Stochastic Bridges of Linear Systems

Yongxin Chen and Tryphon Georgiou

***Abstract*—We consider particles obeying Langevin dynamics while being at known positions and having known velocities at the two end-points of a given interval. Their motion in phase space can be modeled as an Ornstein–Uhlenbeck process conditioned at the two end-points—a generalization of the Brownian bridge. Using standard ideas from stochastic optimal control we construct a stochastic differential equation (SDE) that generates such a bridge that agrees with the statistics of the conditioned process, as a degenerate diffusion. Higher order linear diffusions are also considered. In general, a time-varying drift is sufficient to modify the prior SDE and meet the end-point conditions. When the drift is obtained by solving a suitable differential Lyapunov equation, the SDE models correctly the statistics of the bridge. These types of models are relevant in controlling and modeling distribution of particles and the interpolation of density functions.**

***Index Terms*—Schrödinger bridge, stochastic differential equation (SDE).**

I. Introduction

The theoretical foundations on how molecular dynamics affect large scale properties of ensembles were laid down more than a hundred years ago. A most prominent place among mathematical concepts has been occupied by the Brownian motion which provides a basis for studying diffusion and noise [1]–[4]. The Brownian motion is captured by the mathematical model of a Wiener process, herein denoted by $w(t)$ [5], [6]. Brownian motion represents the random movement of particles suspended in a fluid where their inertia is negligible compared to viscous forces [7]. On the other hand, as in the present note, if we want to take into account inertial effects under a "delta-correlated" stationary Gaussian force field $\eta(t)$ (i.e., white noise, loosely thought of as dw/dt [1, p. 46]), we are led to the "Langevin model"

$$m\frac{d^2x(t)}{dt^2} = -\lambda\frac{dx(t)}{dt} + \eta(t) \tag{1}$$

here x represents position, m mass, t time, and λ viscous friction parameter. The corresponding stochastic differential equations (SDE)

$$\begin{bmatrix} dx(t) \\ dv(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\lambda/m \end{bmatrix}\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} dw(t) \tag{2}$$

where w is a Wiener process and v the velocity, is a degenerate diffusion in that the stochastic term does not affect all degrees of freedom.

Starting with a model for diffusive transport, the problem to condition sample paths at the two ends of a time-interval has been considered as early as 1931, by Schrödinger [8]. A process is often referred to as a "bridge" (as it forms a bridge that links the two end-point conditions). A textbook example is the so-called Brownian bridge [6, p. 35], which has a well-known representation via the SDE (see [3], [5], [9])

$$dx(t) = -\frac{1}{1-t}x(t)dt + dw(t). \tag{3}$$

This represents trajectories of diffusive particles whose position is "pinned" at the end-points of an interval where, e.g., $x(0) = x(1) = 0$, while the drift term

$$u(t) = -\frac{1}{1-t}x(t) \tag{4}$$

can be viewed as a feedback control law added to the "prior evolution" $dx = dw$ so as to ensure the end-point condition. However, the particular form of feedback ensures that the process has the same statistics as the Brownian diffusion conditioned to meet the end-point constraints. In another model, the bridge is to satisfy specified end-point marginal distributions and it is then referred to as a *Schrödinger bridge* [10]. Early impetus for the study of bridges was provided by Schrödinger's insight for an alternative mechanism to explain quantum theory [11], [12]. The construction of a bridge between two given end-point marginal densities, for *non-degenerate* diffusions, turned out to be a stochastic control problem [13]. The study of important connections between bridges of non-degenerate diffusions, large deviations in sample-path spaces, thermodynamics, and stochastic optimal control ensued (see [14], [15] and the references therein).

Earlier literature on stochastic bridges focuses primarily on non-degenerate diffusions where the stochastic term affects *all* directions in the coordinate space. In this present note we have been motivated by questions regarding the transport of particles having *inertia*. Thus, in contrast, we consider bridges of diffusion processes as in (2) where the diffusive coefficient may have a rank that is less than the dimension of the coordinate space. In particular, we are interested in an *Ornstein–Uhlenbeck bridge* pinned at two end-points in *phase space*. Such a model is natural when considering transport of particles in regimes where viscous forces are negligible (e.g., in rarefied gas dynamics) as well as in modeling RLC networks with variables the capacitor charges and the inductor currents and resistors that introduce Johnson-Nyquist thermal noise. The latter play a central role in recent applications of feedback control of nano to meter-sized resonators [16], [17]. Models of bridges also appear in physical sciences, biology, genetics, see e.g., [18]–[20] and are relevant in the problem to interpolate density functions (in many-particle systems, power spectral distributions, etc., cf. [21]–[23]).

In the present note we introduce a model for an Ornstein–Uhlenbeck bridge as well as bridges of general linear time-varying dynamical systems. We show that an SDE representation is available, akin to (3), and that the respective drift term can be obtained by solving the stochastic optimal control problem to ensure end-point conditions (cf., [24]). For didactic purposes, we first explain the Brownian bridge in a way that will be echoed in the construction of the SDE for the Ornstein–Uhlenbeck bridge, followed by the construction of the SDE for bridges of general linear time-varying systems. We finally note that

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in recent work, subsequent to this present correspondence, the subject matter has been extended to Schrödinger bridges where the end-point conditions represent specified marginals [25], [26]—the salient issue in this is the ability to steer the density of a linear stochastic system between the given marginals under general assumption on the directionality of the diffusive and the drift (control) terms. One would think that the ability to steer a linear stochastic system between Dirac delta distributions, as shown in the present correspondence, suggests also the ability to steer between more general marginals, however, the technical details proved to be far from trivial and new insights were necessary [25], [26].

II. Brownian Bridge

The standard Brownian bridge is typically defined as a stochastic process ξ on [0, 1] with $\xi(0) = \xi(1) = 0$, having continuous sample paths, and taking values that are jointly normally distributed with

$$\mathbb{E}\{\xi(t)\xi(s)\} = t(1-s) \text{ for } 0 \le t \le s \le 1.$$

Alternatively, it is often defined as a stochastic process with the same statistics as $w(t) - tw(1)$ and continuous sample paths; this is the Brownian motion conditioned to satisfy the end-point constraints. Below we consider the statistics of the Brownian bridge and outline the construction of an SDE model for it.

A. Statistics of the Brownian Bridge

The Brownian bridge can be viewed as a standard Wiener process w on [0, 1] conditioned on $w(1) = 0$. For $t \le s$, as before, we have that the covariance at two points in time, s and t, is

$$\mathbb{E}\left\{\begin{bmatrix} w(t) \\ w(s) \\ w(1) \end{bmatrix} [w(t),\ w(s),\ w(1)]\right\} = \begin{bmatrix} t & t & t \\ t & s & s \\ t & s & 1 \end{bmatrix}. \tag{5}$$

Therefore, the distribution of $[w(t), w(s)]'$ conditioned on $w(1) = 0$ is normal with zero mean and covariance

$$\begin{bmatrix} t(1-t) & t(1-s) \\ t(1-s) & s(1-s) \end{bmatrix}.$$

The latter is simply the Schur complement of (5) pivoted about its (3,3) entry, which is the covariance of $w(1)$. The covariance of the conditioned process and joint normality of the values provide the law for the Brownian bridge which agrees with those of the aforementioned definitions.

B. Optimal Control and SDE Representation

We now consider the control problem to steer the Brownian motion to a specified terminal point at the end of a given time interval [0, 1]. To this end, we consider the linear-quadratic stochastic optimal control problem to minimize

$$\mathbb{E}\left\{F\xi(1)^2 + \int_0^1 u(\tau)^2 d\tau\right\} \tag{6}$$

with $F > 0$, subject to the dynamics

$$d\xi(t) = u(t)dt + dw(t), \quad \xi(0) = 0 \text{ a.s.}$$

The optimal control is

$$u_{\text{opt}}(t) = -p(t)\xi(t)$$

with $p(t)$ satisfying the Riccati equation $\dot{p}(t) = p^2(t)$ and the boundary condition $p(1) = F$. The value of the weight F impacts the "spread" of the density function at the final time about the origin. In the limit, as $F \to \infty$, we obtain the control strategy of (6)

$$u_{\text{opt}}(t) = -\frac{1}{1-t}\xi(t). \tag{7}$$

The corresponding "controlled" SDE

$$\begin{aligned} d\xi(t) &= u_{\text{opt}}(t)dt + dw(t) \\ &= -\frac{1}{1-t}\xi(t)dt + dw(t) \end{aligned} \tag{8}$$

with $\xi(0) = 0$ a.s., generates a Brownian bridge as can be easily verified [3, p. 132]. Indeed, the state transition of the deterministic time-varying system with input $r(t)$

$$\frac{d\xi}{dt} = -\frac{1}{1-t}\xi(t) + r(t)$$

which, for this first order system, coincides with the response at s to an impulse at t, is

$$\Phi(s,t) = \frac{1-s}{1-t}.$$

It follows that ξ has a representation as a stochastic integral:

$$\xi(t) = \int_0^t \frac{1-t}{1-\tau} dw(\tau)$$

for $t < 1$ and is a martingale. For $t \le s < 1$

$$\begin{aligned} \mathbb{E}\{\xi(t)\xi(s)\} &= \int_0^t \frac{(1-t)(1-s)}{(1-\tau)^2} d\tau \\ &= t(1-s) \end{aligned}$$

as well as $\mathbb{E}\{\xi(1)\} = 0$. By continuity, $\xi(1) = 0$ a.s. and therefore (8) is a Brownian bridge.

III. Ornstein–Uhlenbeck Bridge

We now follow exactly the same steps in order to define a bridge for the Ornstein–Uhlenbeck dynamics. Without loss of generality we assume that there are no viscous forces and the mass is normalized to one. Thus, we begin with the SDE

$$d\xi(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi(t)dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dw(t),\ \xi(0) = 0 \text{ a.s.} \tag{9a}$$

where

$$\xi(t) = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

is the vectorial process composed of the position and velocity components. We condition ξ to satisfy

$$\xi(1) = 0, \text{ a.s.} \tag{9b}$$

Any Gaussian process that shares the same statistics as ξ and has continuous sample paths will be referred to as an *Ornstein–Uhlenbeck bridge*. Below, we will establish in a manner that echoes the construction of the Brownian bridge that an SDE representation exists for such a process.

A. Statistics of the Ornstein–Uhlenbeck Bridge

To determine the statistics dictated by (9) we condition the "velocity" $v(t)$, which in this case is a Wiener process, since $dv(t) = dw(t)$, to satisfy

$$v(0) = 0 \tag{10a}$$

$$v(1) = 0 \tag{10b}$$

$$x(1) = \int_0^1 v(\tau) = 0 \tag{10c}$$

while it is given that $x(0) = 0$. To this end, we first consider the covariance of the vector

$$[v(t) \quad v(s) \quad v(1) \quad x(1)]'$$

readily seen to be

$$\begin{bmatrix} t & t & t & t - \frac{t^2}{2} \\ t & s & s & s - \frac{s^2}{2} \\ t & s & 1 & \frac{1}{2} \\ t - \frac{t^2}{2} & s - \frac{s^2}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}.$$

Therefore, the covariance of $[v(t) \;\, v(s)]'$ when conditioned on $[v(1) \;\, x(1)]'$ being the zero vector, can be evaluated as the Schur complement

$$\begin{bmatrix} t & t \\ t & s \end{bmatrix} - \begin{bmatrix} t & t - \frac{t^2}{2} \\ s & s - \frac{s^2}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}^{-1} \begin{bmatrix} t & s \\ t - \frac{t^2}{2} & s - \frac{s^2}{2} \end{bmatrix}.$$

This is

$$\begin{bmatrix} -t(3t^3 - 6t^2 + 4t - 1) & -t(s-1)(3st - 3s + 1) \\ -t(s-1)(3st - 3s + 1) & -s(3s^3 - 6s^2 + 4s - 1) \end{bmatrix}.$$

B. Optimal Control and SDE Representation

Just like in the case of the Brownian bridge, we now consider the linear-quadratic stochastic optimal control problem to minimize

$$\mathbb{E}\left\{ \xi(1)' F \xi(1) + \int_0^1 u(\tau)' u(\tau) d\tau \right\}$$

subject to

$$d\xi(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi(t) dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dw(t) \tag{11}$$

with initial condition $\xi(0) = 0$ a.s.. By solving the corresponding Riccati equation and taking the limit as $F \to \infty$, we obtain the control

$$u(t) = -\left[\frac{6}{(1-t)^2} \;\; \frac{4}{1-t} \right] \xi(t)$$

and consider the corresponding "controlled" SDE

$$d\xi(t) = \begin{bmatrix} 0 & 1 \\ -\frac{6}{(1-t)^2} & -\frac{4}{1-t} \end{bmatrix} \xi(t) dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dw(t). \tag{12}$$

We claim that (12) realizes the Ornstein–Uhlenbeck bridge. To establish this, we need to show that the statistics of solutions to (12) are consistent with those of the "pinned" process generated by (9) derived earlier. That is, for $\xi(t)' = [x(t), v(t)]$ it suffices to show that for solutions of (12)

$$\mathbb{E}\{v(t)v(t)\} = -t(3t^3 - 6t^2 + 4t - 1)$$

and

$$\mathbb{E}\{v(t)v(s)\} = -t(s-1)(3st - 3s + 1).$$

Since $x(t)$ is $\int_0^t v(\tau) d\tau$ in both cases, the statistics of $x(t)$ will also be consistent. The proof is given in Section IV for the more general case of time-varying linear dynamics.

IV. The Bridge for a Time-Varying Linear System

We consider the linear SDE

$$d\xi(t) = A(t)\xi(t) dt + B(t) dw(t) \tag{13a}$$

with initial condition

$$\xi(0) = 0 \text{ a.s.} \tag{13b}$$

and are interested in solutions that are conditioned to satisfy

$$\xi(1) = 0 \text{ a.s.} \tag{13c}$$

as well. Below, we first determine the statistics of the pinned process and then an SDE that generates the bridge.

A. Statistics of the Bridge

Since (13a) is a linear SDE driven by a Wiener process and $\xi(0) = 0$, it follows that $\xi(t)$ is a zero-mean Gaussian process. Thus, we only need to determine second order statistics of the conditioned process. The covariance of

$$[\xi(t)' \; \xi(s)' \; \xi(1)']'$$

is

$$\begin{bmatrix} P(t) & P(t)\Phi(s,t)' & P(t)\Phi(1,t)' \\ \Phi(s,t)P(t) & P(s) & P(s)\Phi(1,s)' \\ \Phi(1,t)P(t) & \Phi(1,s)P(s) & P(1) \end{bmatrix}' \tag{14}$$

where $\Phi(s,t)$ is the state transition of (13a) and

$$P(t) = \mathbb{E}\{\xi(t)\xi(t)'\}$$

satisfies the Lyapunov equation

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)' + B(t)B(t)'. \tag{15}$$

Since $\xi(0) = 0$ is given, $P(0) = 0$. Taking the Schur complement of (14) gives the covariance of $[\xi(t)' \; \xi(s)']'$ conditioned on $\xi(1) = 0$ as

$$\begin{bmatrix} Q(t,t) & Q(t,s) \\ Q(t,s)' & Q(s,s) \end{bmatrix}$$

where

$$Q(t,s) = P(t)\Phi(s,t)' - P(t)\Phi(1,t)'P(1)^{-1}\Phi(1,s)P(s). \tag{16}$$

Any stochastic process that agrees with these statistics will be referred to as a bridge of (13).

B. SDE Representation

Once again let us consider the linear-quadratic stochastic optimization problem to minimize

$$\mathbb{E}\left\{\xi(1)'F\xi(1)+\int_0^1 u(\tau)'u(\tau)d\tau\right\}$$

subject to the dynamics

$$d\xi(t)=A(t)\xi(t)dt+B(t)u(t)dt+B(t)dw(t),\ \xi(0)=0 \text{ a.s.}$$

The optimal solution is

$$u_{\rm opt}(t)=-B(t)'\hat{P}(t)^{-1}\xi(t)$$

where $\hat{P}(t)$ satisfies the differential Lyapunov equation

$$\dot{\hat{P}}(t)=A(t)\hat{P}(t)+\hat{P}(t)A(t)'-B(t)B(t)' \tag{17}$$

with boundary condition $\hat{P}(1)=F^{-1}$. We consider the limiting case of infinite terminal cost, i.e., $F\to\infty$, corresponding to $\hat{P}(1)=0$ and verify that the corresponding controlled stochastic system realizes a process with the sought statistics.

Proposition 1: Under the earlier notation and assumptions on $A, B, \hat{P}, w$, the SDE

$$d\xi(t)=\left(A(t)-B(t)B(t)'\hat{P}(t)^{-1}\right)\xi(t)dt+B(t)dw(t) \tag{18}$$

with $\xi(0)=0$ a.s. generates a bridge of (13).

Proof: We only need to consider second order statistics of solutions to (18) and establish that these coincide with the statistics computed in Section IV-A. Hence, for $0\le t\le s\le 1$ we denote $\hat{Q}(t,s)=\mathbb{E}\{\xi(t)\xi(s)'\}$ to be the covariance of solutions to (18) and we will show that $\hat{Q}(t,s)=Q(t,s)$. For simplicity we denote $\hat{Q}(t,t)=\hat{Q}(t)$ and the same for Q.

We first begin with

$$Q(t)=P(t)-P(t)\Phi(1,t)'P(1)^{-1}\Phi(1,t)P(t) \tag{19}$$

and show that it also satisfies the differential Lyapunov equation

$$\dot{Q}(t)=\hat{A}(t)Q(t)+Q(t)\hat{A}(t)'+B(t)B(t)' \tag{20}$$

for

$$\hat{A}(t)=\left(A(t)-B(t)B(t)'\hat{P}(t)^{-1}\right)$$

and, since $Q(0)=0$, that indeed $Q(t)=\hat{Q}(t)$. To this end, consider $Q(t)$ as in (19). Then

$$\begin{aligned}\dot{Q}(t)-\hat{A}(t)Q(t)-Q(t)\hat{A}(t)'-B(t)B(t)'\\ =B(t)B(t)'G(t)+G(t)'B(t)B(t)'\end{aligned}$$

where

$$\begin{aligned}G(t)&=\hat{P}(t)^{-1}Q(t)-\Phi(1,t)'P(1)^{-1}\Phi(1,t)P(t)\\ &=\hat{P}(t)^{-1}P(t)-\hat{P}(t)^{-1}T(t)\Phi(1,t)'P(1)^{-1}\Phi(1,t)P(t)\end{aligned}$$

and

$$T(t)=P(t)+\hat{P}(t).$$

From (15) and (17)

$$\dot{T}(t)=A(t)T(t)+T(t)A(t)'$$

and therefore

$$T(t)=\Phi(t,0)T(0)\Phi(t,0)'$$

while $T(0)=\hat{P}(0)$ and $T(1)=P(1)$. Since

$$\begin{aligned}&T(t)\Phi(1,t)'P(1)^{-1}\Phi(1,t)\\ &=\Phi(t,0)T(0)\Phi(t,0)'\Phi(1,t)'P(1)^{-1}\Phi(1,t)\\ &=\Phi(t,1)\Phi(1,0)T(0)\Phi(1,0)'P(1)^{-1}\Phi(1,t)\\ &=\Phi(t,1)T(1)P(1)^{-1}\Phi(1,t)=I\end{aligned}$$

the identity matrix, we deduce that

$$G(t)=\hat{P}(t)^{-1}P(t)-\hat{P}(t)^{-1}IP(t)=0.$$

Therefore (20) holds and $Q(t)=\hat{Q}(t)$.

For general $0\le t\le s\le 1$

$$\hat{Q}(t,s)=\hat{Q}(t,t)\hat{\Phi}(s,t)'$$

where

$$\frac{\partial\hat{\Phi}(s,t)}{\partial s}=\hat{A}(s)\hat{\Phi}(s,t).$$

Therefore

$$\frac{\partial\hat{Q}(t,s)}{\partial s}=\hat{Q}(t,s)\hat{A}(s)'.$$

We now show that $Q(t,s)$ satisfies the same differential equation, i.e., that

$$\frac{\partial Q(t,s)}{\partial s}=Q(t,s)\hat{A}(s)'. \tag{21}$$

From (16) we deduce that

$$\frac{\partial Q(t,s)}{\partial s}-Q(t,s)\hat{A}(s)'=H(t,s)B(s)B(s)'$$

where

$$\begin{aligned}H(t,s)&=Q(t,s)\hat{P}(s)^{-1}-P(t)\Phi(1,t)'P(1)^{-1}\Phi(1,s)\\ &=P(t)\Phi(s,t)'\hat{P}(s)^{-1}-P(t)K(t,s)\hat{P}(s)^{-1}.\end{aligned}$$

But

$$\begin{aligned}K(t,s)&=\Phi(1,t)'P(1)^{-1}\Phi(1,s)T(s)\\ &=\Phi(1,t)'P(1)^{-1}\Phi(1,s)\Phi(s,0)T(0)\Phi(s,0)'\\ &=\Phi(1,t)'P(1)^{-1}T(1)\Phi(s,1)'=\Phi(s,t)'.\end{aligned}$$

Therefore $H(t,s)=0$ and (21) holds. Since we already know that $Q(t,t)=\hat{Q}(t,t)$, it follows that $Q(t,s)=\hat{Q}(t,s)$. This completes the proof. ■

V. Bridge With Arbitrary Boundary Points

So far we have discussed bridges with initial and terminal states being 0. The more general case with nonzero initial and terminal states is straightforward. More specifically, we consider the linear SDE

$$d\xi(t)=A(t)\xi(t)dt+B(t)dw(t) \tag{22a}$$

with initial condition

$$\xi(0)=\xi_0 \text{ a.s.} \tag{22b}$$

while the process $\xi(t)$ is conditioned to satisfy

$$\xi(1)=\xi_1 \text{ a.s.} \tag{22c}$$

Here ξ_0 and ξ_1 are fixed values. Next, we determine the statistics of the pinned process and then the SDE that generates the bridge.

A. Statistics of the Bridge

The second order statistics of (22) coincide with those of (13). Hence, we only need to compute first-order statistics. Considering only (22a) and (22b)

$$\mathbb{E}\{\xi(t)\} = \Phi(t,0)\xi_0.$$

Thus, the conditional expectation of $\xi(t)$, given $\xi(1)=\xi_1$, is

$$L(t) = \Phi(t,0)\xi_0 + P(t)\Phi(1,t)'P(1)^{-1}\left(\xi_1 - \Phi(1,0)\xi_0\right). \quad (23)$$

B. SDE Representation

In order to enforce the terminal constraint (22c), we now penalize the difference between $\xi(1)$ and ξ_1 and consider the linear-quadratic optimal control problem to minimize

$$\mathbb{E}\left\{(\xi(1)-\xi_1)'F(\xi(1)-\xi_1) + \int_0^1 u(\tau)'u(\tau)d\tau\right\}$$

subject to the dynamics

$$d\xi(t) = A(t)\xi(t)dt + B(t)u(t)dt + B(t)dw(t),\ \xi(0)=\xi_0 \text{ a.s.}$$

The optimal solution is

$$u_{\text{opt}}(t) = -B(t)'\hat{P}(t)^{-1}\left(\xi(t) - \Phi(t,1)\xi_1\right)$$

where $\hat{P}(t)$ satisfies the differential Lyapunov equation (17) with boundary condition $\hat{P}(1) = F^{-1}$. Once again the limit as $F \to \infty$ corresponds to $\hat{P}(1) = 0$. We now verify that the resulting "controlled" SDE realizes the sought bridge.

Proposition 2: Under the above assumptions on A, B, $\hat{P}$, and w, the SDE

$$d\xi(t) = \hat{A}(t)\xi(t)dt + B(t)B(t)'\hat{P}(t)^{-1}\Phi(t,1)\xi_1 dt + B(t)dw(t),\ \xi(0)=\xi_0 \text{ a.s.}$$

with

$$\hat{A}(t) = A(t) - B(t)B(t)'\hat{P}(t)^{-1}$$

generates a bridge of (22).

Proof: The second order statistics of (24) coincide with those of (18) and, by Proposition 1 with those of (13) and therefore (22) as well. Next we show that the first order statistics are also consistent. For this, it suffices to show that $L(t)$ in (23) satisfies

$$\dot{L}(t) = \hat{A}(t)L(t) + B(t)B(t)'\hat{P}(t)^{-1}\Phi(t,1)\xi_1.$$

Using the same argument as in the proof of Proposition 1 we obtain

$$\begin{aligned}\dot{L}(t) &- \hat{A}(t)L(t) - B(t)B(t)'\hat{P}(t)^{-1}\Phi(t,1)\xi_1 \\ &= B(t)B(t)'\hat{P}(t)^{-1} \\ &\quad \times\left(\Phi(t,1)\left(\xi_1 - \Phi(1,0)\xi_0\right) + \Phi(t,0)\xi_0 - \Phi(t,1)\xi_1\right) \\ &= 0.\end{aligned}$$

This completes the proof. ■

VI. Illustrative Examples

We consider a double integrator as in Section III with state $\xi(t) = [x(t)\ v(t)]'$, and plot two representative sample paths of (12). More specifically, Figs. 1 and 2 show position and velocity, respectively, while Fig. 3 displays the two paths in phase space. Phase plots of a 2-dimensional Brownian bridge are shown in Fig. 4 for comparison.

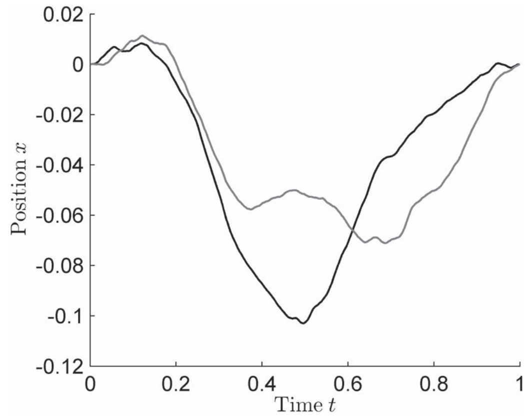

Fig. 1. Position $x(t)$ of Ornstein–Uhlenbeck bridge (for two representative sample paths in black and gray, respectively).

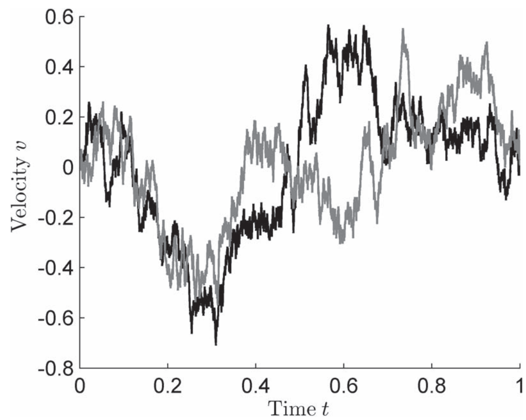

Fig. 2. Velocity $v(t)$ of Ornstein–Uhlenbeck bridge (for two representative sample paths in black and gray, respectively).

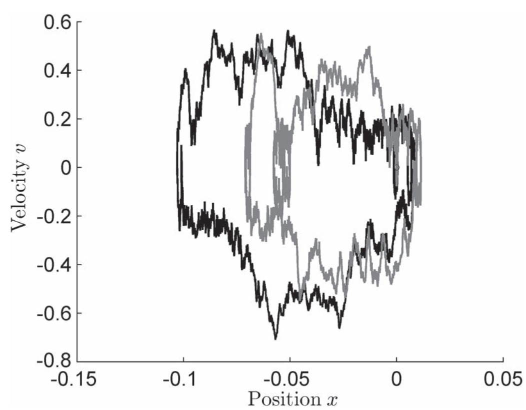

Fig. 3. Phase plots of Ornstein–Uhlenbeck bridge two sample paths with coordinates representing position and velocity (in black and gray, respectively).

VII. Conclusion

The present correspondence was motivated by questions regarding the transport of particles having inertia. In this context, two typical questions arise. How to steer the particles between given end-point constraints and, how to model sample trajectories based on observed end-point measurements. Traditionally such questions have been limited to diffusive particles where the stochastic excitation impacts all directions in the coordinate space as in the Brownian bridge. Herein, our aim has been to draw attention to the possibility of utilizing models with diffusion coefficient of reduced rank. These are more suitable to represent the movement of particles having inertia and be subject only to stochastic forcing/acceleration.

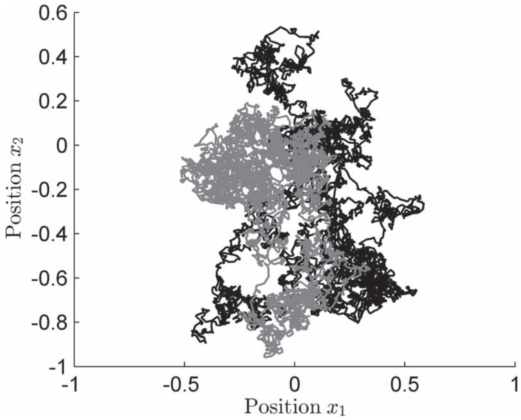

Fig. 4. Phase plots of two 2-dimensional Browian bridge sample paths with coordinates representing two spacial directions (in black and gray, respectively).

Stochastic models for processes with linear dynamics, conditioned on the two end-point constraints, are constructed using ideas from stochastic optimal control. Indeed, the appropriate drift that ensures that the statistics of the diffusion coincide with those of the bridge, is obtained by solving a Lyapunov differential equation. The end-point conditions can alternatively be viewed as Dirac distributions for particles emanating and absorbed at particular points in phase space, and the control problem that we solve here can be seen as steering a corresponding Fokker-Planck equation between the two end-point one-time marginal Dirac distributions.

The example of a pinned process with Dirac marginals was seen as a first step towards a more general Schrödinger bridge and the steering of particles between specified *marginal distributions* (see [10], [13], [15], [27], [28] and the references therein). Extending the framework of this correspondence to this more general setting has been the subject of recent and on-going work [23], [25], [26], [29], [30]. The scope of this work includes applications to steering particle systems and cooling of oscillators by actively damping thermal noise, but also the development of geometric tools for use in problems spectral analysis and system identification, cf. [22], [31].

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