELLOG, LI, and YORKE [1976], and also MEYER [1968]). However, this is very cumbersome and objectionable for almost all practical purposes. Thus, we shall not pursue it here but instead, we shall develop an approximate but efficient solution.

Our first tool is a new representation for partial realizations of $C_s$ of dimension less than or equal to $s$. Let $C_s$ be a positive sequence, and $\Psi_t(z)$, $\Phi_t(z)$, $t = 0, 1, \ldots, s$ be the orthogonal polynomials of $C_s$.

(12.1) LEMMA. Any rational function $c_o\pi(z)/\chi(z)$, $\pi(z), \chi(z) \in \underline{\underline{C}}[z]$, with $\pi(0) = \chi(0) = 1$ and power series expansion that begins with

(12.2) $c_o + 2c_1 z + \ldots + 2c_s z^s$

is of the form

$$c_o \frac{\pi(z)}{\chi(z)} = c_o \frac{\Psi_s(z)* + \alpha_1 z\Psi_{s-1}(z)* + \ldots + \alpha_s z^s \Psi_o(z)*}{\Phi_s(z)* + \alpha_1 z\Phi_{s-1}(z)* + \ldots + \alpha_s z^s \Phi(z)*} .$$

This result was independently of ours utilized by KIMURA [1983] who also argues that it provides a canonical form for partial realization of covariances. In our work a precise use of this lemma is given in Theorem (12.4). We should also mention that this lemma is really a fact about partial realizations and has essentially nothing to do with positivity. Positivity is assumed for the sake of some other properties of this representation that we shall soon discuss.

PROOF of Lemma (12.1). The polynomials $\pi(z)$ and $\chi(z)$ are related through

$$c_o\pi(z) = [\chi(z)(c_o + 2c_1z + \ldots + 2c_sz^s)]_o^s.$$

This represents a nonsingular transformation between polynomials of degree less than or equal to $s$. The two sets of polynomials $\{z^{s-t}\Psi_t(z)*,\ t = 0, 1, \ldots, s\}$ and $\{z^{s-t}\Phi_t(z)*,\ t = 0, 1, \ldots, s\}$ form bases for this space and they are related by

$$c_oz^{s-t}\Psi_t(z)* = [z^{s-t}\Phi_t(z)*(c_o + 2c_1z + \ldots + 2c_sz^s)]_o^s,$$

for $t = 0, 1, \ldots, s$, as it follows from the definition of $\Psi_t(z)$, $t = 0, 1, \ldots$. The proof of the lemma is now immediate. □

This lemma places a system of coordinates, in the linear space whose points represent partial realizations of (12.2), so that the ME solution lies at the origin. Another aspect of this representation is shown in

(12.3) LEMMA. *Let*

$$\pi(z) = \Psi_s(z)* + \alpha_1z\Psi_{s-1}(z)* + \ldots + \alpha_sz^s\Psi_o(z)*,$$

*and*

$$\chi(z) = \Phi_s(z)* + \alpha_1z\Phi_{s-1}(z)* + \ldots + \alpha_az^s\Phi_o(z)*.$$

*The polynomial (in* $z$ *and* $z^{-1}$)

$$d(z, z^{-1}) = \frac{1}{2}\{\pi(z)\bar{\chi}(z^{-1}) + \chi(z)\bar{\pi}(z^{-1})\}$$

*has degree* $t < s$ *if and only if* $\alpha_s = \ldots = \alpha_{t=1} = 0$.

PROOF. This follows from the fact that the degree of

$$\Psi_s(z)\bar{\Phi}_t(z^{-1}) + \Phi_s(z)\bar{\Psi}_t(z^{-1})$$

is equal to $|s - t|$. This can be shown using Lemma (2.6). □

We now proceed to our final

(12.4) THEOREM. Let $C = \{c_t : t = 0, 1, \ldots\}$ be the covariance sequence of a nondeterministic stochastic process, and let

$$\eta(z) = 1 + a_1 z + \ldots + a_u z^u$$

be any polynomial with roots in $\{|z| > 1\}$. Then for $s$ sufficiently large

$$c_o \frac{\pi^{(s)}(z)}{\chi^{(s)}(z)} := c_o \frac{\Psi_s(z)* + a_1 z\Psi_{s-1}(z)* + \ldots + a_u z^u \Psi_{s-u}(z)*}{\Phi_s(z)* + a_1 z\Phi_{s-1}(z)* + \ldots + a_u z^u \Phi_{s-u}(z)*}$$

is a pr-extension of $C_s$, and if $\eta^{(s)}(z)$ denotes the stable spectral factor of the associated dissipation polynomial, then

$$\lim_{s\to\infty} \eta^{(s)}(z) = \eta(z).$$

PROOF. Let $M_t(z)$ be as in Theorem (3.1). Then

$$\begin{pmatrix} \pi^{(s)}(z) \\ \chi^{(s)}(z) \end{pmatrix} = M_{s-u}(z)* \begin{pmatrix} \Psi_u^{s-u}(z)* + a_1 z\Psi_{u-1}^{s-u}(z)* + \ldots + a_u z^u \Psi_o^{s-u}(z)* \\ \Phi_u^{s-u}(z)* + a_1 z\Phi_{u-1}^{s-u}(z)* + \ldots + a_u z^u \Phi_o^{s-u}(z)* \end{pmatrix}$$

where $\Psi_t^{s-u}(z)$, $\Phi_t^{s-u}(z)$ are the orthogonal polynomials of the parameter sequence $R_u^{s-u} = \{r_{s-u+1}, \ldots, r_s\}$.

Since $C$ corresponds to a deterministic process it follows from the result of GERONIMUS [1961, p. 159] (see (4.6)) that the parameters of $C$ are squarely summable. Hence, as $s \to \infty$, both

$$\Psi_t^{s-u}(z)* \to 1$$

and

$$\Phi_t^{s-u}(z)^* \to 1$$

for $t = 0, 1, \ldots, u$. Consequently, as $s \to \infty$

$$\frac{\Psi_u^{s-u}(z)^* + a_1 z \Psi_{u-1}^{s-u}(z)^* + \ldots + a_u z^u \Psi_o^{s-u}(z)^*}{\Phi_u^{s-u}(z)^* + a_1 z \Phi_{u-1}^{s-u}(z)^* + \ldots + a_u z^u \Phi_o^{s-u}(z)^*}$$

tends to 1 uniformly on $|z| \leq 1$, and the associated dissipation polynomial tends to $\eta(z)\bar{\eta}(z^{-1})$. Applying now Theorem (3.1), the proof is complete. □

## APPENDIX. THE MINIMAL DIMENSION PROBLEM

Here we shall consider the following problem: Given a partial covariance sequence $C_s$ find (simple) necessary and sufficient conditions on $C_s$ so that it admits a pr-extension of dimension strictly less than $s$.

This question is certainly the first one in attempting to obtain an explicit answer to the minimal dimension problem of KALMAN [1981], and YOULA and SAITO [1967]. We shall discuss the first two simple cases: $s = 2$, and $s = 3$. The case $s = 2$ is trivial as it requires conditions for the positivity of a degree 1 polynomial in $z$ and $z^{-1}$. The case $s = 3$, that requires conditions for the positivity of a degree 2 polynomial in $z$ and $z^{-1}$, presents already difficulties due to the implicit nature of the conditions that seem to be possible to approach only with the techniques of decision methods (see JACOBSON [1974, V]). Although this appears to be quite elementary, it should be noted that conditions for the first case only is what exists in the current literature (see KALMAN [1981], and also KRISHNAPRASAD [1980] in an equivalent setting).

We begin by considering the case $s = 2$. Unless $c_1 = 0$, the minimal dimension partial realization of

$$1 + 2c_1 z + 2c_2 z^2$$

is of dimension 1 and given by

$$\frac{\pi(z)}{\chi(z)} = \frac{1 + (\alpha + c_1)z}{1 + (\alpha - c_1)z}$$

where $c_2 = c_1(c_1 - \alpha)$ (Here and in the sequel we use the representation introduced in Lemma (12.1).) Applying Proposition (5.2) it is straightforward to check that $\pi(z)/\chi(z)$ is in $\underset{\sim}{C}$ if and only if

$$\text{(A.1)} \quad \begin{aligned} &|c_1| \le 1, \text{ and} \\ &|\alpha| \le 1 - |c_1|. \end{aligned}$$

We now consider the case $s = 3$. In case $c_1 = c_2 = 0$ and $c_3 \neq 0$, then the minimal pr-extension is of dimension 3. In case $c_3 = c_1 c_2$, then the minimal pr-extension is either of dimension 1 or 3 depending on whether the minimal partial realization which is of dimension 1 is in $\underset{\sim}{C}$ or not. In the generic case we consider a general rational function $\pi(z)/\chi(z)$ with $\pi(z)$, $\chi(z)$ polynomials of degree 2, with $\pi(0) = \chi(0) = 1$ and power series that begin with

$$1 + 2c_1 z + 2c_2 z^2 + 2c_3 z^3.$$

We shall restrict our attention to the case where all the scalars take real values, and use in the various expressions the associated parameters $\{r_1, r_2, r_3\}$ instead of $\{c_1, c_2, c_3\}$. The function $\pi(z)/\chi(z)$ is of the form

$$\frac{\Psi_2(z)* + \alpha z \Psi_1(z)* + \beta z^2 \Psi_o(z)*}{\Phi_2(z)* + \alpha z \Phi_1(z)* + \beta z^2 \Phi_o(z)*},$$

where $\alpha$ and $\beta$ satisfy

$$(A.2) \qquad (1 - r_1^2)(1 - r_2^2) r_3 + \alpha(1 - r_1^2) r_2 + \beta r_1 = 0.$$

The conditions

$$\pi(z) + \chi(z) \neq 0 \quad \text{for all } z \text{ in } |z| \leq 1,$$

and

$$\pi(z)\bar{\chi}(z^{-1}) + \bar{\pi}(z^{-1})\chi(z) \geq 0 \quad \text{for all } z \text{ on } |z| = 1,$$

give rise to the following:

$$(A.3) \qquad \alpha(1 + r_1) \leq \beta + (1 + r_1)(1 - r_2)$$

$$(A.4) \qquad - \alpha(1 - r_1) \leq \beta + (1 - r_1)(1 - r_2)$$

(A.5) $\quad -\alpha(1 + r_1) \leq \beta + (1 + r_1)(1 + r_2)$

(A.6) $\quad \alpha(1 - r_1) \leq \beta + (1 - r_1)(1 + r_2)$

(A.7) $\quad \beta \leq 1$

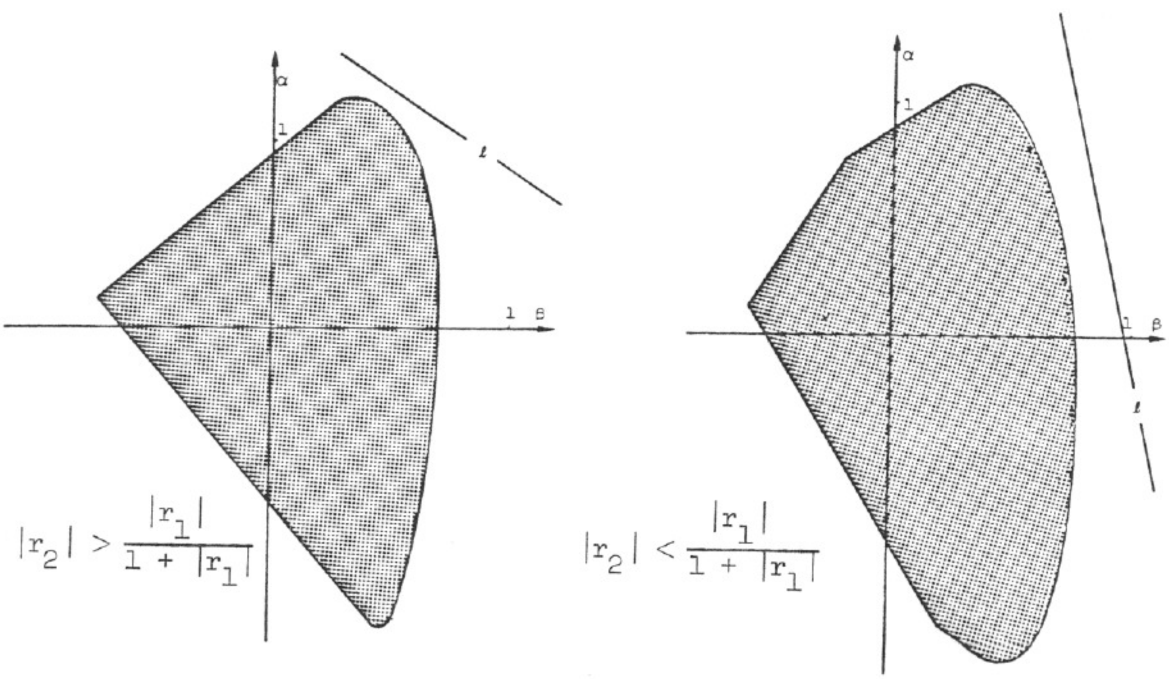
and either

(A.8) $\quad 4\beta \leq |\alpha(1 - r_1^2) + \alpha\beta + \beta r_1 r_2|$

or

(A.9)
$$\alpha^2(1 - r_1^2 - \beta)^2 + \beta^2 r_1^2 r_2^2 + 2\alpha\beta r_1 r_2(1 - r_1^2 + \beta) -$$
$$- 4\beta[\beta^2 - 2\beta + (1 - r_1^2)(1 - r_2^2)] \leq 0.$$

In general, neither of the above conditions is redundant. In the $(\alpha, \beta)$-space, these conditions cut out a set that corresponds to pr-extensions of $C_2$ of dimension 2. This set is illustrated below as a shaded area for the two typical cases.



Whether $C_3$ admits a pr-extension of dimension 2 depends on whether the line $\ell$ given by (A.2) intersects the shaded region. Due to (A.9) which is implicit in $\alpha$ and $\beta$ and is of degree 4, these conditions when expressed in terms of the original parameters of the problem, e.g. $r_1$, $r_2$, and $r_3$, are also implicit and extremely involved.

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