Optimal Steering of a Linear Stochastic System to a Final Probability Distribution, Part I

Yongxin Chen, *Student Member, IEEE*, Tryphon T. Georgiou, *Fellow, IEEE*, and Michele Pavon

***Abstract*—We consider the problem of steering a linear dynamical system with complete state observation from an initial Gaussian distribution in state-space to a final one with minimum energy control. The system is stochastically driven through the control channels; an example for such a system is that of an inertial particle experiencing random "white noise" forcing. We show that a target probability distribution can always be achieved in finite time. The optimal control is given in state-feedback form and is computed explicitly by solving a pair of differential Lyapunov equations that are *nonlinearly coupled* through their boundary values. This result, given its attractive algorithmic nature, appears to have several potential applications such as to quality control, control of industrial processes, as well as to active control of nanomechanical systems and molecular cooling. The problem to steer a diffusion process between end-point marginals has a long history (*Schrödinger bridges*) and the present case of steering a linear stochastic system constitutes such a Schrödinger bridge for possibly degenerate diffusions. Our results provide the first *implementable* form of the optimal control for a general Gauss–Markov process. Illustrative examples are provided for steering inertial particles and for "cooling" a stochastic oscillator. A final result establishes directly the property of Schrödinger bridges as the most likely random evolution between given marginals to the present context of linear stochastic systems. A second part to this work, that is to appear as part II, addresses the general situation where the stochastic excitation enters through channels that may differ from those used to control.**

***Index Terms*—Linear stochastic system, Schrödinger bridge, stochastic control.**

I. Introduction

THE most basic paradigm in optimal control deals with the steering of a dynamical system between two end-points in time, while minimizing a suitable cost functional—here, the expected quadratic integral of the control input. The specifications for the marginal conditions are either explicit, requiring that the value of the state vector belongs to a specified set, or implicit, penalizing the distance of the state vector from a target location. In the presence of a stochastic disturbance, in both cases, the end-point conditions may be further modified accordingly. For instance, a bound on the probability of failing to meet explicit constraints may be specified or, in the second case, the penalty on the distance to a target location averaged. Herein, we consider a natural third alternative where *the marginal end-point probability densities for the state vector are explicitly specified*. This type of "soft conditioning" may be thought of as a relaxed constraint with respect to the requirement that the state vector belongs to a specified set. It leads to a problem which is in a way similar, but also sharply different from the classical LQG problem [1] as well as some of its more recent variants such as the one with *chance constraints*, see, e.g., [2], [3].

This problem is relevant in a wide spectrum of classical as well as emerging control applications. First and foremost, since it represents soft conditioning, it is of importance in applications where a distribution rather than a set of values for the state vector is a natural specification, e.g., in quality control, industrial and manufacturing processes. Applications may also be envisaged to control of aircrafts, UAVs, and autonomous cars. It is also relevant in a host of other applications at the forefront of modern technological developments such as in the control at the molecular and even atomic scale, the shaping of NMR pulse sequences, laser driven molecular reactions, quantum metrology, atomic force microscopy (AFM), dynamic force microscopy (DFM) and many others; see [4]–[9]. For example, in surface topography using AFM, feedback control is used to limit the fluctuations of the tip of the cantilever. Similarly, in other applications, the distribution of particles in phase space is shaped using a time-varying control potential, and the energy profile of molecules and polymers is shaped using suitable energy sources. Steering a thermodynamic system to a desired steady state corresponding to a lower *effective* temperature is referred to as "cooling." Cooling via feedback control is of great interest in both, microscopic as well as macroscopic electro-mechanical systems. For instance, cooling to ultra-low temperatures is indispensable to investigate decoherence—see [10], [11] for a feedback cooling technique of a ton-scale resonant-bar gravitational wave detector, and [12] for a survey of cooling techniques for both meter-sized detectors as well as nano-mechanical systems. For these diffusion-mediated devices, which are often called *Brownian motors* since work can be extracted from them [13], *motor efficiency* can be cast as the optimal control problem of steering the distribution of a diffusion process [14].

Interestingly, the special case of steering a Brownian diffusion between an initial and a final distribution relates to a seemingly disparate problem that was posed by Erwin Schrödinger in 1931/1932 [15]. In his quest for a more classical formulation

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Y. Chen and T. T. Georgiou are with the Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis, MN 55455 USA (e-mail: chen2468@umn.edu; tryphon@umn.edu).

M. Pavon is with the Dipartimento di Matematica, Università di Padova, Padova 35121, Italy (e-mail: pavon@math.unipd.it).

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of quantum mechanics[1] Schrödinger sought to identify the "most likely" evolution that a cloud of Brownian particles may have taken so as to give rise to observed initial and final-point empirical marginal distributions. Schrödinger's question, which was rather unorthodox at the time, can be properly understood in the modern language of *large deviations*.[2] In essence, he asked for a probability law on the space of sample paths of the process, that is closest to the prior law (Wiener measure) in the relative entropy sense and agrees with the two end-point marginals. It turns out [19] that the sought probability law is the law of another diffusion with a suitable drift term,[3] and the relative entropy between the two laws turns out to be the quadratic integral of this drift! The new law/diffusion on path spaces in known as a Schrödinger bridge (connecting the given end-point marginals) [24]. The fact that the quadratic integral of the drift is identical to the entropy between the prior and the Schrödinger bridge is the link to the optimal control problems that we consider in this paper.

In spite of the substantial literature on Schrödinger bridges and related stochastic control problems, the situation is far from satisfactory. One drawback is that the connection between optimal control and the Schrödinger bridges has been developed for non-degenerate diffusions where the noise acts on all components of the state vector, whereas in many control applications, such as the Nyquist-Johnson model of a circuit with noisy resistor in phase space, this is not the case. A much more serious problem is that the solution, excepting very special cases [14], [23], is in general not given in a form amenable to computations. Indeed, computing the optimal control requires solving a pair of partial differential equations nonlinearly coupled through their boundary values [24].

The purpose of this paper is to partially remedy this situation. In this and in the second part [25], we provide what can be regarded as the first *computable* and *implementable* solution in the important case of a Gauss–Markov process. Gauss–Markov process have been discussed in the discrete-time setting in [26]. However, the existence and an implementable form of the optimal control are missing in [26] and, moreover, the noise intensity is restricted to be nonsingular. Similarly, [27], [28] consider continuous-time Gauss–Markov processes also with nonsingular noise intensity. Another related line of research in [29] sought to assign the *asymptotic* closed-loop state-covariance with dynamic output feedback. In spite of the fact that control takes place over an infinite time interval, here too, computational aspects and conditions for "assignability" of steady state covariance are non-trivial (see also [25]).

In this paper, we show that a linear dynamical system can be optimally steered from *any* initial Gaussian distribution for the initial state to *any* final one, over any finite interval $[0,T]$. In the present, we assume that the control input and the stochastic disturbance share the same input channels (i.e., the same input matrix); the general case is treated in [25]. The unique minimum-energy state-feedback control is explicitly constructed by solving two linear Lyapunov differential equations. These are *nonlinearly coupled* through boundary conditions at the two end points of the interval. Moreover, we show that the optimal choice for these boundary values can be expressed in *closed form* as (nonlinear) functions of the covariances for the initial and target Gaussian distributions.

The paper is structured as follows. The formulation of the main problem and the variational analysis that shows the form of the optimal control are given in Section II. The existence and the explicit construction of the optimal control is given in Section III. Although the state process may be a degenerate diffusions (since, typically, the rank of the input matrix is less than the dimension of the state vector), the law of the controlled dynamics is closest in the relative entropy sense to that of the uncontrolled dynamics, just as in the theory of the Schrödinger bridges; this is shown in Section IV. Finally, in Section V we present two illustrative examples. The first one is on *inertial* particles experiencing random (white) acceleration, and the second, on active damping of an oscillator driven by Nyquist–Johnson thermal noise.

II. Problem Formulation and Variational Analysis

Consider a "prior" evolution given by the vector Gauss–Markov process $\{x(t) \mid 0 \le t \le T\}$ satisfying the n-dimensional linear stochastic differential equation

$$dx(t) = A(t)x(t)dt + B(t)dw(t) \text{ with } x(0) = \xi \text{ a.s.} \tag{1}$$

and ξ an n-dimensional random vector independent of $\{w(t) \mid 0 \le t \le T\}$ with density

$$\rho_0(x) = (2\pi)^{-n/2}\det(\Sigma_0)^{-1/2}\exp\left(-\frac{1}{2}x'\Sigma_0^{-1}x\right). \tag{2}$$

Throughout, $\{w(t) \mid 0 \le t \le T\}$ is a standard, m-dimensional Wiener process and $A(\cdot)$ and $B(\cdot)$ are continuous matrix functions taking values in $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times m}$, respectively. Consider also the controlled evolution

$$\begin{aligned} dx^u(t) &= A(t)x^u(t)dt + B(t)u(t)dt + B(t)dw(t)\\ x^u(0) &= \xi \text{ a.s.} \end{aligned} \tag{3}$$

and a "target" end-point distribution

$$\rho_T(x) = (2\pi)^{-n/2}\det(\Sigma_T)^{-1/2}\exp\left(-\frac{1}{2}x'\Sigma_T^{-1}x\right) \tag{4}$$

which is Gaussian with zero mean with covariance $\Sigma_T > 0$. We let $\mathcal{U}$ be the family of *adapted*, *finite-energy* control functions such that (3) has a strong solution and $x^u(T)$ is distributed according to (4). More precisely, $u \in \mathcal{U}$ is such that $u(t)$ only depends on t and on $\{x^u(s); 0 \le s \le t\}$ for each $t \in [0,T]$, satisfies

$$\mathbb{E}\left\{\int_0^T u(t)'u(t)dt\right\} < \infty$$

and effects $x^u(T)$ to be distributed according to (4). The family $\mathcal{U}$ represents *admissible* control inputs which achieve the desired probability density transfer from ρ_0 to ρ_T. Thence we formulate the following *Schrödinger Bridge Problem*:

[1] The quest, in a sense, culminated in 1966/67 with Nelson's *Stochastic Mechanics* [16].

[2] Large deviations theory has various applications in hypothesis testing, rate distortion theory, etc, see, e.g., [17, Ch.11], [18, Ch. 2,3,7].

[3] We note that the drift term of the corresponding process can be obtained by the so called "logarithmic transformation" of stochastic control of Fleming, Holland, Mitter *et al.*, see [14], [19]–[22].

Problem 1: Determine whether $\mathcal{U}$ is non-empty and if so, determine $u^* := \arg\min_{u\in\mathcal{U}} J(u)$ where

$$J(u) := \mathbb{E}\left\{\int_0^T u(t)'u(t)dt\right\}.$$

In the next section we will prove that a minimizing control u^* always exists. The stochastic process $\{x^*(t) = x^{u^*}(t) \mid 0 \leq t \leq T\}$ will be referred to as the *Schrödinger bridge from* ρ_0 *to* ρ_T *over the prior* $\{x(t) = x^0(t) \mid 0 \leq t \leq T\}$.

Notice that in the "controlled" (3) the control variables $u(t)$ act through the same input "channels" which are subject to noise, i.e., both $u(t)$ and $dw(t)$ affect the state through the same $B(\cdot)$ matrix. The theory that follows can accordingly be relaxed to the case where the control has more "authority" (i.e., the range of the corresponding B-matrix contains the range of the B-matrix for the noise). Evidently the case where the control and noise enter through different channels is of great importance as well, and in particular, the case where the control authority is less than that of the stochastic noise. This is the subject of a sister paper [25], presented as part II and following the present one.

In the remaining of the section we identify a candidate structure for the optimal controls and reduce the problem to an algebraic condition involving two differential Lyapunov equations that are nonlinearly coupled through split boundary conditions.

Let us start by observing that this problem resembles a standard stochastic linear quadratic regulator problem except for the boundary conditions. The usual variational analysis can in fact be carried out, up to a point, namely the expression for the optimal control, in a similar fashion. Of the several ways in which the form of the optimal control can be obtained, we choose a most familiar one, namely the so-called "completion of squares."[4] Let $\{\Pi(t) \mid 0 \leq t \leq T\}$ be a differentiable function taking values in the set of symmetric, $n \times n$ matrices. Observe that Problem 1 is equivalent to minimizing over $\mathcal{U}$ the modified index

$$\tilde{J}(u)=\mathbb{E}\left\{\int_0^T u(t)'u(t)dt+x(T)'\Pi(T)x(T)-x(0)'\Pi(0)x(0)\right\}. \tag{5}$$

Indeed, as the two end-point marginals densities ρ_0 and ρ_T are fixed when u varies in $\mathcal{U}$, the two boundary terms are constant over $\mathcal{U}$. We can now rewrite $\tilde{J}(u)$ as follows:

$$\tilde{J}(u) = \mathbb{E}\left\{\int_0^T u(t)'u(t)dt + \int_0^T d\left(x(t)'\Pi(t)x(t)\right)\right\}.$$

Assuming that on $[0,T]$ $\Pi(t)$ satisfies the matrix Riccati equation

$$\dot{\Pi}(t) = -A(t)'\Pi(t) - \Pi(t)A(t) + \Pi(t)B(t)B(t)'\Pi(t) \tag{6}$$

a standard argument using It ō's rule (e.g., see [1]) shows that

$$\tilde{J}(u) = \mathbb{E}\left\{\int_0^T \|u(t) + B(t)'\Pi(t)x(t)\|^2\, dt + \int_0^T \frac{1}{2}\text{trace}\left(\Pi(t)B(t)B(t)'\right) dt\right\}.$$

Observe that the second integral is finite and invariant over $\mathcal{U}$. Hence, we obtain a candidate for the optimal control in the familiar form

$$u^*(t) = -B(t)'\Pi(t)x(t). \tag{7}$$

Such a choice of control will be possible provided we can find a solution $\Pi(t)$ of (6) such that the process

$$dx^*(t) = \left(A(t) - B(t)B(t)'\Pi(t)\right)x^*(t)dt + B(t)dw(t)$$
$$\text{with } x^*(0) = \xi \text{ a.s.} \tag{8}$$

leads to $x^*(T)$ with density ρ_T. If this is indeed possible, then we have solved Problem 1. It is important to observe that the optimal control, if it exists, is in a *state feedback* form. Consequently, the new optimal evolution is a *Gauss–Markov* process just as the prior evolution.

Finding the solution of the Riccati equation which achieves the density transfer is nontrivial. In the classical linear quadratic regulator theory, the terminal cost of the index would provide the boundary value $\Pi(T)$ for (6). However, here there is no boundary value and the two analyses sharply bifurcate. Therefore, we need to resort to something quite different as we have information concerning both initial and final densities, namely Σ_0 and Σ_T.

Let $\Sigma(t) := \mathbb{E}\{x^*(t)x^*(t)'\}$ be the state covariance of the sought optimal evolution. From (8) we have that $\Sigma(t)$ satisfies

$$\dot{\Sigma}(t) = \left(A(t) - B(t)B(t)'\Pi(t)\right)\Sigma(t) + \Sigma(t)\left(A(t) - B(t)B(t)'\Pi(t)\right)' + B(t)B(t)'. \tag{9}$$

It must also satisfy the two boundary conditions

$$\Sigma(0) = \Sigma_0, \quad \Sigma(T) = \Sigma_T \tag{10}$$

and, provided $(A(t), B(t))$ is controllable (see Section III), $\Sigma(t)$ is positive definite on $[0,T]$. Thus, we seek a solution pair $(\Pi(t), \Sigma(t))$ of the *coupled* system of these two equations (6) and (9) with split boundary conditions (10).

Interestingly, if we define the new matrix-valued function

$$H(t) := \Sigma(t)^{-1} - \Pi(t)$$

then a direct calculation using (9) and (6) shows that $H(t)$ satisfies the homogeneous Riccati equation

$$\dot{H}(t)=-A(t)'H(t)-H(t)A(t)-H(t)B(t)B(t)'H(t). \tag{11}$$

This equation is dual to (6) and the system of the two coupled matrix equations (6) and (9) can be replaced by (6) and (9). The new system is *decoupled*, except for the coupling through their boundary conditions

$$\Sigma_0^{-1} = \Pi(0) + H(0) \tag{12a}$$
$$\Sigma_T^{-1} = \Pi(T) + H(T). \tag{12b}$$

[4] Although it might be the most familiar to control engineers, the completion of the square argument for stochastic linear quadratic regulator control is not the most elementary. Indeed, a derivation which does not employ It ō's rule was presented in [30].

These boundary conditions (12) are sufficient for meeting the two end-point marginals ρ_0 and ρ_T provided of course that $\Pi(t)$ remains finite. We have therefore established the following result.

Proposition 2: Suppose $\Pi(t)$ and $H(t)$ satisfy equations (6)–(11) on $[0,T]$ with boundary conditions (12). Then the feedback control u^* given in (7) is optimal for Problem 1 and the optimal evolution of the Schrödinger bridge is given by (8).

Since (6) and (11) are homogeneous, they always admit the zero solution. The case $\Pi(t)\equiv 0$ corresponds to the situation where the prior evolution satisfies the boundary marginals conditions and, in that case, $H(t)^{-1}$ is simply the prior state covariance. In addition, it is also possible that $\Pi(t)$ vanishes in certain directions. Clearly, such directions remain invariant in that, if $\Pi(t)v=0$ for a value of $t\in[0,T]$, then $\Pi(t)v=0$ for all $t\in[0,T]$ as well. In such cases, it suffices to consider (6) and (11) in the orthogonal complement of null directions.

Thus, in general, Problem 1 reduces to the atypical situation of *two* Riccati equations (6) and (11) coupled through their boundary values. This might still at first glance appear to be a formidable problem. However, (6)–(11) are homogeneous and, as far as their non singular solutions, they reduce to *linear* differential Lyapunov equations. The latter, however, are still coupled through their boundary values in a *nonlinear* way. Indeed, suppose $\Pi(t)$ exists on the time interval $[0,T]$ and is invertible. Then $Q(t)=\Pi(t)^{-1}$ satisfies the linear equation

$$\dot Q(t)=A(t)Q(t)+Q(t)A(t)'-B(t)B(t)'. \tag{13a}$$

Likewise, if $H(t)$ exists on the time interval $[0,T]$ and is invertible, $P(t)=H(t)^{-1}$ satisfies the linear equation

$$\dot P(t)=A(t)P(t)+P(t)A(t)'+B(t)B(t)'. \tag{13b}$$

The boundary conditions (12) for this new pair $(P(t),Q(t))$ now read

$$\Sigma_0^{-1}=P(0)^{-1}+Q(0)^{-1} \tag{14a}$$

$$\Sigma_T^{-1}=P(T)^{-1}+Q(T)^{-1}. \tag{14b}$$

Conversely, if $Q(t)$ solves (13a) and is nonsingular on $[0,T]$, then $Q(t)^{-1}$ is a solution of (6), and similarly for $P(t)$. We record the following immediate consequence of Proposition 2.

Corollary 3: Suppose $P(t)$ and $Q(t)$ are nonsingular on $[0,T]$ and satisfy the equations (13a), (13b) with boundary conditions (12). Then the feedback control

$$u^*(t)=-B(t)'Q(t)^{-1}x(t). \tag{15}$$

is optimal for Problem 1. The evolution of the optimal Gauss–Markov process is given by

$$dx^*(t)=\left(A(t)-B(t)B(t)'Q(t)^{-1}\right)x^*(t)dt+B(t)dw(t)\quad \text{with } x^*(0)=\xi \text{ a.s.} \tag{16}$$

Thus, the system (13), (14) or, equivalently, the system (6), (11), and (12), appears as the *bottleneck* of the Schrödinger bridge problem. In the next section, we prove that this Schrödinger system always has solution $(\Pi(t),H(t))$, with both $\Pi(t)$ and $H(t)$ bounded on $[0,T]$, that satisfies (6), (11), and (12) and that this solution is unique.

III. Existence and Uniqueness of Optimal Control for the Linear Gaussian Bridge

We assume throughout that the system (1) (or equivalently the pair $(A(t),B(t))$) is controllable in the sense that the reachability Gramian

$$M(t_1,t_0):=\int_{t_0}^{t_1}\Phi(t_1,\tau)B(\tau)B(\tau)'\Phi(t_1,\tau)'d\tau$$

is nonsingular for all $t_0<t_1$ (with $t_0,t_1\in[0,T]$). As usual, $\Phi(t,s)$ denotes the state-transition matrix of (1) determined via

$$\frac{\partial}{\partial t}\Phi(t,s)=A(t)\Phi(t,s) \text{ and } \Phi(t,t)=I$$

and this is nonsingular for all $t,s\in[0,T]$. It is worth noting that for $t_1>0$ the reachability Gramian $M(t_1,0)=P(t_1)>0$ satisfies the differential Lyapunov equation (13b) with $P(0)=0$. The controllability Gramianmian

$$N(t_1,t_0):=\int_{t_0}^{t_1}\Phi(t_0,\tau)B(\tau)B(\tau)'\Phi(t_0,\tau)'d\tau$$

is necessarily also nonsingular for all $t_0<t_1$ $(t_0,t_1\in[0,T])$ and if, we similarly set $Q(t_0)=N(T,t_0)$, then $Q(t)$ satisfies (13a) with $Q(T)=0$.

However, as suggested in the previous section, we need to consider solutions $P(\cdot)$, $Q(\cdot)$ of these two differential Lyapunov equations (13) that satisfy boundary conditions that are coupled through (14). In general, $P(t)$ and $Q(t)$ do not need to be sign definite, but in order for

$$\Sigma(t)^{-1}=P(t)^{-1}+Q(t)^{-1}. \tag{17}$$

to qualify as a covariance of the controlled process (3) $P(t)$ and $Q(t)$ need to be invertible. This condition is also sufficient and $\Sigma(t)$ satisfies the corresponding differential Lyapunov equation for the covariance of the controlled process (16)

$$\dot\Sigma(t)=A_Q(t)\Sigma(t)+\Sigma(t)A_Q(t)'+B(t)B(t)' \tag{18}$$

with

$$A_Q(t):=(A(t)-B(t)B(t)'Q(t)^{-1}). \tag{19}$$

Next, we present our main technical result on the existence and uniqueness of an admissible pair $(P_-(t),Q_-(t))$ of solutions to (13), (14) that are invertible on $[0,T]$. Interstingly, there is always a second solution $(P_+(t),Q_+(t))$ to the nonlinear problem (13), (14) which is not admissible as it fails to be invertible on $[0,T]$.

Proposition 4: Consider $\Sigma_0,\Sigma_T>0$ and a controllable pair $(A(t),B(t))$ as before. The system of the two differential Lyapunov equations (13) has two sets of solutions $(P_\pm(\cdot), Q_\pm(\cdot))$ over $[0,T]$ that simultaneously satisfy the coupling boundary conditions (14). These two solutions are specified by

$$Q_\pm(0)=N(T,0)^{\frac12}S_0^{\frac12}\left(S_0+\frac12 I\pm\left(S_0^{\frac12}S_TS_0^{\frac12}+\frac14 I\right)^{\frac12}\right)^{-1}\times S_0^{\frac12}N(T,0)^{\frac12}$$

$$P_\pm(0)=\left(\Sigma_0^{-1}-Q_\pm(0)^{-1}\right)^{-1}$$

and the two differential equations (13), where

$$S_0 = N(T,0)^{-1/2}\Sigma_0 N(T,0)^{-1/2},$$
$$S_T = N(T,0)^{-1/2}\Phi(0,T)\Sigma_T\Phi(0,T)N(T,0)^{-1/2}.$$

The two pairs $(P_\pm(t), Q_\pm(t))$ with subscript $-$ and $+$, respectively, are distinguished by the following:

i) $Q_-(t)$ and $P_-(t)$ are both nonsingular on $[0,T]$, whereas
ii) $Q_+(t)$ and $P_+(t)$ become singular for some $t\in[0,T]$, possibly not for the same value of t.

Proof: Apply the time-varying change of coordinates

$$\xi(t) = N(T,0)^{-1/2}\Phi(0,t)x(t).$$

Then, in this new coordinates the dynamical system (1) becomes

$$d\xi(t) = \underbrace{N(T,0)^{-1/2}\Phi(0,t)B(t)}_{B_{\rm new}(t)}dw(t).$$

We will prove the statement in this new set of coordinates for the state, where the state matrix $A_{\rm new}=0$ and the state equation is simply $d\xi(t) = B_{\rm new}(t)dw(t)$, and then revert back to the original set of coordinates at the end. Accordingly

$$\dot P_{\rm new}(t) = B_{\rm new}(t)B_{\rm new}(t)'$$
$$\dot Q_{\rm new}(t) = -B_{\rm new}(t)B_{\rm new}(t)'$$

along with

$$M_{\rm new}(T,0) = N_{\rm new}(T,0) = I$$
$$\Sigma_{0,\rm new} = N(T,0)^{-1/2}\Sigma_0 N(T,0)^{-1/2} \tag{20a}$$

while

$$\Sigma_{T,\rm new} = N(T,0)^{-1/2}\Phi(0,T)\Sigma_T\Phi(0,T)'N(T,0)^{-1/2}. \tag{20b}$$

The relation between $Q_{\rm new}(t)$ and $Q(t)$ is given by

$$Q_{\rm new}(t) = N(T,0)^{-1/2}\Phi(0,t)Q(t)\Phi(0,t)'N(T,0)^{-1/2}.$$

This can be seen by taking the derivative of both sides

$$\begin{aligned}\dot Q_{\rm new}(t) =& -N(T,0)^{-1/2}\Phi(0,t)A(t)Q(t)\Phi(0,t)'N(T,0)^{-1/2}\\ &- N(T,0)^{-1/2}\Phi(0,t)Q(t)A(t)'\Phi(0,t)'N(T,0)^{-1/2}\\ &+ N(T,0)^{-1/2}\Phi(0,t)\dot Q(t)\Phi(0,t)'N(T,0)^{-1/2}\\ =& -N(T,0)^{-1/2}\Phi(0,t)B(t)B(t)'\Phi(0,t)'N(T,0)^{-1/2}\\ =& -B_{\rm new}(t)B_{\rm new}(t)'.\end{aligned}$$

In the next paragraph, for notational convenience, we drop the subscript "new" and prove the statement assuming that $A(t)=0$ as well as $N(T,0)=I$. We will return to the notation that distinguishes the two sets of coordinates with the subscript "new" and relate back to the original ones at the end of the proof.

Since $A(t)=0$, then $\Phi(t,x)=I$ for all $s,t\in[0,T]$. Further, $M(T,0)=N(T,0)=I$. Thus

$$P(T) = P(0) + I$$
$$Q(T) = Q(0) - I.$$

Substituting in (14), we obtain that

$$Q(0)^{-1} + P(0)^{-1} = \Sigma_0^{-1}$$
$$(Q(0)-I)^{-1} + (P(0)+I)^{-1} = \Sigma_T^{-1}.$$

Solving the first for $P(0)$ as a function of $Q(0)$ and substituting in the second, we have

$$\begin{aligned}\Sigma_T^{-1} &= \left(\left(\Sigma_0^{-1} - Q(0)^{-1}\right)^{-1} + I\right)^{-1} + (Q(0)-I)^{-1}\\ &= \left(\left(\Sigma_0^{-1} - Q(0)^{-1}\right)^{-1} + I\right)^{-1}\\ &\quad\times\left(Q(0) + \left(\Sigma_0^{-1} - Q(0)^{-1}\right)^{-1}\right)(Q(0)-I)^{-1}\\ &= \left(\left(\Sigma_0^{-1} - Q(0)^{-1}\right)^{-1} + I\right)^{-1}\\ &\quad\times\left(\Sigma_0^{-1} - Q(0)^{-1}\right)^{-1}\Sigma_0^{-1}Q(0)\left(Q(0)-I\right)^{-1}\\ &= \left(\Sigma_0^{-1} + I - Q(0)^{-1}\right)^{-1}\Sigma_0^{-1}\left(I - Q(0)^{-1}\right)^{-1}.\end{aligned}$$

After inverting both terms, simple algebra leads to

$$(I - Q(0)^{-1})\Sigma_0(I - Q(0)^{-1}) + (I - Q(0)^{-1}) = \Sigma_T.$$

This is a quadratic expression and has two Hermitian solutions

$$I - Q(0)^{-1} = \Sigma_0^{-1/2}\left(-\frac{1}{2}I \mp \left(\Sigma_0^{\frac12}\Sigma_T\Sigma_0^{\frac12} + \frac14 I\right)^{\frac12}\right)\Sigma_0^{-1/2}. \tag{21}$$

This gives that

$$Q(0) = \Sigma_0^{\frac12}\left(\Sigma_0 + \frac12 I \pm \left(\Sigma_0^{\frac12}\Sigma_T\Sigma_0^{\frac12} + \frac14 I\right)^{\frac12}\right)^{-1}\Sigma_0^{\frac12}.$$

To see that i) holds evaluate (in these simplified coordinates where there is no drift and $M(T,0)=I$)

$$\begin{aligned}Q_-(t)^{-1} &= (Q_-(0) - M(t,0))^{-1}\\ &= -M(t,0)^{-1} - M(t,0)^{-1}\\ &\quad\times\left(Q_-(0)^{-1} - M(t,0)^{-1}\right)^{-1}M(t,0)^{-1}\\ &= -M(t,0)^{-1} - M(t,0)^{-1}\Sigma_0^{\frac12}\\ &\quad\times\left(\Sigma_0 + \frac12 I - \left(\Sigma_0^{\frac12}\Sigma_T\Sigma_0^{\frac12} + \frac14 I\right)^{\frac12}\right.\\ &\quad\left. - \Sigma_0^{\frac12}M(t,0)^{-1}\Sigma_0^{\frac12}\right)^{-1}\Sigma_0^{\frac12}M(t,0)^{-1}\end{aligned}$$

for $t>0$. For $t\in(0,T]$, the expression in parenthesis

$$\Sigma_0 + \frac12 I - \left(\Sigma_0^{\frac12}\Sigma_T\Sigma_0^{\frac12} + \frac14 I\right)^{\frac12} - \Sigma_0^{\frac12}M(t,0)^{-1}\Sigma_0^{\frac12}$$

is clearly maximal when $t=T$. However, for $t=T$ when $M(T,0)=I$, this expression is seen to be

$$\frac12 I - \left(\Sigma_0^{\frac12}\Sigma_T\Sigma_0^{\frac12} + \frac14 I\right)^{\frac12} < 0.$$

Therefore, the expression in parenthesis is never singular and we deduce that $Q_-(t)^{-1}$ remains bounded for all $t \in (0,T]$, i.e., $Q_-(t)$ remains non-singular. For $t=0$, $Q(0)^{-1}$ is seen to be finite from (21). The argument for $P_-(t)$ is similar. Regarding ii), it suffices to notice that $0 < Q_+(0) < I$ while $Q_+(T) = Q_+(0) - I < 0$. The statement ii) follows by continuity of $Q_+(t)$, and similarly for $P_+(t)$.

We now revert back to the set of coordinates where the drift is not necessarily zero and where $N(T,0)$ may not be the identity. We see that

$$\begin{aligned} Q_\pm(0) &= N(T,0)^{\frac{1}{2}} (Q_\pm(0))_{\rm new} N(T,0)^{\frac{1}{2}} \\ &= N(T,0)^{\frac{1}{2}} \Sigma_{0,\rm new}^{\frac{1}{2}} \\ &\quad \times \left(\Sigma_{0,\rm new} + \frac{1}{2} I \pm \left(\Sigma_{0,\rm new}^{\frac{1}{2}} \Sigma_{T,\rm new} \Sigma_{0,\rm new}^{\frac{1}{2}} + \frac{1}{4} I \right)^{\frac{1}{2}} \right)^{-1} \\ &\quad \Sigma_{0,\rm new}^{\frac{1}{2}} N(T,0)^{\frac{1}{2}} \end{aligned}$$

where $\Sigma_{0,\rm new}, \Sigma_{T,\rm new}$ as in (20a) and (20b), which for compactness of notation in the statement of the proposition we rename S_0 and S_T, respectively. ■

Remark 5: We have numerically observed that the iteration

$$\begin{aligned} & P(0) \\ & \downarrow \\ & P(T) = \Phi(T,0) P(0) \Phi(T,0)' + M(T,0) \\ & \downarrow \\ & Q(T) = \left(\Sigma_T^{-1} - P(T)^{-1} \right)^{-1} \\ & \downarrow \\ & Q(0) = \Phi(0,T) \left(Q(T) + M(T,0) \right) \Phi(0,T)' \\ & \downarrow \\ & P(0) = (\Sigma_0^{-1} - Q(0)^{-1})^{-1} \end{aligned}$$

using (14), converges to $Q_-(0)$, $P_-(0)$, $Q_-(T)$, $P_-(T)$, starting from a generic choice for $Q(0)$. The choice with a "$-$" is the one that generates the Schrödinger bridge as explained below. It is interesting to compare this property with similar properties of iterations that lead to solutions of Schrödinger systems in [31] and [32]. A proof of convergence is, at present, not available.

Remark 6: Besides the expression in the proposition, another equivalent "closed form" formula for $Q_\pm(0)$ is

$$\begin{aligned} Q_\pm(0) = \Sigma_0^{\frac{1}{2}} \Bigg(& \frac{1}{2} I + \Sigma_0^{\frac{1}{2}} \Phi(T,0)' M(T,0)^{-1} \Phi(T,0) \Sigma_0^{\frac{1}{2}} \\ & \pm \Bigg(\frac{1}{4} I + \Sigma_0^{\frac{1}{2}} \Phi(T,0)' M(T,0)^{-1} \Sigma_T M \\ & (T,0)^{-1} \Phi(T,0) \Sigma_0^{\frac{1}{2}} \Bigg)^{\frac{1}{2}} \Bigg)^{-1} \Sigma_0^{\frac{1}{2}} \end{aligned}$$

Remark 7: Interestingly, the solution $\Pi_+(t) = Q_+(t)^{-1}$ of the Riccati (6) corresponding to the choice "$+$" in $Q_\pm$ has a *finite escape time*.

We are now in a position to state the full solution to the Schrödinger Bridge Problem 1.

Theorem 8: Assuming that the pair $(A(t), B(t))$ is controllable and that $\Sigma_0, \Sigma_T > 0$, Problem 1 has a unique optimal solution

$$u^\star(t) = -B(t)' Q_-(t)^{-1} x(t) \tag{22}$$

where $Q_-(\cdot)$ [together with a corresponding matrix function $P_-(\cdot)$] solves to the pair of coupled Lyapunov differential equations in Proposition 4.

Proof: Since Proposition 4 has established existence and uniqueness of nonsingular solutions $(P_-(\cdot), Q_-(\cdot))$ to the system (13), the result now follows from Corollary 3. ■

Thus, the controlled process (16) with $\Pi(t) = Q_-(t)^{-1}$

$$dx^* = (A(t) - B(t)B(t)'Q_-(t)^{-1}) x^*(t) dt + B(t) dw(t) \tag{23}$$

steers the beginning density ρ_0 to the final one, ρ_T, with the least cost. Alternatively, it forms a least-effort bridge between the two given marginals. It turns out that this controlled stochastic differential equation specifies the random evolution which is closest to the prior in the sense of relative entropy among those with the two given marginal distributions. This will be explained in the next section.

Remark 9: The variant of Problem 1 where the two marginals have a non-zero mean is of great practical significance. The formulae for the optimal control easily extend to this case as follows. Assuming that the Gaussian marginals ρ_0 and ρ_T have mean m_0 and m_T, respectively, a deterministic term is needed in (23) for the bridge to satisfy the means. The controlled process becomes

$$\begin{aligned} dx^* = (A(t) - B(t)B(t)'Q_-(t)^{-1}) x^*(t) dt + B(t)B(t)' m(t) dt \\ + B(t) dw(t) \end{aligned} \tag{24}$$

where

$$m(t) = \hat\Phi(0,t)' \hat M(T,0)^{-1} (m_T - \hat\Phi(T,0) m_0)$$

and $\hat\Phi(t,s), \hat M(t,s)$ satisfy

$$\frac{\partial \hat\Phi(t,s)}{\partial t} = (A(t) - B(t)B(t)'Q_-(t)^{-1}) \hat\Phi(t,s), \hat\Phi(t,t) = I$$

$$\hat M(t,s) = \int_s^t \hat\Phi(t,\tau) B(t) B(t)' \hat\Phi(t,\tau)' d\tau.$$

It is easy to verify that (24) meets the condition on the two marginal distributions. To see (24) is in fact optimal, observe that Problem 1 is equivalent to minimizing over $\mathcal{U}$ the augmented cost functional

$$\begin{aligned} \tilde J(u) = \mathbb{E} \Bigg\{ & \int_0^T u(t)' u(t) dt + x(T)' Q_-(T)^{-1} x(T) \\ & - 2m(T)' x(T) - x(0)' Q_-(0)^{-1} x(0) + 2m(0)' x(0) \Bigg\}. \end{aligned}$$

But

$$\tilde{J}(u) = \mathbb{E}\left\{\int_0^T u(t)'u(t)dt + \int_0^T d\left(x(t)'Q_-(t)^{-1}x(t) - 2m(t)'x(t)\right)\right\}$$
$$= \mathbb{E}\left\{\int_0^T \left\|u(t) + B(t)'Q_-(t)^{-1}x(t) - B(t)'m(t)\right\|^2 dt + \int_0^T \left[\frac{1}{2}\text{trace}\left(Q_-(t)^{-1}B(t)B(t)'\right) - m(t)'B(t)B(t)'m(t)\right]dt\right\}.$$

Hence

$$u(t) = -B(t)'Q_-(t)^{-1}x(t) + B(t)'m(t)$$

is indeed the optimal control law and (24) is the sought bridge.

IV. Minimum Relative Entropy Interpretation of Optimal Control

As noted earlier, there is a close relationship between the theory of large deviations, maximum entropy problems for random evolutions and stochastic optimal control [19], [21], [22], [33], [34]. In particular, classical Schrödinger bridges can be interpreted as both, a solution to a stochastic optimal control problem, as well as one inducing a probability law on path space that is consistent with given marginals and closest to the prior in the sense of relative entropy. In other words, in effect, they answer the question of what the most likely path distribution is after "conditioning" the stochastic evolution on the two end-point marginals. Below we show that the same property holds for the present case of general stochastic linear system, i.e., of possibly degenerate linear diffusions.

For the purposes of this section we denote by $\mathcal{X} = C([0,T];\mathbb{R}^n)$ the space of continuous, n-dimensional sample paths of a linear diffusion as in (1) and by $\mathcal{P}(\cdot)$ the induced probability measure on $\mathcal{X}$. One can describe $\mathcal{P}(\cdot)$ as a mixture of measures pinned at the two ends of the interval $[0,T]$, that is

$$\mathcal{P}(\cdot) = \int \mathcal{P}(\cdot \mid x(0) = x_0, x(T) = x_T)\mathcal{P}_{0,T}(dx_0 dx_T)$$

where $\mathcal{P}(\cdot \mid x(0) = x_0, x(T) = x_T)$ is the conditional probability and $\mathcal{P}_{0,T}(\cdot)$ is the joint probability of $(x(0), x(T))$. The two end-point joint measure $\mathcal{P}_{0,T}(\cdot)$, which is Gaussian, has a (zero-mean) probability density function $g_{S_{0,T}}(x_0, x_T)$ with covariance

$$S_{0,T} = \begin{bmatrix} S_0 & S_0\Phi(T,0)' \\ \Phi(T,0)S_0 & S_T \end{bmatrix} \tag{25}$$

where

$$S_0 = \mathbb{E}\{x_0 x_0'\}$$
$$S_t = \Phi(t,0)S_0\Phi(t,0)' + \int_0^t \Phi(t,\tau)B(\tau)B(\tau)'\Phi(t,\tau)'d\tau.$$

In view of Sanov's theorem, see [24, Section 3], Schrödinger's question reduces to identifying a probability law $\tilde{\mathcal{P}}(\cdot)$ on $\mathcal{X}$ that minimizes the relative entropy

$$\mathcal{S}(\tilde{\mathcal{P}}, \mathcal{P}) := \int_{\mathcal{X}} \log\left(\frac{d\tilde{\mathcal{P}}}{d\mathcal{P}}\right) d\tilde{\mathcal{P}}$$

among those that have the prescribed marginals. Evidently, this is an abstract problem on an infinite-dimensional space. However, since

$$\tilde{\mathcal{P}}(\cdot) = \int \tilde{\mathcal{P}}(\cdot \mid x(0) = x_0, x(T) = x_T)\tilde{\mathcal{P}}_{0,T}(dx_0 dx_T)$$

the relative entropy can be readily written as the sum of two nonnegative terms, the relative entropy between the two end-point joint measures

$$\int \log\left(\frac{d\tilde{\mathcal{P}}_{0,T}}{d\mathcal{P}_{0,T}}\right)\tilde{\mathcal{P}}_{0,T}$$

plus

$$\int \log\left(\frac{d\tilde{\mathcal{P}}(\cdot \mid x(0) = x_0, x(T) = x_T)}{d\mathcal{P}(\cdot \mid x(0) = x_0, x(T) = x_T)}\right)\tilde{\mathcal{P}}.$$

The second term becomes zero (and therefore minimal) when the conditional probability $\tilde{\mathcal{P}}(\cdot \mid x(0) = x_0, x(T) = x_T)$ is taken to be the same as $\mathcal{P}(\cdot \mid x(0) = x_0, x(T) = x_T)$. Thus, the solution is in the same *reciprocal class* [35] as the prior evolution and, as already observed by Schrödinger [15] in a simpler context, the problem reduces to the finite-dimensional problem of minimizing relative entropy of the joint initial-final distribution among those that have the prescribed marginals.

It turns out that the probability law induced by (23), is the closest in the relative entropy sense to the law of (1), that agrees with the two end-point marginal distributions at $t = 0$ and $t = T$. Below we show this by verifying directly that the densities between the two are identical when conditioned at the two end points, i.e., they share the same bridges, and that the end-point joint marginal for (23) is indeed closest to the corresponding joint marginal for (1).

In order to show that two linear systems share the same bridges, we need the following lemma which is based on [36].

Lemma 10: The probability law of the SDE (1), when conditioned on $x(0) = x_0, x(T) = x_T$, for any x_0, x_T, reduces to the probability law induced by the SDE

$$dx = (A - BB'R(t)^{-1})xdt + BB'R(t)^{-1}\Phi(t,T)x_T dt + Bdw$$

where $R(t)$ satisfies

$$\dot{R}(t) = AR(t) + R(t)A' - BB'$$

with $R(T) = 0$.

The stochastic process specified by this conditioning, or the latter SDE, will be referred to as the *pinned process associated to* (1). Thus, in order to establish that the probability laws of (1) and (23) conditioned on $x(0) = x_0$, $x(T) = x_T$ are identical, it suffices to show that they have the same pinned processes for any x_0, x_T. This is done next.

Theorem 11: The probability law induced by (23) represents the minimum of the relative entropy with respect to the law of (1) over all probability laws on $\mathcal{X}$ that have Gaussian marginals with zero mean and covariances Σ_0 and Σ_T, respectively, at the two end-points of the interval $[0, T]$.

Proof: We show that i) the joint distribution between the two end-points of $[0, T]$ for (23) is the minimizer of the relative entropy, with respect to the corresponding two-end-point joint distribution of (1), over distributions that satisfy the end-point constraint that the marginals are Gaussian with specified covariances, and ii) the probability laws of these two SDEs on sample paths, conditioned on $x(0) = x_0, x(T) = x_T$ for any x_0, x_T, are identical by showing that they have the same pinned processes. We use the notation

$$g_S(x) := (2\pi)^{-n/2} \det(S)^{-1/2} \exp\left[-\frac{1}{2} x' S^{-1} x\right]$$

to denote the standard Gaussian probability density function with mean zero and covariance S.

We start with i). In general, the relative entropy between two Gaussian distributions $g_S(x)$ and $g_\Sigma(x)$ is

$$\begin{aligned}\int_{\mathbb{R}^n} g_\Sigma(x) \log\left(\frac{g_\Sigma}{g_S}\right) dx &= \int_{\mathbb{R}^n} g_\Sigma \log\left(\frac{\det(S)^{\frac{1}{2}}}{\det(\Sigma)^{\frac{1}{2}}}\right) dx \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^n} g_\Sigma(x) \left(x' S^{-1} x - x \Sigma x\right) dx \\ &= \frac{1}{2}\log\left(\det(S)\right) - \frac{1}{2}\log\left(\det(\Sigma)\right) \\ &\quad + \frac{1}{2}\text{trace}(S^{-1}\Sigma) - \frac{1}{2}\text{trace}(I).\end{aligned} \tag{26}$$

If p_Σ is a probability density function, not necessarily Gaussian, having covariance Σ, then

$$\begin{aligned}\int_{\mathbb{R}^n} p_\Sigma(x) \log\left(\frac{p_\Sigma}{g_S}\right) dx &= \int_{\mathbb{R}^n} p_\Sigma(x) \log\left(\frac{p_\Sigma}{g_S}\frac{g_\Sigma}{g_\Sigma}\right) dx \\ &= \int_{\mathbb{R}^n} p_\Sigma(x) \log\left(\frac{p_\Sigma}{g_\Sigma}\right) dx + \int_{\mathbb{R}^n} p_\Sigma(x) \log\left(\frac{g_\Sigma}{g_S}\right) dx\end{aligned} \tag{27}$$

where we multiplied and divided by g_Σ and then partitioned accordingly. We observe that

$$\int_{\mathbb{R}^n} p_\Sigma(x) \log\left(\frac{g_\Sigma}{g_S}\right) dx = \int_{\mathbb{R}^n} g_\Sigma(x) \log\left(\frac{g_\Sigma}{g_S}\right) dx.$$

since $\log(g_\Sigma / g_S)$ is a quadratic form in x. Thus, the minimizer of relative entropy to g_S among probability density functions with covariance Σ is Gaussian since the first term in (27) is positive unless $p_\Sigma = g_\Sigma$, in which case it is zero.

We consider two-point joint Gaussian distributions with covariances $S_{0,T}$ as in (25) with $S_0 = \Sigma_0$, and

$$\Sigma_{0,T} := \begin{bmatrix} \Sigma_0 & Y' \\ Y & \Sigma_T \end{bmatrix}$$

and evaluate Y that minimizes the relative entropy. To this end we focus on

$$\text{trace}(S_{0,T}^{-1}\Sigma_{0,T}) - \log\det(\Sigma_{0,T}). \tag{28}$$

Since

$$S_{0,T} = \begin{bmatrix} I \\ \Phi(T,0) \end{bmatrix} \Sigma_0 \left[\, I, \quad \Phi(T,0)' \,\right] + \begin{bmatrix} 0 & 0 \\ 0 & M(T,0) \end{bmatrix}$$

it follows that:

$$S_{0,T}^{-1} = \begin{bmatrix} \Sigma_0^{-1} + \Phi' M^{-1} \Phi & -\Phi' M^{-1} \\ -M^{-1}\Phi & M^{-1} \end{bmatrix}$$

where we simplified notation by setting $\Phi := \Phi(T, 0)$ and $M := M(T, 0)$. Then, the expression in (28) becomes

$$\begin{aligned}&\text{trace}\big((\Sigma_0^{-1} + \Phi' M^{-1}\Phi)\Sigma_0 - \Phi' M^{-1} Y - Y' M^{-1}\Phi + M^{-1}\Sigma_T\big) \\ &\qquad - \log\det(\Sigma_0) - \log\det(\Sigma_T - Y\Sigma_0^{-1}Y').\end{aligned}$$

Retaining only the terms that involve Y leads us to seek a maximizing choice for Y in

$$f(Y) := \log\det(\Sigma_T - Y\Sigma_0^{-1}Y') + 2\,\text{trace}(\Phi' M^{-1} Y).$$

Equating the differential of this last expression as a function of Y to zero gives

$$-2\Sigma_0^{-1} Y' (\Sigma_T - Y\Sigma_0^{-1}Y')^{-1} + 2\,\Phi' M^{-1} = 0. \tag{29}$$

To see this, denote by Δ a small perturbation of Y and retain the linear terms in Δ in

$$\begin{aligned}&f(Y + \Delta) - f(Y) \\ &\quad = \log\det(I - (\Sigma_T - Y\Sigma_0^{-1}Y')^{-1}(\Delta\Sigma_0^{-1}Y' + Y\Sigma_0^{-1}\Delta')) \\ &\qquad + 2\,\text{trace}(\Phi' M^{-1}\Delta) \\ &\quad \simeq -\text{trace}((\Sigma_T - Y\Sigma_0^{-1}Y')^{-1}(\Delta\Sigma_0^{-1}Y' + Y\Sigma_0^{-1}\Delta')) \\ &\qquad + 2\,\text{trace}(\Phi' M^{-1}\Delta) \\ &\quad = -2\,\text{trace}(\Sigma_0^{-1}Y'(\Sigma_T - Y\Sigma_0^{-1}Y')^{-1}\Delta) \\ &\qquad + 2\,\text{trace}(\Phi' M^{-1}\Delta).\end{aligned}$$

Let now

$$\Sigma_{0,T} = \begin{bmatrix} \Sigma_0 & \Sigma_0\Phi_{Q_-}(T,0)' \\ \Phi_{Q_-}(T,0)\Sigma_0 & \Sigma_T \end{bmatrix}$$

where $\Phi_{Q_-}(T, 0)$ is the state-transition matrix of $A_{Q_-}(t)$, i.e., it satisfies

$$\begin{aligned}\frac{\partial}{\partial t}\Phi_{Q_-}(t,s) &= A_{Q_-}(t)\Phi_{Q_-}(t,s) \\ -\frac{\partial}{\partial s}\Phi_{Q_-}(t,s) &= \Phi_{Q_-}(t,s)A_{Q_-}(s)\end{aligned}$$

with $\Phi_{Q_-}(s,s)=I$. We need to show that $\Sigma_{0,T}$ here is the solution of the relative entropy minimization problem above. By concavity of $f(Y)$, it suffices to show that $Y=\Phi_{Q_-}(T,0)\Sigma_0$ satisfies the first-order condition (29), that is

$$\begin{aligned}&\Phi_{Q_-}(T,0)'(\Sigma_T-\Phi_{Q_-}(T,0)\Sigma_0\Phi_{Q_-}(T,0)')^{-1}\\&\quad=\Phi(T,0)'M(T,0)^{-1}\\&\quad=\Phi(T,0)'(S_T-\Phi(T,0)S_0\Phi(T,0)')^{-1}\end{aligned}$$

where S_t is as in (25) with $S_0=\Sigma_0$. By taking inverse of both sides we obtain an equivalent formula

$$\Sigma_T\Phi_{Q_-}(0,T)'-\Phi_{Q_-}(T,0)\Sigma_0=S_T\Phi(0,T)'-\Phi(T,0)\Sigma_0. \tag{30}$$

We claim

$$\Sigma_t\Phi_{Q_-}(0,t)'-\Phi_{Q_-}(t,0)\Sigma_0=S_t\Phi(0,t)'-\Phi(t,0)\Sigma_0$$

then (30) follows by taking $t=T$. We now prove our claim. For convenience, denote

$$\begin{aligned}F_1(t)&=\Sigma_t\Phi_{Q_-}(0,t)'-\Phi_{Q_-}(t,0)\Sigma_0\\F_2(t)&=S_t\Phi(0,t)'-\Phi(t,0)\Sigma_0\\F_3(t)&=Q_-(t)\left(\Phi_{Q_-}(0,t)'-\Phi(0,t)'\right).\end{aligned}$$

We will show that $F_1(t)=F_2(t)=F_3(t)$. First we show $F_2(t)=F_3(t)$. Since $F_2(0)=F_3(0)=0$, we only need to show that they satisfy the same differential equation. To this end, compare

$$\begin{aligned}\dot F_2(t)&=\dot S_t\Phi(0,t)'-S_tA'\Phi(0,t)'-A\Phi(t,0)\Sigma_0\\&=(AS_t+S_tA'+BB')\Phi(0,t)'-S_tA'\Phi(0,t)'\\&\quad-A\Phi(t,0)\Sigma_0\\&=AF_2(t)+BB'\Phi(0,t)'\end{aligned}$$

with

$$\begin{aligned}\dot F_3(t)&=\dot Q_-(t)(\Phi_{Q_-}(0,t)'-\Phi(0,t)')\\&\quad+Q_-(t)(-A_{Q_-}(t)'\Phi_{Q_-}(0,t)'+A'\Phi(0,t)')\\&=(AQ_-(t)+Q_-(t)A'-BB')(\Phi_{Q_-}(0,t)'-\Phi(0,t)')\\&\quad-Q_-(t)A'(\Phi_{Q_-}(0,t)'-\Phi(0,t)')+BB'\Phi_{Q_-}(0,t)'\\&=AF_3(t)+BB'\Phi(0,t)'\end{aligned}$$

which proves the claim $F_2(t)=F_3(t)$. We next show that $F_1(t)=F_3(t)$. Let

$$\begin{aligned}H(t)&=Q_-(t)^{-1}(F_3(t)-F_1(t))\\&=-(Q_-(t)^{-1}-\Sigma_t^{-1})\Sigma_t\Phi_{Q_-}(0,t)'\\&\quad+Q_-(t)^{-1}\Phi_{Q_-}(t,0)\Sigma_0-\Phi(0,t)'\\&=P(t)^{-1}\Sigma_t\Phi_{Q_-}(0,t)'\\&\quad+Q_-(t)^{-1}\Phi_{Q_-}(t,0)\Sigma_0-\Phi(0,t)'\end{aligned}$$

then

$$\begin{aligned}\dot H(t)&=\dot P(t)^{-1}\Sigma_t\Phi_{Q_-}(0,t)'+P(t)^{-1}\dot\Sigma_t\Phi_{Q_-}(0,t)'\\&\quad-P(t)^{-1}\Sigma_tA_{Q_-}(t)'\Phi_{Q_-}(0,t)'+\dot Q_-(t)^{-1}\Phi_{Q_-}(t,0)\\&\quad\times\Sigma_0+Q_-(t)^{-1}A_{Q_-}(t)\Phi_{Q_-}(t,0)\Sigma_0+A'\Phi(0,t)'\\&=-A'H(t).\end{aligned}$$

Since $H(0)=Q_-(0)^{-1}(F_3(0)-F_1(0))=0$, it follows that $H(t)=0$ for all t, and therefore, $F_1(t)=F_3(t)$. This completes the proof of the first part.

We now prove ii). According to Lemma 10, the pinned process corresponding to (1) satisfies

$$dx=(A-BB'R_1(t)^{-1})xdt+BB'R_1(t)^{-1}\Phi(t,T)x_Tdt+Bdw \tag{31}$$

where $R_1(t)$ satisfies

$$\dot R_1(t)=AR_1(t)+R_1(t)A'-BB'$$

with $R_1(T)=0$, while the pinned process corresponding to (23) satisfies

$$\begin{aligned}dx&=(A_{Q_-}(t)-BB'R_2(t)^{-1})xdt\\&\quad+BB'R_2(t)^{-1}\Phi_{Q_-}(t,T)x_Tdt+Bdw\end{aligned} \tag{32}$$

where $R_2(t)$ satisfies

$$\dot R_2(t)=A_{Q_-}(t)R_2(t)+R_2(t)A_{Q_-}(t)'-BB'$$

with $R_2(T)=0$. We next show (31) and (32) are identical. It suffices to prove that

$$A-BB'R_1(t)^{-1}=A_{Q_-}(t)-BB'R_2(t)^{-1} \tag{33}$$

$$R_1(t)^{-1}\Phi(t,T)=R_2(t)^{-1}\Phi_{Q_-}(t,T). \tag{34}$$

Equation (33) is equivalent to

$$R_1(t)^{-1}=R_2(t)^{-1}+Q_-(t)^{-1}.$$

Multiply $R_1(t)$ and $R_2(t)$ on both sides to obtain

$$R_2(t)=R_1(t)+R_1(t)Q_-(t)^{-1}R_2(t).$$

Now let

$$J(t)=R_1(t)+R_1(t)Q_-(t)^{-1}R_2(t)-R_2(t).$$

Then

$$\begin{aligned}\dot J(t)&=\dot R_1(t)+\dot R_1(t)Q_-(t)^{-1}R_2(t)+R_1(t)\dot Q_-(t)^{-1}R_2(t)\\&\quad+R_1(t)Q_-(t)^{-1}\dot R_2(t)-\dot R_2(t)\\&=AJ+JA_{Q_-}(t)'.\end{aligned}$$

Since

$$J(T)=R_1(T)+R_1(T)Q_-(T)^{-1}R_2(T)-R_2(T)=0$$

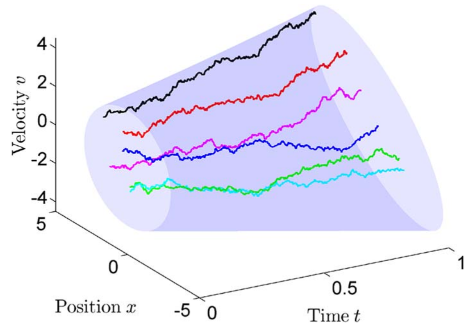

Fig. 1. Inertial particles: state trajectories without control.

it follows that $J(t) = 0$. This completes the proof of (33). Equation (34) is equivalent to

$$\Phi(T,t)R_1(t) = \Phi_{Q_-}(T,t)R_2(t).$$

Let

$$K(t) = \Phi(T,t)R_1(t) - \Phi_{Q_-}(T,t)R_2(t)$$

and then

$$\begin{aligned}\dot{K}(t) =& -\Phi(T,t)AR_1(t) + \Phi(T,t)\dot{R}_1(t)\\ &+ \Phi_{Q_-}(T,t)A_{Q_-}(t)R_2(t) - \Phi_{Q_-}(T,t)\dot{R}_2(t)\\ =& K(t)(A' - R_1(t)^{-1}BB').\end{aligned}$$

Since

$$K(T) = \Phi(T,T)R_1(T) - \Phi_{Q_-}(T,T)R_2(T) = 0$$

it follows that $K(t) = 0$ as well for all t. This completes the proof. ■

V. Illustrative Examples

We present two examples that illustrate the effect of optimal probability density steering. The first is based on inertial particles experiencing random accelerations and the second on an electrical circuit experiencing Nyquist-Johnson thermal noise from a resistor.

A. Inertial Particles

Consider inertial particles experiencing random acceleration according to the model

$$\begin{aligned}dx(t) &= v(t)dt\\ dv(t) &= u(t)dt + dw(t)\end{aligned}$$

where $u(t)$ is a control force at our disposal, $x(t)$ represents position and $v(t)$ represents velocity. We want to squeeze the spread of the particles from an initial Gaussian distribution with $\Sigma_0 = I$ at $t = 0$ to a terminal marginal $\Sigma = (1/4)I$ at $t = 1$. Figs. 1 and 2 show sample paths in the phase space of (x, v) as functions of time in two cases, first in Fig. 1 when no control is being applied and the sample paths diverge and, second, in Fig. 2 when using the optimal strategy for feedback control that

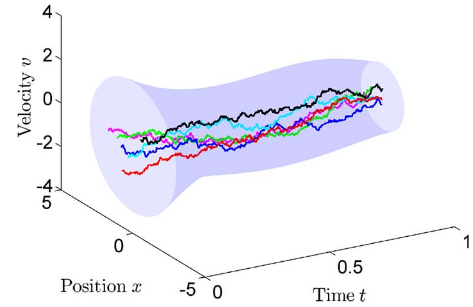

Fig. 2. Inertial particles: state trajectories for $\Sigma_1 = (1/4)I$.

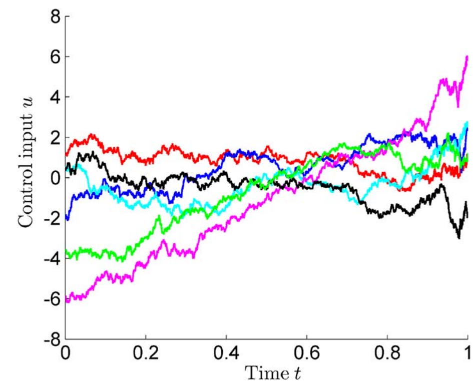

Fig. 3. Inertial particles: control inputs for $\Sigma_1 = (1/4)I$.

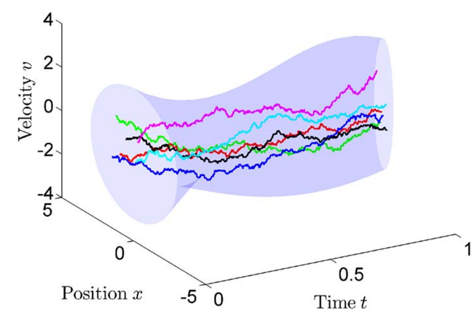

Fig. 4. Inertial particles: state trajectories for $\Sigma_1 = \mathrm{diag}(.05, 1)$.

was explained earlier (Theorem 8). For the latter case, Fig. 3 displays the corresponding control action for each trajectory.

We provide two additional situations where the final distribution is localized in space and in velocity, respectively. The limit may be thought to approximate singular marginals, in each case, and it is of interest to compare the two since in one case the stochastic excitation affects directly the component of interest (velocity) whereas in the other after integration. Thus, we again take $\Sigma_0 = I$ while we take Σ_1 to equal to $\mathrm{diag}(.05, 1)$ and $\mathrm{diag}(1, .05)$, respectively, for the two cases. Sample paths in phase space under the optimal control law are shown in Figs. 4 and 5, respectively.

In all phase plots in Figs. 1, 2, 4, and 5, the transparent blue "tube" represents the "3σ" tolerance interval. More specifically, the intersection ellipsoid between the tube and the slice plane t is the set

$$[\,x \quad v\,]\,\Sigma_t^{-1}\begin{bmatrix} x \\ v \end{bmatrix} \le 3^2.$$

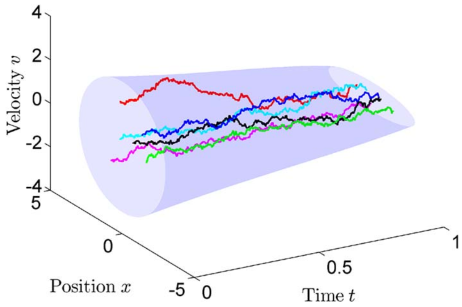

Fig. 5. Inertial particles: state trajectories for $\Sigma_1 = \mathrm{diag}(1, .05)$.

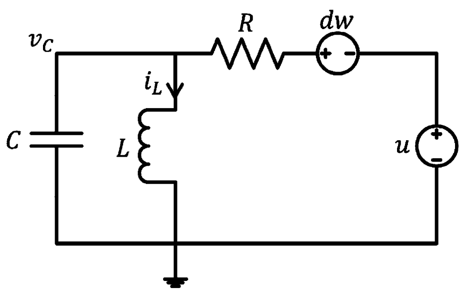

Fig. 6. Circuit.

B. Nyquist–Johnson Resistor Noise

Consider the circuit in Fig. 6 that includes a resistor with a Nyquist–Johnson thermal noise voltage source. A model for the circuit is

$$\begin{aligned} Ldi_L(t) &= \quad v_C(t)dt \\ RCdv_C(t) &= -v_C(t)dt - Ri_L(t)dt + u(t)dt + dw(t) \end{aligned}$$

with all parameters $R = L = C = 1$ in compatible units. Without any active control, i.e., when $u(t) \equiv 0$, the RLC circuit is driven by the thermal noise and reaches a steady state where the covariance matrix of the state vector $(i_L, v_C)'$ is $(1/2)I$. Thus, we begin with random initial conditions for the state having an initial Gaussian distribution with $\Sigma_0 = (1/2)I$ at $t = 0$. Our aim is to specify the control voltage input $u(t)$ so as to reduce the effect of the thermal noise on the oscillator. As before, our target covariance at the end of a pre-specified interval [0,1] is set to to a terminal value; here this is $\Sigma_1 = (1/16)I$. Fig. 7 shows the evolution of (i_L, v_C) as a function of time under the effect of the least energy regulating input voltage $u(t)$ that aims to actively "cool" the resonator to its target final distribution. As before, Fig. 8 displays the corresponding control inputs. Once again, in Fig. 7, the transparent blue "tube" represents the "3σ" tolerance interval.

VI. Conclusion

The problem to steer linear stochastic systems from a starting probability Gaussian density to a target one with minimum effort has an explicit solution in feedback form. The minimum-energy control is computed by solving a pair of Lyapunov equations which are coupled through their boundary values at the two end-points of the interval. The optimal stochastic process turns out to coincide with a solution to a seemingly different problem, that of seeking the most likely random evolution that connects the two marginals given a prior law in the form of the uncontrolled diffusion. Both of these properties, the minimum energy and minimum relative entropy distance to the prior, generalize corresponding properties of classical Schrödinger bridges for nondegenerate diffusions.

Optimal steering of a stochastic system to a final distribution and, in particular, the explicit form of solution in the present setting, appears quite attractive for applications in quality control, process control, manufacturing, vehicle control, etc. It can also be effectively applied to active damping of nano- and macro-mechanical systems. In a follow-up work [37], we studied cooling of nonlinear oscillators while in [38] we considered the steering of a cloud of particles diffusing anisotropically with losses. Finally, we note that our control problem of "probability density transfer" resembles that of resource allocation and optimal mass transport [39]–[41]. Thereby, it has the potential to provide well-conditioned numerical schemes for optimal transport problems, a subject that is taken up in [42].

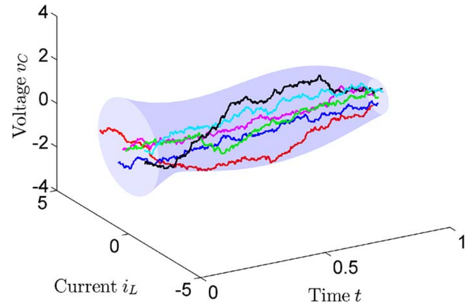

Fig. 7. Nyquist–Johnson noise: trajectories for $\Sigma_1 = (1/16)I$.

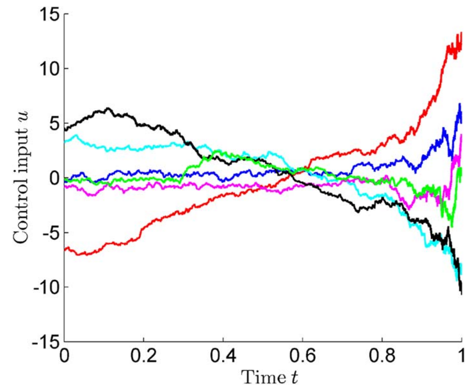

Fig. 8. Nyquist–Johnson noise: controls for $\Sigma_1 = (1/16)I$.

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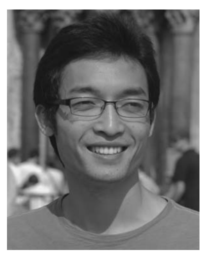
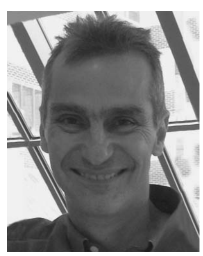


Yongxin Chen (S’13) received the B.Sc. degree in mechanical engineering from Shanghai Jiao Tong University, Shanghai, China, in 2011 and is currently pursuing the Ph.D. degree in Mechanical Engineering at the University of Minnesota.

He is interested in the application of mathematics in engineering and theoretical physics. His current research focuses on linear dynamical systems, stochastic processes and optimal mass transport theory.



Tryphon T. Georgiou (F’00) received the Diploma in Mechanical and Electrical Engineering from the National Technical University of Athens, Greece, in 1979 and the Ph.D. degree from the University of Florida, Gainesville, in 1983.

He is a faculty in the Department of Electrical and Computer Engineering, University of Minnesota and the Vincentine Hermes-Luh Chair.

Dr. Georgiou received the George S. Axelby Outstanding Paper Award from the IEEE Control Systems Society, in 1992, 1999, and 2003. He is a Foreign Member of the Royal Swedish Academy of Engineering Sciences (IVA).

Michele Pavon was born in Venice, Italy, on October 12, 1950. He received the Laurea degree from the University of Padova, Padova, Italy, in 1974, and the Ph.D. degree from the University of Kentucky, Lexington, in 1979, both in mathematics.

After service in the Italian Army, he was on the research staff of LADSEB-CNR, Padua, Italy, for six years. Since July 1986, he has been a Professor at the School of Engineering, University of Padova. He has visited several institutions in Europe, Northern America and Asia. His present research interests include maximum entropy and optimal transport problems.