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Remarks on Control Design With Degree Constraint

Tryphon T. Georgiou and Anders Lindquist

***Abstract*—The purpose of this note is to highlight similarities and differences between two alternative methodologies for feedback control design under constraints on the McMillan degree of the feedback system. Both sets of techniques focus on uniformly optimal designs. The first is based on the work of Gahinet and Apkarian and that of Skelton *et al.*, while the other is based on earlier joint work of the authors with C. I. Byrnes.**

***Index Terms*—H^∞ control, linear matrix inequalities (LMIs), McMillan degree constraint, weighted entropy.**

I. Introduction

The McMillan degree of a feedback system is often of key importance in analysis, design, and implementation. At the analysis and design stages, numerics are adversely affected by large degrees. Similarly, for a feedback controller, a large dimension could lead to computational delays as well as to problems in implementation and robustness. Finally, a high order dynamical response may be undesirable to a human who is called to operate a piece of machinery. Thus, it is of interest to explore effective control design techniques that are capable of incorporating degree constraints.

Gahinet and Apkarian [11], and Skelton *et al.* [19] have introduced a technique for feedback design that allows such a constraint on the degree of the controller. Their approach relies on expressing performance and robustness by linear matrix inequalities. These inequalities involve matrices of parameters that specify a class of controllers, while an added rank condition bounds their dimension. On the other hand, historically, H_∞-control theory [9] casts feedback design as an analytic interpolation problem. Based on this formulation, our recent work together with C. I. Byrnes on analytic interpolation with degree constraint (see, e.g., [4] and [5]) is especially relevant and provides an alternative handle on McMillan degrees in feedback design. The authors have been repeatedly asked to explain possible contact points between these two alternative formalisms and sets of techniques. The purpose of the present note is precisely to address this issue and explain similarities and differences between the two approaches. We have chosen to contrast the two by working out explicitly the paradigm of sensitivity minimization in a single-input–single-output setting.

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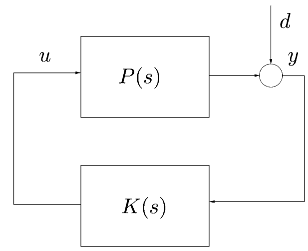

Fig. 1. Feedback loop.

Consider the feedback interconnection in Fig. 1 and let d represent an external disturbance whose effect on the output is to be minimized. When the dynamical system is linear, this can be formulated as a standard H_∞-minimization problem. The controller is chosen to ensure internal stability and minimize the gain of the sensitivity function

$$S = \frac{1}{1 - PK} \tag{1}$$

over selected frequency bands. Throughout, P, K represent the transfer functions of plant and controller and $\deg(P), \deg(K)$ represent their respective McMillan degrees. In the standard H_∞-control formalism the performance is encapsulated in a weighting function $W(s)$ and the design specifications cast in the form of ensuring a bound on the weighted norm

$$\|WS\|_\infty < \gamma \tag{2}$$

subject to internal stability.

Typically, $\deg(K), \deg(S)$ depend on $\deg(P)$ and $\deg(W)$ (and, in fact, they depend on the sum of these two McMillan degrees). Aside from the resulting "inflation" of the degree for the controller, the choice of the weight $W(s)$ is a delicate task since it is not at all transparent how it affects feasibility of the performance specification (2). Indeed, small changes in the desired bandwidth of the system and the desired "shape" of $S(s)$ (dictated by our choice of $W(s)$) may render the performance specification unattainable. Although the task of choosing weights in H_∞-design is somewhat intuitive and more accepted than that of choosing design parameters in, say, linear quadratic problems, it is far from straightforward and often a challenging task [10], [23].

Starting from a state–space formalism to H_∞-control problems [8] and via a clever use of the bounded real lemma, Gahinet and Apkarian [11] (see also [19]) expressed the conditions for the existence of a controller that guarantees performance and has a given McMillan degree,

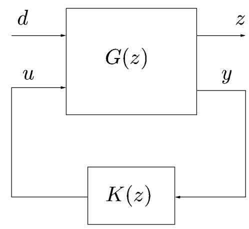

Fig. 2. Standard feedback interconnection.

in the form of a linear matrix inequality (LMI) with a rank constraint. Typically, weighting functions are incorporated into the plant description ("inflating" the degree of the "new" plant accordingly). With that in place, a case which is particularly appealing is when, in our context, $\deg(K) = \deg(P)$. Then the approach in [11], [19] leads to a set of ordinary LMIs. Requiring further degree reduction is a highly nontrivial problem in general.

An alternative viewpoint is to consider the totality of sensitivity functions of a given degree that meet a possibly conservative bound, and then select a particular one within this class. This hinges upon an effective parametrization of sensitivity functions of a given degree. Such a parametrization is in place for the precise class of sensitivity functions that do not exceed in dimension the sum of unstable plant poles and non-minimum phase plant zeros [4]. Each such solution is the unique minimizer of a suitably weighted entropy functional (10). The central object of interest is therefore the sensitivity function and its dimension (or, more generally, the transfer function of any closed loop mapping). Provided the plant is strictly proper and we select the sensitivity function within this class, it also holds that $\deg(K) < \deg(P)$ (see e.g., [17]). To determine such controllers by the approach in [11] and [19], one needs to impose a rank condition, thus destroying the LMI structure of the problem.

Next, in turn, we compare the two methodologies and illustrate how they apply on a first-order single-input–single-output example. This example is sufficiently simple to allow for analytic expressions that clearly display the similarities and differences between the two, without blurring the picture.

II. LMI-Based Design

We begin by explaining the pertinent formalism and key findings in [11], [19]. Assume the standard setting of a dynamical system G with two sets of inputs and outputs d, u and z, y, respectively, as in Fig. 2, and transfer function

$$G(z) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (zI - A)^{-1} (B_1 \quad B_2)$$

with (A, B_2, C_2) stabilizable and detectable, and $D_{22} = 0$.The search for dynamic controllers

$$K(z) = D_K + C_K (zI - A_K)^{-1} B_K$$

having input y, output u, dimension $\deg(K)$, and ensuring an H_∞-gain from d to y less than γ, proceeds as follows. Determine a pair of symmetric matrices X, Y of dimensions $\deg(P) \times \deg(P)$ satisfying

$$\begin{pmatrix} X & \gamma I \\ \gamma I & Y \end{pmatrix} \geq 0 \tag{3}$$

and

$$\begin{pmatrix} B_2 \\ D_{12} \end{pmatrix}^{\perp} \left[\begin{pmatrix} X & 0 \\ 0 & \gamma^2 I \end{pmatrix} - \begin{pmatrix} A & B_1 \\ C_1 & D_{11} \end{pmatrix} \right. \\ \left. \times \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B_1 \\ C_1 & D_{11} \end{pmatrix}' \right] \begin{pmatrix} B_2 \\ D_{12} \end{pmatrix}^{\perp'} > 0 \tag{4}$$

$$\begin{pmatrix} C_2' \\ D_{21}' \end{pmatrix}^{\perp} \left[\begin{pmatrix} Y & 0 \\ 0 & \gamma^2 I \end{pmatrix} - \begin{pmatrix} A & B_1 \\ C_1 & D_{11} \end{pmatrix}' \right. \\ \left. \times \begin{pmatrix} Y & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B_1 \\ C_1 & D_{11} \end{pmatrix} \right] \begin{pmatrix} C_2' \\ D_{21}' \end{pmatrix}^{\perp'} > 0 \tag{5}$$

where $M^{\perp}$ denotes any matrix whose rows form a basis of the left null space of a matrix M. The previous conditions are linear matrix inequalities and can be easily solved by standard methods.

For any such solution (X, Y), we have $X > 0, Y > 0$ and, hence

$$\text{rank} \begin{pmatrix} X & \gamma I \\ \gamma I & Y \end{pmatrix} = \text{rank}(Y - \gamma^2 X^{-1}) + \deg(P). \tag{6}$$

Now, compute a factorization

$$N M^{-1} N' = Y - \gamma^2 X^{-1}$$

with M a $k \times k$ invertible matrix, and form the positive–definite matrices

$$\hat{Y} := \begin{pmatrix} Y & N \\ N' & M \end{pmatrix}$$
$$\hat{X} = \gamma^2 \hat{Y}^{-1}.$$

For each such $\hat{X}$ there is a ball of controllers defined by

$$\begin{pmatrix} D_K & C_K \\ B_K & A_K \end{pmatrix} = K_0 + R_{\text{left}}^{1/2} L R_{\text{right}}^{1/2}$$

with L any matrix having norm $\|L\| < 1$. The center K_0 and the radii R_{left} and R_{right} can be computed as in ([19, p. 174]) and the dimension of the controller is $\deg(K) = k$. Clearly

$$\deg(K) \geq \text{rank}(Y - \gamma^2 X^{-1}) \tag{7}$$

i.e., generically, $\deg(K) \geq \deg(P)$. If one desires a controller of lower dimension, one needs to choose (X, Y) so that $Y - \gamma^2 X^{-1}$ has lower rank, which destroys the LMI structure of the solution set.

The class of all controllers of dimension $\deg(K)$ is the union of all (possibly overlapping) controller balls obtained by varying X, Y over the solution set of the earlier LMIs together with the rank constraint.

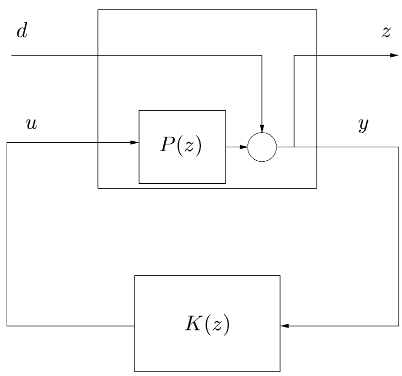

Fig. 3. Feedback loop of Fig. 1 in the standard form.

We illustrate this with a simple example, to which we return in the next section using a different approach.

Consider once again the sensitivity minimization problem with

$$P(z) = \frac{1}{z-2}$$

and the feedback loop redrawn in Fig. 3 in the standard form. Then, the parameters of the nominal system $G(z)$ are

$$\left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{22} \\ C_2 & D_{21} & D_{22} \end{array}\right) = \left(\begin{array}{c|cc} 2 & 0 & 1 \\ \hline 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right).$$

The linear matrix inequalities (4)–(5) become

$$\begin{aligned} &(\,0 \quad 1\,)\left[\begin{pmatrix} x & 0 \\ 0 & \gamma^2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \right.\\ &\quad \left. \times \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} 0 \\ 1 \end{pmatrix} > 0 \\ &(\,1 \quad -1\,)\left[\begin{pmatrix} y & 0 \\ 0 & \gamma^2 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \right.\\ &\quad \left. \times \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\right]\begin{pmatrix} 1 \\ -1 \end{pmatrix} > 0 \end{aligned}$$

yielding, together with the positivity condition,

$$0 < x < \gamma^2 - 1 \quad \text{and} \quad 0 < y < \frac{\gamma^2}{3}.$$

Inequality (3) now implies that $\gamma > 2$. Let us choose

$$\gamma = \frac{5}{2}.$$

At this point x, y are independent of each other, and in general (7) will require a controller of degree $\deg(K) = 1$. If we want controllers of degree $\deg(K) = 0$, i.e., constant gain, we need to take the lower bound in (7) to be zero, i.e.,

$$xy = \gamma^2 = \frac{25}{4}.$$

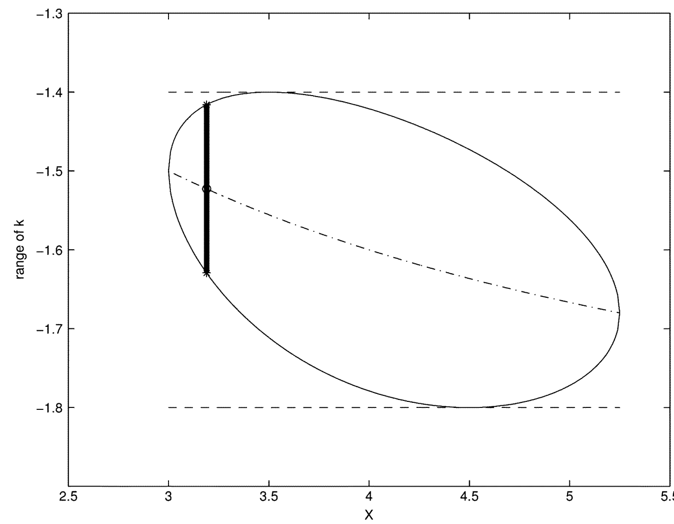

Fig. 4. Range of admissible values for the pair (x, K).

This gives

$$3 < x < \gamma^2 - 1 = \frac{21}{4}.$$

For each value of x in this range, we compute

$$\begin{aligned} K_0 &= -\frac{2x}{1+x} \\ R_{\text{left}} &= \frac{x^2 - 3x}{x+1} \\ R_{\text{right}} &= \frac{\gamma^2 - (x+1)}{x+1}. \end{aligned}$$

Hence, the range of controller gains for a given x is

$$K_0 - R < K < K_0 + R$$

where

$$\begin{aligned} R :&= \frac{\sqrt{x(x-3)(\gamma^2 - (x+1))}}{\gamma(x+1)} \\ &= \frac{\frac{2}{5}\sqrt{x(x-3)(\frac{21}{4} - x)}}{x+1}. \end{aligned}$$

Fig. 4 displays the range of controller gains over each admissible value of x. The dashed–dotted curve represents K_0 and the solid curves represent $K_0 \pm R$ as functions of x. A typical interval of admissible controller gains for $x = 3.2$ is highlighted with a thick vertical line (the center indicated with a circle and the end points with an asterisk). Then, the union of all control-gain intervals over the admissible range of x is

$$-1.8 < K < -1.4. \tag{8}$$

This range is indicated by dashed lines accross the values of x in Fig. 4. Figs. 5 and 6, display the range of sensitivity functions (by showing their respective Bode plots) that can be obtained by choosing K in the admissible ranges corresponding to values $x = 3.2$ and $x = 5.2$, respectively, for comparison.

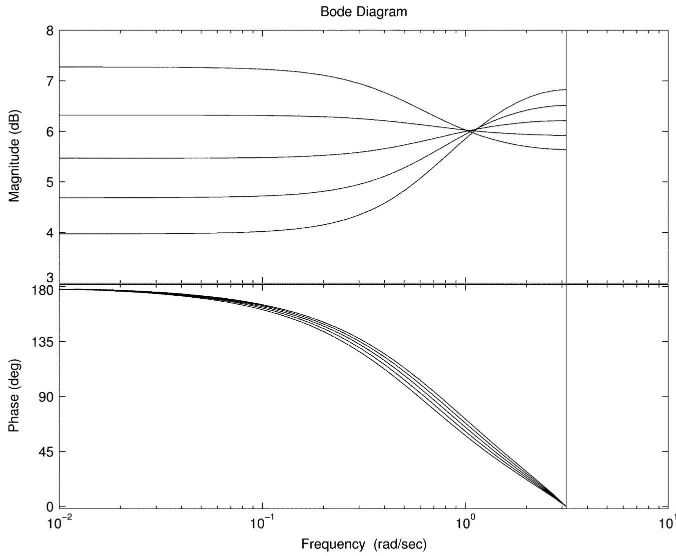

Fig. 5. Range of sensitivity shapes for $x = 3.2$.

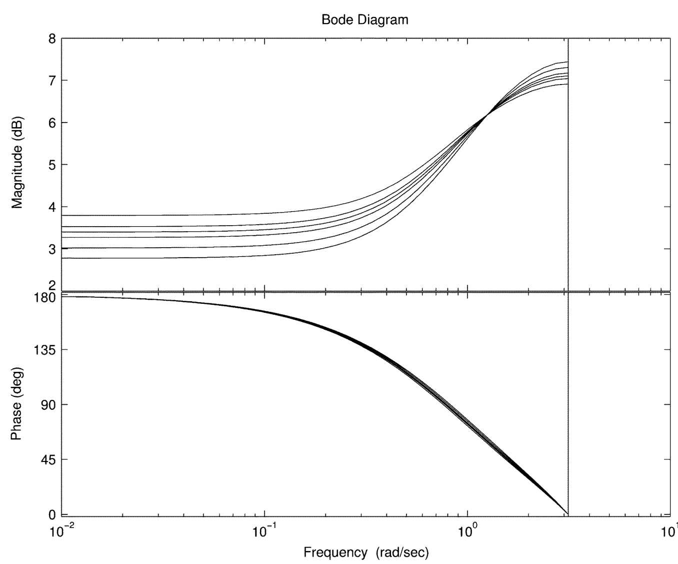

Fig. 6. Range of sensitivity shapes for $x = 5.2$.

III. Entropy-Based Design

We continue by explaining a formalism based on analytic interpolation with degree constraint developed over the last decade by the authors together with several co-workers; see, e.g., [4], [5], and the references therein.

It is well known that H_∞-control problems in the most general form discussed in Section II, can be cast as analytic interpolation problems [18], [1]. In fact, using the Youla–Kucera parametrization of stabilizing controllers for (A, B_2, C_2) (see [9]), the standard control problem in Fig. 2 can be brought into the form of a so-called four-block interpolation problem of selecting Q so as to minimize or bound

$$\|T_1 - T_2 Q T_3\|_\infty$$

where $T_i, i = 1, 2, 3$, and Q are H_∞-matrix functions of compatible dimension, T_i, $i = 1, 2, 3$, obtained from the problem data, while Q specifies the controller. The class of possible functions $T = T_1 - T_2 Q T_3$ is constrained at the singularities of the T_k's, $k = 2, 3$, where the value of T is independent of Q and agrees with T_1, giving rise to "tangential interpolation" constraints. In turn, analytic interpolation constraints of a most general nature can be cast as moment constraints [15], [1], and the theory of analytic interpolation with degree constraint can be extended to cover the generalized moment problem [7]. While most of the theory can be carried out in considerable generality (incorporating degree constraints, e.g., see [3], and also [13], and [14]), we restrict our attention to the scalar sensitivity shaping problem.

For disturbance attenuation one needs to prescribe the gain of the sensitivity function of the feedback system over different frequency bands according to specification. This was formulated by Zames [22] as an optimization problem. A stable and stably invertible transfer function W_1 is selected, and the controller K is chosen to minimize the H^∞ norm of the "weighted sensitivity" $W_1 S$. This choice of controller, namely

$$K_{\rm opt} := \operatorname{arginf}\{\|W_1 S\|_\infty \, : \, K \text{ stabilizing}\}$$

yields a lower bound

$$\gamma_{\rm opt} = \inf_{K \text{stabilizing}} \|W_1 S\|_\infty$$

on the norm of the weighted sensitivity, where S is given by (1). The optimal weighted sensitivity, whenever it is attained (e.g., see, [22] and [9]), turns out to be an all-pass function. In fact, because internal stability can be expressed in the form of the interpolation conditions

$$S(z_i) = 1, \qquad i = 1, \ldots, \nu \quad \text{and} \quad S(p_j) = 0, \qquad j = 1, \ldots, \mu \tag{9}$$

where $z_1, \ldots, z_\nu$ are the nonminimum phase zeros of the plant $P(z)$ and $p_1, \ldots, p_\mu$ are the unstable poles [20], it can be shown that

$$W_1 S_{\rm opt} = \gamma_{\rm opt} \frac{\beta}{\alpha}$$

where α is a Schur polynomial of degree $n = \nu + \mu - 1$ and $\beta(z) = z^n \alpha(z^{-1})$.

Such optimization problems are often very sensitive to the problem data [21] and, therefore, one could focus instead on stabilizing controllers in the bigger class

$$\mathcal{K}_\gamma := \{K \text{stabilizing} \, : \, \|W_1 S\|_\infty < \gamma\}$$

with $\gamma > \gamma_{\rm opt}$. The so-called "central solution" in this class of suboptimal controllers

$$K_{\rm ME} = \operatorname*{argmax}_{K \in \mathcal{K}_\gamma} \int_{-\pi}^{\pi} \log(\gamma^2 - \|W_1(e^{i\theta}) S(e^{i\theta})\|^2) d\theta$$

is easily computed in a state–space formalism [16]. This is known as the maximum entropy solution, and the corresponding weighed sensitivity function again takes the form

$$W_1 S_{\rm ME} = \gamma \frac{\beta}{\alpha}$$

where α is a Schur polynomial of degree n and β is a polynomial of degree at most n.

It turns out that there is an efficient characterization of *all* admissible sensitivity functions of degree not exceeding n, which of course includes the maximum entropy one, in terms of a weighted entropy functional

$$\int_{-\pi}^{\pi} W_2(e^{i\theta}) \log(\gamma^2 - |W_1(e^{i\theta})S(e^{i\theta})|^2) d\theta \tag{10}$$

with

$$W_2 = \left|\frac{\rho}{\tau}\right|^2 \tag{11}$$

where

$$\tau(z) = \prod_{j=1}^{\nu} \left(z - \bar{z}_j^{-1}\right) \prod_{j=1}^{\mu} \left(z - \bar{p}_j^{-1}\right)$$

and ρ ranges over the class $\mathcal{S}_n$ of all monic Schur polynomials of degree at most n [5]. For each $\rho \in \mathcal{S}_n$, the corresponding optimization problem has a unique solution

$$K_\rho = \underset{K \in \mathcal{K}_\gamma}{\operatorname{argmax}} \int_{-\pi}^{\pi} W_2(e^{i\theta}) \log(\gamma^2 - |W_1(e^{i\theta})S(e^{i\theta})|^2) d\theta$$

and the weighed sensitivity function again takes the form

$$W_1 S_\rho = \gamma \frac{\beta}{\alpha}$$

where $\alpha \in \mathcal{S}_n$ and $\deg \beta$ is also bounded by n.

These polynomials can be computed via convex optimization [4], [5]. In fact, the map

$$\varphi : \mathcal{S}_n \to \mathcal{S}_n : \rho \mapsto \alpha$$

is a homeomorphism onto its image $\mathcal{A} := \varphi(\mathcal{S}_n)$ [6], and β can be computed from the interpolation conditions once α is determined. Furthermore

$$|\alpha|^2 - |\beta|^2 = \lambda |\rho|^2 \tag{12}$$

on the unit circle, for some $\lambda > 0$. The correspondence φ provides a complete and smooth parametrization of all solutions in terms of $\rho \in \mathcal{S}_n$. The roots of ρ can be given the interpretation as being either transmission zeros of $\frac{1}{\gamma} S_\rho$ thought as the scattering function of a passive circuit or, as being spectral zeros of an associated spectral density [4].

The weighted entropy functional in (10) suggests that the polynomial ρ in (11) can be thought of as a "tuning parameter" in controller design. For example, ρ can be chosen to yield large values for $|W_2|$ in a frequency range where low sensitivity is desired. This added degree of freedom does not increase the degree of $W_1 S_\rho$. This may often permit a choice of W_1 of low degree, or simply the choice $W_1 \equiv 1$. The McMillan degree of the controller K_ρ is bounded by

$$\deg K_\rho \leq \deg P + \deg W_1 - 1$$

provided the plant P is strictly proper.

We illustrate this approach on the elementary example discussed in Section II, where

$$P(z) = \frac{1}{z-2}.$$

Of course, all computations can be done by hand and the full power of convex optimization is not needed. Nevertheless, this example highlights the differences with the approach of Section II.

Since P has a nonminimum-phase zero at $z = \infty$ and an unstable pole at $z = 2$, the interpolation conditions (9) are $S(\infty) = 1$ and $S(z) = 0$. Consequently, the sentitivity function must take the form

$$S(z) = \frac{z-2}{z-a}$$

where we must have $-1 < a < 1$ for S to be analytic in $|z| \geq 1$. It is easy to see that

$$\gamma_{\text{opt}} = \inf_{-1<a<-1} |S|_\infty = 2.$$

We select $\gamma = 5/2$ as before, and write

$$S_\rho(z) = \gamma \frac{\beta(z)}{\alpha(z)} = \gamma \frac{\frac{2}{5}z - \frac{4}{5}}{z-a}.$$

As $\rho(z)$ ranges over

$$\mathcal{S}_1 = \{z - r : -1 \leq r \leq 1\}$$

the polynomial $\alpha(z)$ ranges over

$$\mathcal{A} = \left\{z - a : \frac{1}{5} \leq a \leq \frac{3}{5}\right\}.$$

This can be readily verified from (12) without the need to solve the optimizaton problem. Indeed, substituting α, β and ρ in (12) and eliminating λ, we obtain that

$$\frac{r}{1+r^2} a^2 - a + \left(\frac{1}{5}\frac{r}{1+r^2} + \frac{8}{25}\right) = 0.$$

The value a as a function of r is plotted in Fig. 7 and represents a smooth and complete parametrization of all sensitivity functions of degree 1 (and controllers of degree 0). In general, (12) represent quadratic equations which are difficult to solve directly. In view of (1),

$$K = (1 - S^{-1})P^{-1} = a - 2$$

is of degree zero. Consequently,the range of constant gains that satisfy $\|S_\rho\|_\infty < \gamma$ lie in the interval

$$-1.8 \leq K \leq -1.4 \tag{13}$$

in bijective correspondence with elements in $\rho \in \mathcal{S}_1$. (This can also be verified directly, e.g., by computing $\|(z-2)/(z-2-K)\|_\infty$ over the range of K). Comparing (8) and (13), we see that the current approach

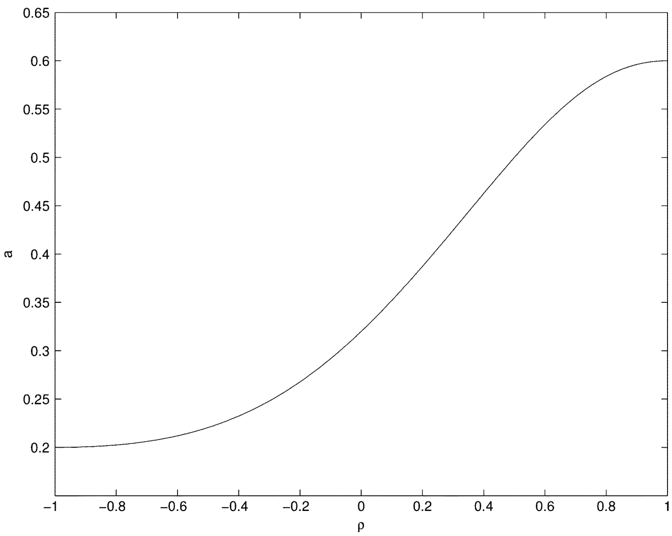

Fig. 7. a as a function of r.

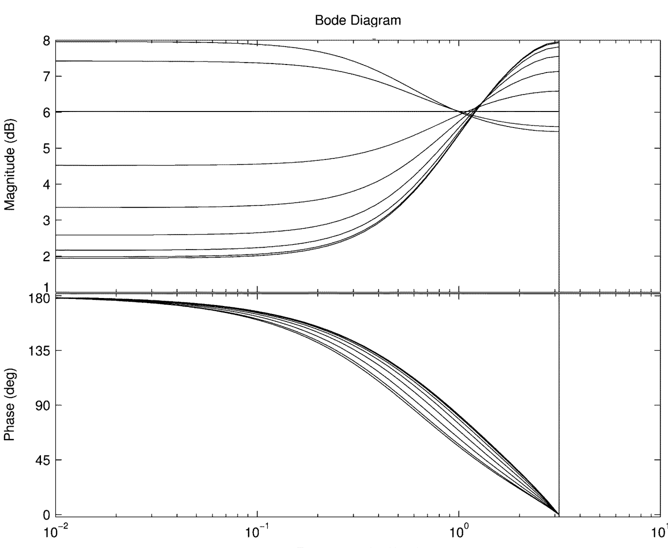

Fig. 8. Complete range of sensitivity shapes under the degree constraint.

yields the closed interval of admissible controllers. Fig. 8 shows Bode plots of S_ρ for $\rho = z - r$ and $r \in [-1, 1]$ at intervals of 0.25 apart.

A comparison of Figs. 5, 6, and 8 reveals a fundamental difference between the approach in Section II and that of Section III. In Section III we parameterize *all* solutions of degree one in terms of the tuning parameter r, and consequently the whole range of possible sensitivity functions are depicted in Fig. 8. In the approach of Section II, the range of sensitivity functions will depend on the particular solution of the LMIs (4)–(5) and the rank condition (7). The choice $x = 5.2$ (Fig. 6) yields a very narrow subclass of possible sensitivity functions and controllers and the choice $x = 3.2$ (Fig. 5) a somewhat wider. However, as is clear from Fig. 4, no choice of x will yield the complete class of sensitivity functions depicted in Fig. 8 and the corresponding interval of controllers given in (13). In this respect, note the difference of scale in Figs. 5 and 8.

IV. Concluding Remarks

It is important to point out that, in Section III, the range of values for the controller and the closed-loop sensitivity function is provided at the outset via the parameterizing set $\mathcal{S}_n$ and the smooth mapping φ. By way of contrast, in Section II, each particular solution of the LMIs (4)–(5) yields only a subset of possible controllers. Moreover, for the sensitivity shaping problem in particular, the controller degree is larger than that obtained in Section III if one wants to avoid imposing the rank condition (7); i.e., insisting on solving only LMIs. The framework of Section III handles in the same way interior as well as boundary points and provides a complete parametrization. This is not the case for the LMI-based approach and in fact, the radii for the balls of controller gains shrink to zero as the solutions of the LMIs tend to boundary values.

Of course, essentially any solution obtained by the complete parameterization of Section III corresponds to a solution that can also be obtained using the techniques in Section II for a suitable choice of X. The methodology of Section II has been fully developed in [19] for multivariable problems as well. The basic framework in [4] and [5] extends to the matricial setting, e.g., see [3], [14]. Detailed studies of various robust control problems using this formalism have been carried out in [17], [2] and a general methodology for multivariable problems is currently under development. The question as to which approach is most suitable for tuning and selection of controllers remains to be established.

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A Theoretical Study of Different Leader Roles in Networks

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***Abstract*—We study synchronization conditions for distributed dynamic networks with different types of leaders. The role of a "power" leader specifying a desired global state trajectory through local interactions has long been recognized and modeled. This note introduces the complementary notion of a "knowledge" leader holding information on the target dynamics, which is propagated to the entire network through local adaptation mechanisms. Different types of leaders can coexist in the same network. For instance, in a network of locally connected oscillators, the power leader may set the global phase while the knowledge leader may set the global frequency and the global amplitude. Knowledge-based leader-followers networks have many analogs in biology, e.g., in evolutionary processes and disease propagation.**

***Index Terms*—Adaptation, group cooperation, leaders, networks, synchronization.**

I. Introduction

Recent results in the theoretical study of synchronization and group cooperation [10], [13], [22], [5], [24], [30], [32], [28], [33] have greatly helped understand distributed networks in the natural world and emulate them in artificial systems. In these networks, each element only gets local information from a set of neighbors but the whole system exhibits a collective behavior. Examples of such networked systems pervade nature at every scale, including for instance neural networks, pacemaker cells, flashing fireflies, chirping crickets, and the aggregate motions of bird flocks, fish schools, animal herds and bee swarms. For diffusion-coupled networks with arbitrary size and general structure, explicit conditions on the coupling strengths can be derived for synchronization to occur, based on network connectivity and uncoupled element dynamics [28], [33].

In a network composed of peers, the phase of the collective behavior is hard to predict, since it depends on the initial conditions of all the coupled elements. To let the whole network converge to a specific trajectory, a "leader" can be added [10], [13]. Here, the leader is an element whose dynamics is independent from and thus followed by all the others. Such leader–followers network occurs in natural aggregate motions, with the leader specifying "where to go." We will refer to this kind of leader as the *power leader*. A synchronization condition for a dynamic network with a power leader was derived in [28] and [33], and will be briefly reviewed here (Section II).

The main goal of this note is to introduce a different type of leader, which we shall refer to as a *knowledge leader* (Section III). In this case, the network members' dynamics can all be different. The knowledge leader is the one whose dynamics properties are fixed (or changes comparatively slowly), with the followers obtaining dynamic knowledge from the leader through local adaptation mechanisms. In this sense, a knowledge leader can be understood as the one who indicates "how to go." Such knowledge leaders may exist in many natural processes. For instance, in evolutionary biology [21], [23], the adaptive model we describe could represent genotype–phenotype mappings. Similar mechanisms occur in infectious-disease dynamics [16]. Knowledge leaders may also exist in animal aggregate motion as a junior or injured member with limited capacities. Using Lyapunov analysis, we shall derive conditions of synchronization and also for dynamics convergence for networks with knowledge leaders.

Both types of leaders may coexist (Section IV), and be located anywhere in the network. In a circuit of electronic oscillators, the power leader may be a local clock setting global phase while the knowledge leader sets global frequency or amplitude, for instance. Both types of leaders can be virtual (as, e.g., in [13] in the case of power leaders) and may be used for instance to coordinate behaviors in groups of robots of different types.

II. Power Leader

Consider the dynamics of a coupled network containing one power leader and n power followers

$$\begin{aligned}\dot{\mathbf{x}}_0 &= \mathbf{f}(\mathbf{x}_0, t)\\ \dot{\mathbf{x}}_i &= \mathbf{f}(\mathbf{x}_i, t) + \sum_{j\in\mathcal{N}_i}\mathbf{K}_{ji}(\mathbf{x}_j - \mathbf{x}_i)\\ &\quad + \gamma_i\mathbf{K}_{0i}(\mathbf{x}_0 - \mathbf{x}_i).\end{aligned}\tag{1}$$

Here, vector $\mathbf{x}_0 \in \mathbb{R}^m$ is the state of the leader whose dynamics is independent, and $\mathbf{x}_i$ the state of the ith follower, $i = 1,\ldots,n$. The vector function $\mathbf{f}$ represents the uncoupled dynamics, which is assumed to be identical for each element. For notational simplicity, the coupling forces are set to be diffusive, where all coupling gains are symmetric positive definite, and the couplings between the followers are bidirectional with $\mathbf{K}_{ji} = \mathbf{K}_{ij}$ if both $i, j \neq 0 \cdot \mathcal{N}_i$ denotes the set of peer-neighbors of element i, which for instance could be defined as the set of the followers within a certain distance around element $i \cdot \gamma_i$ is equal to either 0 or 1, representing the connection from the leader to the followers. In our model, the network connectivity can be very general. Thus, $\mathcal{N}_i$ and γ_i can be defined arbitrarily. An example is illustrated in Fig. 1(a).

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