On Achievable Rates of the Two-user Symmetric Gaussian Interference Channel

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• Capacity region for the simplest two user symmetric Gaussian interference channel is not yet fully characterized

• Best known achievability strategy was proposed by Han and Kobayashi (1981)
  • Split each user’s transmitted message into private and common portions
  • Time sharing between multiples of such splits
State-of-the-Art

- *Low interference regime*: [Annapureddy *et al.*] [Motahari *et al.*] [Shang *et al.*]
  ⇒ Send only private message and treat interference as noise
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  Etkin, Tse and Wang showed that Han-Kobayashi with Gaussian inputs, no time sharing and equal fixed power splitting ratios can achieve to within a single bit of the capacity
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- **Our Contributions** (Gaussian inputs):
  - Best power splitting ratios with no time sharing
  - The corresponding maximum achievable HK sum-rate
  - Comparison with orthogonal signaling (TDMA/FDMA)
  - Study of some time sharing schemes
Two-user Gaussian interference channel:

\[ y_1 = h_{11} x_1 + h_{21} x_2 + \tilde{z}_1 \]
\[ y_2 = h_{12} x_1 + h_{22} x_2 + \tilde{z}_2 \]
Network Model

Two-user Gaussian interference channel:

\[ y_1 = h_{11} x_1 + h_{21} x_2 + \bar{z}_1 \]
\[ y_2 = h_{12} x_1 + h_{22} x_2 + \bar{z}_2 \]

Normalized symmetric channel

\[ |h_{11}| = |h_{22}|, |h_{12}| = |h_{21}|, a = \frac{|h_{21}|^2}{|h_{11}|^2} = \frac{|h_{12}|^2}{|h_{22}|^2}, 0 < a < 1, \]

\[ z_i \sim C N(0, 1), P_1 = P_2 = P = \text{SNR} \]

\[ y_1 = x_1 + \sqrt{a} x_2 + z_1, \quad y_2 = \sqrt{a} x_1 + x_2 + z_2 \]

Channel fully characterized by interference coefficient \( a \) and \( P \)
Virtual 3-user MAC Channel

Private/common power splitting ratio $\lambda_i$ ($0 \leq \lambda_i \leq 1$)

$\Rightarrow$ private message $u_i$: $P_{u_i} = \lambda_i P$

$\Rightarrow$ common message $w_i$: $P_{w_i} = (1 - \lambda_i)P = \bar{\lambda}_i P$
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Symmetric Gaussian interference channel + rate splitting:

$$y_1 = \underbrace{u_1 + w_1 + \sqrt{a}w_2 + \sqrt{a}u_2}_\text{signals} + \underbrace{z_1}_\text{noise}$$

$\Rightarrow R_{1u}, R_{1w}, R_{2w} \in C_{\text{MAC-}1}(\lambda_1, \lambda_2)$

$$y_2 = \underbrace{u_2 + w_2 + \sqrt{a}w_1 + \sqrt{a}u_1}_\text{signals} + \underbrace{z_2}_\text{noise}$$

$\Rightarrow R_{2u}, R_{2w}, R_{1w} \in C_{\text{MAC-}2}(\lambda_1, \lambda_2)$
For fixed $\lambda_1$ and $\lambda_2$, the maximum sum-rate with rate splitting is:

$$R_{HK}(\lambda_1, \lambda_2) \triangleq \max_{R_{1u}, R_{1w}, R_{2u}, R_{2w}} \left( \frac{R_{1u} + R_{1w} + R_{2u} + R_{2w}}{R_1} \right)$$

$$= \gamma \left( \frac{\lambda_1 P}{1 + a\lambda_2 P} \right) + \gamma \left( \frac{\lambda_2 P}{1 + a\lambda_1 P} \right) +$$

$$\min \left\{ \gamma \left( \frac{a\lambda_2 P}{1 + \lambda_1 P + a\lambda_2 P} \right) + \gamma \left( \frac{a\lambda_1 P}{1 + \lambda_2 P + a\lambda_1 P} \right), \right.$$}

$$\left. \frac{1}{2} \gamma \left( \frac{\lambda_1 P + a\lambda_2 P}{1 + \lambda_1 P + a\lambda_2 P} \right) + \frac{1}{2} \gamma \left( \frac{\lambda_2 P + a\lambda_1 P}{1 + \lambda_2 P + a\lambda_1 P} \right) \right\}$$

where $\gamma(x) \triangleq \log_2(1 + x)$
**HK Rate Optimization**

For fixed $\lambda_1$ and $\lambda_2$, the maximum sum-rate with rate splitting is:

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$$\left. \frac{1}{2} \gamma \left( \frac{\bar{\lambda}_1 P + a\bar{\lambda}_2 P}{1 + \lambda_1 P + a\lambda_2 P} \right) + \frac{1}{2} \gamma \left( \frac{\bar{\lambda}_2 P + a\bar{\lambda}_1 P}{1 + \lambda_2 P + a\lambda_1 P} \right) \right\}$$

where $\gamma(x) \triangleq \log_2(1 + x)$

The maximum HK sum-rate without time sharing is the solution to the following optimization problem:

$$R_{RS}(a, P) \triangleq \max_{0 \leq \lambda_1, \lambda_2 \leq 1} R_{HK}(\lambda_1, \lambda_2)$$
Symmetric Power Split

Theorem

If we only consider symmetric power splits (i.e., $\lambda_1 = \lambda_2 = \lambda_{sym}$), the maximum symmetric sum rate achievable with rate splitting is:

$$R_{sym}(a, P) = \max_{0 \leq \lambda_1 = \lambda_2 \leq 1} R_{HK}(\lambda_1, \lambda_2)$$

$$= \begin{cases} 
2 \gamma \left( \frac{P}{1+aP} \right) & \text{if } P \leq \frac{1-a}{a^2} \\
2 \gamma \left( \frac{(a^2 P + a - 1)(1-a) + aP}{1 + a(a^2 P + a - 1)} \right) & \text{if } \frac{1-a}{a^2} < P \leq \frac{1-a^3}{a^3(a+1)} \\
\gamma \left( \frac{1-a}{2a} \right) + \gamma \left( \frac{(1+a)^2 P - (1-a)}{2} \right) & \text{if } P > \frac{1-a^3}{a^3(a+1)}
\end{cases}$$

and the corresponding optimal power split ratio is:

$$\lambda^*_\text{sym} = \begin{cases} 
1 & \text{if } P \leq \frac{1-a}{a^2} \\
\frac{a^2 P + a - 1}{aP} & \text{if } \frac{1-a}{a^2} < P \leq \frac{1-a^3}{a^3(a+1)} \\
\frac{1-a}{(1+a)(aP)} & \text{if } P > \frac{1-a^3}{a^3(a+1)}
\end{cases}$$
Asymmetric Power Split

If we constrain one of the users to send only a common message (i.e., \( \lambda_1 = 0 \)), the corresponding maximum sum rate is:

\[
R_{asym} = \max_{0 \leq \lambda_2 \leq 1} R_{HK}(\lambda_1 = 0, \lambda_2) = \log_2 \left( \frac{(1 + \lambda_2^* P + aP)(1 + aP)}{1 + a\lambda_2^* P} \right)
\]

where \( \lambda_2^* \) is the solution to the following equation:

\[
\sqrt{\frac{1 + \lambda_2^* P}{1 + a\lambda_2^* P}} (1 + P + aP) = \frac{(1 + \lambda_2^* P + aP)(1 + aP)}{1 + a\lambda_2^* P}.
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**Conjecture**

The maximum HK sum-rate is achieved either using symmetric power splits or constraining one of the users send only a common message (i.e., $R_{RS} = \max\{R_{sym}, R_{asym}\}$)
Symmetric Rate

\[ R_{sym} = \max_{0 \leq \lambda_1 = \lambda_2 \leq 1} R_{HK}(\lambda_1, \lambda_2) \]
Summary

Symmetric Rate

\[ R_{sym} = \max_{0 \leq \lambda_1 = \lambda_2 \leq 1} R_{HK}(\lambda_1, \lambda_2) \]

Asymmetric Rate

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### Summary

#### Symmetric Rate

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R_{sym} = \max_{0 \leq \lambda_1 = \lambda_2 \leq 1} R_{HK}(\lambda_1, \lambda_2)
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R_{asym} = \max_{0 \leq \lambda_2 \leq 1} R_{HK}(\lambda_1 = 0, \lambda_2)
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#### Etkin, Tse and Wang

\[
R_{ETW} = R_{HK}(\lambda_1 = \lambda_2 = \frac{1}{aP}) \leq R_{sym}
\]
## Summary

### Symmetric Rate

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R_{sym} = \max_{0 \leq \lambda_1 = \lambda_2 \leq 1} R_{HK}(\lambda_1, \lambda_2)
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R_{asym} = \max_{0 \leq \lambda_2 \leq 1} R_{HK}(\lambda_1 = 0, \lambda_2)
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### Etkin, Tse and Wang

\[
R_{ETW} = R_{HK}(\lambda_1 = \lambda_2 = \frac{1}{aP}) \leq R_{sym}
\]

### Orthogonal signaling (TDMA/FDMA)

\[
R_{orth} = \log_2(1 + 2P)
\]
Sum Rate vs. Interference Coefficient $a$

![Graph showing sum rate vs. interference coefficient](image)

- $R_{sym}$
- $R_{orth}$
- $R_{asym}$
- $R_{UB}$
- $R_{ETW}$

SNR = 10 dB

- Interference Coefficient $a$:
  - 0.1
  - 0.2
  - 0.3
  - 0.4
  - 0.5
  - 0.6
  - 0.7
  - 0.8
  - 0.9
  - 1

Sum Rate:
- 3.5
- 4
- 4.5
- 5
- 5.5
- 6
- 6.5
Sum Rate vs. Interference Coefficient $a$

![Graph showing the relationship between sum rate and interference coefficient $a$ for different rates and interference levels.]

- $R_{orth}$
- $R_{sym}$
- $R_{asym}$
- $R_{UB}$
- $R_{ETW}$

SNR = 20 dB
Rate maximizing strategy
Rate maximizing strategy

\[ \text{Interference Coefficient } a \]

\[ \text{P (SNR) [dB]} \]

\[ \text{Low SNR: } R_{\text{all-private}}, R_{\text{orth}} \quad - \quad \text{High SNR: } R_{\text{sym}}, R_{\text{asym}} \]
**SNR vs. INR**

\[
\text{INR}_{dB} = \text{SNR}_{dB} + 10 \log_{10} a
\]
SNR vs. INR

\[ \text{INR}_{dB} = \text{SNR}_{dB} + 10 \log_{10} \alpha \]

\[ \alpha \triangleq \frac{\text{INR}_{dB}}{\text{SNR}_{dB}} \]

\[
\begin{align*}
\text{all-private} & : 0 < \alpha < \frac{1}{2} \\
\text{orth} & : \alpha = \frac{1}{2} \\
\text{sym} & : \frac{1}{2} < \alpha < 1 \\
\text{asym} & : \alpha = 1
\end{align*}
\]

\(\alpha\) only explains behavior above 20 dB
Asymptotic sum-rate offset

Fix the interference coefficient $a$ and take $P \to \infty$

$$\Delta R(a) \triangleq \lim_{P \to \infty} (R - \log_2(P))$$
Asymptotic sum-rate offset

Fix the interference coefficient $a$ and take $P \rightarrow \infty$

$$\Delta R(a) \triangleq \lim_{P \rightarrow \infty} (R - \log_2(P))$$

$$\Delta R_{sym}(a) = \log_2 \left( \frac{(1 + a)^3}{4a} \right)$$

$$\Delta R_{asym}(a) = \log_2 \left( \frac{1 + a}{\sqrt{a}} \right)$$

$$\Delta R_{ETW}(a) = \log_2 \left( \frac{(2a + 1)(a + 1)}{4a} \right)$$

$$\Delta R_{orth}(a) = 1$$
High SNR Behavior

[Graph showing the high SNR offset behavior with labels for $\Delta R_{\text{asym}}$, $\Delta R_{\text{orth}}$, $\Delta R_{\text{sym}}$, and $\Delta R_{\text{ETW}}$.]
High SNR Behavior

\[ R_{asym} > R_{sym} \text{ for } a > 0.087 \]
Time Sharing Schemes

Scheme I: 2 equal time slots (optimization over $\alpha_1$, $\alpha_2$, $\lambda_1$ and $\lambda_2$)

<table>
<thead>
<tr>
<th>TS_1</th>
<th>TS_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1(1)$ P, $\lambda_1(1)$</td>
<td>$\alpha_1(2)$ P, $\lambda_1(2)$</td>
</tr>
<tr>
<td>$\alpha_2(1)$ P, $\lambda_2(1)$</td>
<td>$\alpha_2(2)$ P, $\lambda_2(2)$</td>
</tr>
</tbody>
</table>

$\alpha_1(1) = \alpha_2(2)$, $\alpha_2(1) = \alpha_1(2)$
$\alpha_1(1) + \alpha_1(2) = 2$
$\lambda_1(1) = \lambda_2(2)$, $\lambda_2(1) = \lambda_1(2)$
**Time Sharing Schemes**

Scheme I: 2 equal time slots (optimization over $\alpha_1$, $\alpha_2$, $\lambda_1$ and $\lambda_2$)

<table>
<thead>
<tr>
<th>TS₁</th>
<th>TS₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1(1)P$, $\lambda_1(1)$</td>
<td>$\alpha_1(2)P$, $\lambda_1(2)$</td>
</tr>
<tr>
<td>$\alpha_2(1)P$, $\lambda_2(1)$</td>
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$\alpha_1(1) + \alpha_1(2) = 2$

$\lambda_1(1) = \lambda_2(2)$, $\lambda_2(1) = \lambda_1(2)$

Scheme II: 4 time slots (optimization over $\beta$, $\lambda_1$ and $\lambda_2$) - Sason (04)

$TS_1 = TS_2 = \beta$

<table>
<thead>
<tr>
<th>TS₁</th>
<th>TS₂</th>
<th>TS₃ = TS₄ = (1-2 $\beta$)/2</th>
</tr>
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<td>$\alpha_1(1) = 2\beta$, $\lambda_1(1)$</td>
<td>$\alpha_1(2) = 2\beta$, $\lambda_1(2)$</td>
<td>$\alpha_1(3)=2(1+2\beta)$</td>
</tr>
<tr>
<td>$\alpha_2(1) = 2\beta$, $\lambda_2(1)$</td>
<td>$\alpha_2(2) = 2\beta$, $\lambda_2(2)$</td>
<td>$\alpha_2(3)=0$</td>
</tr>
<tr>
<td>$\alpha_2(4)=2(1+2\beta)$</td>
<td>$\alpha_2(4)=2(1+2\beta)$</td>
<td>$\alpha_1(4)=0$</td>
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$\lambda_1(1) = \lambda_2(2)$, $\lambda_2(1) = \lambda_1(2)$
Numerical Results

SNR = 20 dB

Similar behavior at other SNR’s
• Derived expressions for the maximum achievable HK sum-rate with no time sharing and corresponding optimal power split ratios $\Rightarrow$ tighter capacity lower bound
Conclusions

• Derived expressions for the maximum achievable HK sum-rate with no time sharing and corresponding optimal power split ratios ⇒ tighter capacity lower bound

• Despite the fact that the channel is symmetric, allowing for asymmetric power split ratio at both users (i.e., asymmetric rates) provides larger sum rate for a wide range of $a$ and $P$ values ($a > 0.087$ at the high SNR regime)
Conclusions

- Derived expressions for the maximum achievable HK sum-rate with no time sharing and corresponding optimal power split ratios ⇒ tighter capacity lower bound.

- Despite the fact that the channel is symmetric, allowing for asymmetric power split ratio at both users (i.e., asymmetric rates) provides larger sum rate for a wide range of $a$ and $P$ values ($a > 0.087$ at the high SNR regime).

- Orthogonal signaling is good for $\frac{\text{INR}_{dB}}{\text{SNR}_{dB}} \approx \frac{1}{2}$ and low SNR’s.
Conclusions

- Derived expressions for the maximum achievable HK sum-rate with no time sharing and corresponding optimal power split ratios $\Rightarrow$ tighter capacity lower bound

- Despite the fact that the channel is symmetric, allowing for asymmetric power split ratio at both users (i.e., asymmetric rates) provides larger sum rate for a wide range of $a$ and $P$ values ($a > 0.087$ at the high SNR regime)

- Orthogonal signaling is good for $\frac{INR_{dB}}{SNR_{dB}} \approx \frac{1}{2}$ and low SNR’s

- Advantage of using time sharing schemes is quite small