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Effect of Topological Dimension on Rigidity of Vehicle Formations: Fundamental Limitations of Local Feedback

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Abstract— **We consider the role of topological dimension in problems of network consensus and vehicular formations where only local feedback is available. In particular, we consider the simple network topologies of regular lattices in 1, 2 and higher dimensions. Performance measures for consensus and formation problems are proposed that measure the deviation from average and rigidity or tightness of formations respectively. A common phenomenon appears where in dimensions 1 and 2, consensus is impossible in the presence of any amount of additive stochastic perturbations, and in the limit of large formations. In dimensions 3 and higher, consensus is indeed possible. We show that microscopic error measures that involve only neighboring sites do not suffer from this effect. This phenomenon reflects the fact that in dimensions 1 and 2, local stabilizing feedbacks can not suppress long spatial wavelength "meandering" motions. These effects are significantly more pronounced in vehicular problems than in consensus, and yet they are unrelated to string stability issues.**

I. INTRODUCTION

The control problem for strings of vehicles (the so-called platooning problem) has been extensively studied since the 90's, with original problem formulations and studies dating back to the 60's [1], [2], [3], [4], [5]. These problems are intimately related to more recent formation flying and formation control problems as well [6].

It has long been observed in platooning problems that to achieve reasonable performance, certain global information such as leader's position or state need to be broadcast to the entire formation. A precise analysis of the limits of performance associated with localized versus global control strategies does not appear to exist in the formation control literature. In this paper we study the platooning problem as the 1 dimensional version of a more general formation control problem on regular lattices. For such problems, we investigate the limits of performance of any local feedback law that is globally stabilizing. In particular, we propose and study certain measure of the "tightness" or coherence of the formation. These are measures that capture the notion of how well the formation resembles a rigid lattice.

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The tightness of a formation is a different concept from, and often unrelated to, string instability. In the platooning case (i.e. 1 dimensional formations) which is most problematic, a localized feedback control law may posses string stability in the sense that the effects of any injected disturbance do not grow with spatial location. However, as we show in this paper, it is impossible to achieve a large tight formation with only localized feedback if all vehicles are subject to any amount of stochastic disturbance. The net effect is that with the best localized feedback, a 1 dimensional formation will appear to behave well on a "microscopic" scale in the sense that distances between neighboring vehicles will be well regulated. However, if a large formation is observed in its entirety, it will appear to have temporally slow, long spatial wavelength modes that are unregulated, i.e. a "meandering" type of motion. This is not a safety issue, since the formation is microscopically well regulated, but it might effect throughput performance in a platooning arrangement since that does depend on the tightness or rigidity of the formation.

The phenomenon that we discuss occurs in both consensus algorithms and vehicular formation problems, and we therefore treat both. Both problems are set up in the d-dimensional torus $\mathbb{Z}_N^d$. The asymptotic results (in the limit of large size) hold for the same problems set up in d-dimensional regular lattices with boundaries. This follows from the correspondence between the asymptotics of circulant and Toeplitz multi-dimensional operators, but will be reported elsewhere.

Notation

The multidimensional Discrete Fourier Transform is used throughout. All states are multidimensional arrays which we define as real or complex vector-valued functions on the Torus $\mathbb{Z}_N^d$. The Fourier transform of an array a is denoted with $\hat{a}$. Multi-index notation is also used, as in $a_k = a_{(k_1,\ldots,k_d)}$ to denote individual entries of an array. We refer to indices of spatial Fourier transforms as wavenumbers. Generally, we use k and l for spatial indices and n and m for wavenumbers.

II. PROBLEM FORMULATION

We begin with formulating consensus problems with additive stochastic perturbations in the dynamics [7], [8]. As opposed to standard consensus algorithms without additive noise, nodes do not achieve equilibrium asymptotically but

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fluctuate around the average equilibrium, and the variance of this fluctuation is what we study. This formulation can be used to model scenarios such as load balancing over a distributed file system, where the additive nosie represents file insertion and deletion, or parallel processing systems where the noise processes model job creation and completion.

A. Consensus with random insertions/deletions

We consider a consensus algorithm over an undirected (connected) tori network $\mathbb{Z}_N^d$, where each node exchanges information with neighboring nodes in the network. One possible nearest neighbor averaging scheme looks like

$$\dot{x}_k = \alpha\, x_k + \beta \big(x_{(k_1-1,\ldots,k_d)} + x_{(k_1+1,\ldots,k_d)} + \cdots + x_{(k_1,\ldots,k_d-1)} + x_{(k_1,\ldots,k_d+1)} \big) + w_k, \tag{1}$$

where w is a mutually uncorrelated white stochastic process. Each node averages with its $2d$ nearest neighbors (two nearest neighbors along each of the d axes). In this paper, we call this the *standard consensus algorithm* since it is essentially the same as other well studied consensus algorithms [9], [10], [11], [12].

For consensus to be an equilibrium, the numbers α and β then must satisfy

$$\alpha + 2d\,\beta = 0.$$

The sum in the equation above can be written as a multidimensional convolution by defining the array

$$a_{(k_1,\ldots,k_d)} = \begin{cases} \alpha & k_1 = \cdots = k_d = 0, \\ \beta & k_i = \pm 1, \text{ and } k_j = 0 \text{ for } i \neq j, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The system (1) can then be written in operator notation as

$$\dot{x} = T_a\, x + w, \tag{3}$$

where T_a is the circulant operator of convolution with the array a.

B. Vehicular Formations

Consider N^d vehicles arranged in a d-dimensional torus ($\mathbb{Z}_N^d$) with the simple double integrator dynamics

$$\ddot{x}_{(k_1,\ldots,k_d)} = u_{(k_1,\ldots,k_d)} + w_{(k_1,\ldots,k_d)}, \tag{4}$$

where $(k_1, \ldots, k_d)$ is a multi-index with each $k_i \in \mathbb{Z}_N$, u is the control input and w is a mutually uncorrelated white stochastic process. w can be considered to model random forcing. Each position vector x_k is a d-dimensional vector with components $x_k = \begin{bmatrix} x_k^1 & \cdots & x_k^d \end{bmatrix}^T$. The objective is to have the k'th vehicle in the formation follow the desired trajectory $\bar{x}_k$

$$\bar{x}_k := vt + k\Delta \;\Leftrightarrow\; \begin{bmatrix} x_k^1 \\ \vdots \\ x_k^d \end{bmatrix} = \begin{bmatrix} v^1 \\ \vdots \\ v^d \end{bmatrix} t + \begin{bmatrix} k_1 \\ \vdots \\ k_d \end{bmatrix} \Delta,$$

which means that all vehicles are to move with constant heading velocity v while maintaining their respective position in a $\mathbb{Z}_N^d$ grid with spacing Δ.

Define the deviations from desired trajectory as

$$\tilde{x}_k := x_k - \bar{x}_k, \qquad \tilde{v}_k := \dot{x}_k - v.$$

We assume the control input to be full state feedback and linear in the variables $\tilde{x}$ and $\tilde{v}$ (therefore affine linear in x and v), i.e. $u = K\tilde{x} + F\tilde{v}$, where K and F are the linear feedback operators. The equations of motion for the controlled system are thus

$$\frac{d}{dt}\begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ K & F \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w. \tag{5}$$

We note that the above equations are written in operator form, i.e. by suppressing the spatial index of all the variables.

We now make further restrictions on the operators K and F as follows. Note that by definition of the consensus algorithm (1) these properties hold for T_a as well.

1) **Spatial Invariance** The feedback operators K and F are spatially invariant with respect to $\mathbb{Z}_N^d$. This implies that they are convolution operators. For instance, the operation Kx can be written as the convolution (over $\mathbb{Z}_N^d$) of the array x with an array $\{K_{(k_1,\ldots,k_d)}\}$

$$(Kx)_k = \sum_{l \in \mathbb{Z}_N^d} K_{k-l}\; x_l,$$

where the arithmetic for $k - l$ is done in $\mathbb{Z}_N^d$.

2) **Relative Measurements** The feedbacks involve only differences between positions and velocities respectively, i.e. for each term of the form $\alpha x_{(k_1,\ldots,k_d)}$ in the convolution, another term of the form $-\alpha x_{(l_1,\ldots,l_d)}$ occurs for some other multi-index l. This implies that the arrays K and F have the property

$$\sum_{k \in \mathbb{Z}_N^d} K_k = \sum_{k \in \mathbb{Z}_N^d} F_k = 0. \tag{6}$$

3) **Locality** The feedbacks use only local information from a neighborhood of width q, where q is independent of N, the size of formation. Specifically

$$K_{(k_1,\ldots,k_d)} = 0, \quad \text{if for any } i \in \{1,\ldots,d\}, \;\; |k_i| > q. \tag{7}$$

The same condition holds for F.

4) **Isotropy** The interactions between vehicles have mirror symmetry. This has the consequence that the arrays representing K and F have even symmetry, e.g. for each nonzero term like $\alpha K_{(k_1,\ldots,k_d)}$ in the array there is a corresponding term $\alpha K_{(-k_1,\ldots,-k_d)}$. This in particular implies that the Fourier transform of the arrays representing K and F are real valued.

This assumption is made to simplify subsequent calculations, but does not appear to be essential to the main result.

III. Performance Measures

We will consider how various performance measures scale with system size for the consensus and vehicle formations problems. Some of these measure can be quantified as steady state variances of outputs of linear systems driven by stochastic inputs, so we consider some generalities first. Consider first a general linear system driven by zero mean white noise with unit covariance

$$\begin{aligned} \dot{x} &= Ax + Bw, \\ y &= Hx. \end{aligned}$$

Since we are interested in cases where A is not necessarily Hurwitz (typically due to a single unstable mode at zero representing motion of the mean), the state x may not have finite steady state variances. However, all cases we consider are ones in which the outputs y do have finite variances, i.e. the unstable modes of A are not observable from y. In such cases, the output does have a finite steady state variance, and the H^2 norm of the system from w to y is by definition

$$V := \sum_k \lim_{t\to\infty} E\{y_k^*(t)y_k(t)\}, \qquad (8)$$

where the sum is taken over all outputs.

We are interested in spatially invariant problems over discrete Tori. This invariance implies that the variances of all outputs are equal, i.e. $E\{y_k^* y_k\}$ is independent of k. Thus, if the output variance at a given site is to be computed, it is simply the total H^2 norm divided by the system size

$$E\{y_k^* y_k\} = \frac{1}{M}\sum_{l\in\mathbb{Z}_N^d} E\{y_l^* y_l\} = \frac{V}{M},$$

where M is the size of the system (N^d for d-dimensional Tori).

We define several different performance measures and give the corresponding output operators for each measure for both the consensus and vehicular formation problems. In the vehicular formation problem, we assume for simplicity that the output involves positions only, and thus the output equation has the form

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix},$$

i.e. $H = \begin{bmatrix} C & 0 \end{bmatrix}$, where C is a circulant operator. A consensus problem with the same performance measure has a corresponding output equation of the form (with the same C operator)

$$y = Cx.$$

We now list the three different performance measures we use in each case.

1) **Local error** This is a measure of the difference between neighboring nodes or vehicles. For the consensus problem, the k'th output (in the case of one dimension) is defined by

$$y_k := x_k - x_{k-1}.$$

For the case of vehicular formations, local error is the difference of neighboring vehicles positions from desired spacing, which can equivalently be written as

$$y_k := \tilde{x}_k - \tilde{x}_{k-1}.$$

The output operator is then given by $C := (I - D)$, where D is the right shift operator, $(Dx)_k := x_{k-1}$.

2) **Long range deviation** In the consensus problem, this corresponds to measuring the disagreement between the two furthest nodes in the network graph. Assume for simplicity that N is even and we are in dimension 1. Then, the most distant node from node k is $\frac{N}{2}$ hops away, and we define

$$y_k := x_k - x_{k+\frac{N}{2}}.$$

In the vehicular formation problem, long range deviation corresponds to measuring the deviation of the distance between the two most distant vehicles from what it should be. The most distant vehicle to the k'th one is the $k+\frac{N}{2}$ vehicle. The desired distance between them is $\Delta\frac{N}{2}$, and the deviation from that is

$$y_k := x_k - x_{k+\frac{N}{2}} - \Delta\frac{N}{2} = \tilde{x}_k - \tilde{x}_{k+\frac{N}{2}}. \qquad (9)$$

We consider the variance of this quantity to be a measure of the "end-to-end tightness" of the vehicle formation.

Generalizing this measure to d dimensions yields an output operator of the form

$$C := T_{(\boldsymbol{\delta}_0 - \boldsymbol{\delta}_{(N/2,\ldots,N/2)})},$$

i.e. the operator of convolution with the array [1] $\boldsymbol{\delta}_0 - \boldsymbol{\delta}_{(N/2,\ldots,N/2)}$.

3) **Deviation from average** For the consensus problem, this measures the deviation of each state from the average of all states,

$$y_k := x_k - \frac{1}{M}\sum_{l\in\mathbb{Z}_N^d} x_l.$$

In operator form we have $y = (I - T_{\mathbf{1}})x$, where $\mathbf{1}$ is the array of all elements equal to $1/M$.

In vehicular formations, this measure can be interpreted the deviation of each vehicle's position error from the average of the overall position error $y = (I - T_{\mathbf{1}})\tilde{x}$.

We note that all performance measures are such that C can be represented as a convolution with an array $\{C_k\}$ which has the property $\sum_{k\in\mathbb{Z}_N^d} C_k = 0$. This condition causes the mean mode at zero to be unobservable, and thus guarantees that all outputs defined above have finite variances.

[1] By a slight abuse of notation, we define the shifted Kronecker delta $\boldsymbol{\delta}_{l,(k)} := \delta_{(k-l)}$, where $\delta_k = 1$ for $k = 0$, and zero otherwise, is the standard Kronecker delta. With this notation, $\boldsymbol{\delta}_0$ is also the standard Kronecker delta.

Formulae for variances: Since we consider spatially invariant systems and in particular systems on the discrete Tori $\mathbb{Z}_N^d$, it is possible to derive formulae for the above defined measures in terms of the Fourier symbols of the operators K, F and C. Recall the state space formula for the H^2 norm V defined in (8)

$$V = \text{tr}\left(\int_0^\infty B^* e^{A^* t} H^* H e^{At} B \, dt\right).$$

When A, B and H are circulant operators, the above can be written in terms of their respective Fourier symbols as

$$\begin{aligned} V &= \text{tr}\left(\sum_n \int_0^\infty \hat{B}_n^* e^{\hat{A}_n^* t} \hat{H}_n^* \hat{H}_n e^{\hat{A}_n t} \hat{B}_n \, d\omega\right) \quad (10) \\ &= \sum_n \text{tr}\left(\hat{B}_n^* \hat{P}_n \hat{B}_n\right), \quad (11) \end{aligned}$$

where we have defined the individual integrals

$$\hat{P}_n := \int_0^\infty e^{\hat{A}^* t} \hat{H}^* \hat{H} e^{\hat{A} t} dt. \quad (12)$$

If $\hat{A}_n$ is Hurwitz, then $\hat{P}_n$ can be obtained by solving the Lyapunov equation

$$\hat{A}_n^* \hat{P}_n + \hat{P}_n \hat{A}_n = -\hat{H}_n^* \hat{H}_n. \quad (13)$$

For wavenumbers n for which $\hat{A}_n$ is not Hurwitz, $\hat{P}_n$ is still finite if the non-Hurwitz modes of $\hat{A}_n$ are not observable from $\hat{H}_n$. In this case we can analyze the integral in (12) on a case by case basis.

The Lyapunov equations are easy to solve in the Fourier domain. Equation (13) is a scalar equation in the Consensus case and a 2×2 matrix equation in the Vehicular case. The two calculations are summarized in the next lemma.

Lemma 3.1: The variances for the consensus and vehicular problems are given by

$$V_c = -\frac{1}{2} \sum_{n \neq 0 \; n \in \mathbb{Z}_N^d} \frac{|\hat{C}_n|^2}{\Re(\hat{a}_n)}, \quad (14)$$

$$V_v = -\frac{1}{2} \sum_{n \neq 0 \; n \in \mathbb{Z}_N^d} \frac{|\hat{C}_n|^2}{\hat{K}_n \hat{F}_n}. \quad (15)$$

These expressions can then be worked out for the variety of output operators C representing the different performance measures defined earlier. The next theorem presents a summary of those calculations for the six different cases.

Theorem 3.2: The following are the performance measures expressed in terms of $\hat{K}$, $\hat{F}$ and $\hat{a}$, the Fourier symbols of the operators K, F, and T_a respectively.

- *Consensus*
  1) Local Error: $V_c^{loc} = \frac{1}{2d\beta}$
  2) Long Range Deviation:

$$V_c^{lrd} = -2 \sum_{n_1 + \cdots + n_d \text{ odd}, \; n \in \mathbb{Z}_N^d} \frac{1}{\Re(\hat{a}_n)}$$

  3) Deviation from Average:

$$V_c^{dav} = -\frac{1}{2} \sum_{n \neq 0, \; n \in \mathbb{Z}_N^d} \frac{1}{\Re(\hat{a}_n)}. \quad (16)$$

- *Vehicular Formations*
  1) Local Error:

$$V_v^{loc} = -\frac{1}{M} \sum_{n \neq 0 \; n \in \mathbb{Z}_N^d} \frac{\left(1 - \cos\left(\frac{2\pi}{N} n\right)\right)}{\hat{K}_n \hat{F}_n}$$

  2) Long Range Deviation:

$$V_v^{lrd} = -\sum_{n_1 + \cdots + n_d \text{ odd}, \; n \in \mathbb{Z}_N^d} \frac{1}{\hat{K}_n \hat{F}_n} \quad (17)$$

  3) Deviation from Average:

$$V_v^{dav} = -\frac{1}{2} \sum_{n \neq 0, \; n \in \mathbb{Z}_N^d} \frac{1}{\hat{K}_n \hat{F}_n}. \quad (18)$$

IV. Asymptotic Bounds

In this section we derive bounds on the asymptotic behavior (in the size of formation M) of consensus algorithms in some detail. The corresponding bounds for vehicular case are summarized and their details will be presented elsewhere. For the consensus problem, we derive lower bounds on any stabilizing algorithm that uses only local information, and then exhibit specific algorithms that achieves these bounds.

The key step in all of the derivations is the formation of bounds that involve sums of reciprocals of quadratic quantities, and then the use of the asymptotic bounds (23). To begin with, consider the expression (14). Noting that the Fourier symbol $\hat{a}$ is

$$\hat{a}_{(n_1,\ldots,n_d)} = \sum_{k \in \mathbb{Z}_N^d} a_{(k_1,\ldots,k_d)} \; e^{-i \frac{2\pi}{N}(k_1 n_1 + \cdots + k_d n_d)},$$

yields the following expression for the H^2 norm of a general consensus system

$$V = -\frac{1}{2} \sum_{\substack{n \neq 0 \\ n \in \mathbb{Z}_N^d}} \frac{1}{\sum_{k \in \mathbb{Z}_N^d} a_k \; \cos\left(\frac{2\pi}{N}(k_1 n_1 + \cdots + k_d n_d)\right)}. \quad (19)$$

A. Bounds for the consensus problem

We first show that any consensus algorithm using only local information has certain lower bounds for each case of dimension $d = 1, 2, 3, \cdots$. We then show that the standard consensus algorithm (1) achieves these bounds by showing that they are upper bounds on that algorithm.

Lower bounds: Beginning with the expression (19) and using the property (21)

$$\begin{aligned} V &\geq -\frac{1}{2} \sum_{\substack{n\neq 0 \\ n\in\mathbb{Z}_N^d}} \frac{1}{\sum_{k\in\mathbb{Z}_N^d} a_k \left(1 - \frac{4\pi^2}{N^2}(k_1 n_1 + \cdots + k_d n_d)^2\right)} \\ &= \frac{N^2}{8\pi^2} \sum_{\substack{n\neq 0 \\ n\in\mathbb{Z}_N^d}} \frac{1}{\sum_{k\in\mathbb{Z}_N^d} a_k \ (k_1 n_1 + \cdots + k_d n_d)^2}, \end{aligned}$$

where the last equation follows from the property $\sum_{k\in\mathbb{Z}_N^d} a_k = 0$. The next lower bound follows from the locality property (7) which has the consequence

$$\begin{aligned} &\sum_{k\in\mathbb{Z}_N^d} a_k \ (k_1 n_1 + \cdots + k_d n_d)^2 \\ &\qquad = \sum_{k\in\mathbb{Z}_N^d,\ |k_i|\leq q} a_k \ (k_1 n_1 + \cdots + k_d n_d)^2 \\ &\qquad \leq \sum_{k\in\mathbb{Z}_N^d,\ |k_i|\leq q} a_k \ (q n_1 + \cdots + q n_d)^2 \\ &\qquad = q^2 (n_1 + \cdots + n_d)^2 \sum_{k\in\mathbb{Z}_N^d} a_k. \end{aligned}$$

Putting the above together gives

$$\begin{aligned} V &\geq \frac{N^2}{8\pi^2 q \sum_{k\in\mathbb{Z}_N^d} a_k} \sum_{n\neq 0,\ n\in\mathbb{Z}_N^d} \frac{1}{(n_1 + \cdots + n_d)^2} \\ &\geq C\ N^2 \sum_{n\neq 0,\ n\in\mathbb{Z}_N^d} \frac{1}{(n_1^2 + \cdots + n_d^2)}, \end{aligned}$$

where the last inequality follows from (22), and the constant C is independent of n

$$C := \frac{1}{8\pi^2 q (2d-1) \sum_{k\in\mathbb{Z}_N^d} a_k}.$$

Finally, utilizing (23) and observing that the number of sites in $\mathbb{Z}_N^d$ is N^d, the variance of each element is given by

$$\begin{aligned} \frac{V}{N^d} &\geq C\ N^{2-d} \sum_{n\neq 0,\ n\in\mathbb{Z}_N^d} \frac{1}{(n_1^2 + \cdots + n_d^2)} \\ &\approx \begin{cases} \frac{1}{d-2}(1 - N^{2-d}) & d \neq 2 \\ \log(N) & d = 2 \end{cases}, \end{aligned} \tag{20}$$

asymptotically. Note that in dimensions 3 and higher, the lower bound is constant in N.

Upper bounds: The H^2 norm of the standard consensus problem (1) is given by the general expression (14), where the array a is specified by (2). We begin the derivation of an upper bound for this problem by assuming without loss of generality that N is odd, defining $\bar{N} := (N+1)/2$, and using the even symmetry of $\cos(x)$ about $x = \pi$. This gives an equivalent expression for V as

$$\begin{aligned} V &= \frac{1}{4\beta} \sum_{\substack{n\neq 0 \\ n\in\mathbb{Z}_N^d}} \frac{1}{\left(d - \cos\left(\frac{2\pi}{N}n_1\right) - \cdots - \cos\left(\frac{2\pi}{N}n_d\right)\right)} \\ &= \frac{2^d}{4\beta} \sum_{\substack{n\neq 0 \\ n\in\mathbb{Z}_{\bar{N}}^d}} \frac{1}{\left(d - \cos\left(\frac{2\pi}{N}n_1\right) - \cdots - \cos\left(\frac{2\pi}{N}n_d\right)\right)}. \end{aligned}$$

Now an upper bound on this quantity can be derived by using (21), and noting that the denominator above is made up of d terms of the form

$$1 - \cos\left(\frac{2\pi}{N}n_i\right) \geq \frac{2}{\pi^2}\left(\frac{2\pi}{N}n_i\right)^2 = \frac{8}{N^2}n_i^2,$$

where the above inequality is valid in the range $n_i \in [0, (\bar{N}-1)]$. Putting this together with the above expression for V, we obtain an upper bound for the variance of each site

$$\begin{aligned} \frac{V}{N^d} &= \frac{2^d}{32\beta} N^{2-d} \sum_{n\neq 0,\ n\in\mathbb{Z}_{\bar{N}}^d} \frac{1}{(n_1^2 + \cdots + n_d^2)} \\ &\approx \frac{2^d}{32\beta} N^{2-d} \begin{cases} \frac{1}{d-2}(\bar{N}^{d-2} - 1) & d \neq 2 \\ \bar{N}^{d-2}\log(\bar{N}) & d = 2 \end{cases} \\ &\leq \frac{2^d}{32\beta} \begin{cases} \frac{1}{d-2}(1 - N^{2-d}) & d \neq 2 \\ \log(N) & d = 2 \end{cases}, \end{aligned}$$

where we have used $\bar{N} \leq N$. Note that these upper bounds are scaled versions of the lower bounds.

B. Bounds for Vehicular Formations

Due to space limitations, we do not include the derivations and arguments for the vehicular formations results here. We only state them for the case of dimension one ($d = 1$). The results for higher dimensions will be reported elsewhere. The derivations of both the lower and upper bounds for the vehicle formation problem is very similar to that of the consensus problem. However, one important difference is that the denominator of the sums involves a product of terms $\hat{K}_n \hat{F}_n$. Under our assumptions, the behavior of those terms for small n in the limit of large N is of a higher order than terms like $\hat{a}_n$. This yields a more severe scaling for the performance measures in the vehicular case versus the consensus case. The scalings in the single dimensional case are summarized in Table I.

TABLE I

BOUNDS FOR 1 DIMENSIONAL TORUS WITH SIZE N

Performance Measure	Consensus	Vehicular Formations
Local Error	1	N
Long Range Deviation	N	N^3
Deviation from Average	N	N^3

V. Discussion

The results presented here have a strong resemblance to results on performance limitations of distributed estimation algorithms based on network topology [13], where the arguments are based on an analogy with effective resistance in resistive lattices. In this paper we have avoided this analogy by directly using the multi-dimensional Fourier transform and reducing all calculations to sums of the form (23).

It was observed in [14] that optimal LQR designs for vehicular platoons suffer from a fundamental problem as the platoon size increases to infinity. These optimal feedback laws are almost local in a sense described by [15], where control gains decay exponentially as a function of distance. The resulting optimal feedbacks [14] suffer from the problem of having underdamped slow modes with long spatial wavelengths. This is yet another manifestation of the phenomenon we have studied in the present paper. Local or almost local feedback can not apparently compensate well for these temporally slow, long spatial wavelength modes. Thus, when additive stochastic disturbances are present, most of their energy appears to accumulate in those modes, causing a slow, spatially meandering motion of the entire formation. In one or two dimensions, one needs global feedback to suppress these modes, while local feedback is sufficient for three dimensional formations.

References

[1] S. Shladover, C. Desoer, J. Hedrick, M. Tomizuka, J. Walrand, W.-B. Zhang, D. McMahon, H. Peng, S. Sheikholeslam, and N. McKeown, "Automated vehicle control developments in the path program," *IEEE Transactions on Vehicular Technology*, vol. 40, no. 1, pp. 114–130, February 1991.

[2] D. Swaroop and J. K. Hedrick, "Constant spacing strategies for platooning in automated highway systems," *Transactions of the ASME. Journal of Dynamic Systems, Measurement and Control*, vol. 121, no. 3, pp. 462–470, September 1999.

[3] ——, "String stability of interconnected systems," *IEEE Transactions on Automatic Control*, vol. 41, no. 2, pp. 349–357, March 1996.

[4] S. M. Melzer and B. C. Kuo, "Optimal regulation of systems described by a countably infinite number of objects," *Automatica*, vol. 7, no. 3, pp. 359–366, May 1971.

[5] W. S. Levine and M. Athans, "On the optimal error regulation of a string of moving vehicles," *IEEE Transactions on Automatic Control*, vol. AC-11, no. 3, pp. 355–361, July 1966.

[6] S. Martinez, J. Cortes, and F. Bullo, "Motion coordination with distributed information," *Control Systems Magazine*, vol. 27, no. 4, pp. 75–88, 2007.

[7] L. Xiao, S. Boyd, and S.-J. Kim, "Distributed average consensus with least-mean-square deviation," *Journal of Parallel and Distributed Computing*, vol. 67, pp. 33–46, 2007.

[8] S. Patterson, B. Bamieh, and A. El Abbadi, "Distributed average consensus with stochastic communication failures," *Submitted to IEEE Trans. Aut. Cont.*, 2008.

[9] J. N. Tsitsiklis, "Problems in decentralized decision making and computation," Ph.D. dissertation, Massachusetts Institute of Technology, 1985.

[10] J. E. Boillat, "Load balancing and poisson equation in a graph," *Concurrency: Practice and Experience*, vol. 2, no. 4, pp. 289–313, 1990.

[11] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agnets using neaqrst neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[12] S. Patterson, B. Bamieh, and A. El Abbadi, "Brief announcement: Convergence analysis of scalable gossip protocols," in *20th Int. Symp. on Distributed Computing*, 2006, pp. 540–542.

[13] P. Barooah and J. P. Hespanha, "Graph effective resistance and distributed control: Spectral properties and applications," in *Proc. of the 45th IEEE Conf. on Decision and Control*, December 2006, pp. 3479–3485.

[14] M. Jovanovic and B. Bamieh, "On the ill-posedness of certain vehicular platoon control problems," *IEEE Trans. Automatic Control*, vol. 50, no. 9, 2005.

[15] B. Bamieh, F. Paganini, and M. A. Dahleh, "Distributed control of spatially-invariant systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 7, pp. 1091–1107, July 2002.

Appendix

Useful Facts

Fact 1: The following bounds are easy to establish

1) For any x and any $y \in [0, \pi]$ we have the following bounds

$$1 - \cos(x) \;\le\; x^2, \qquad\qquad 1 - \cos(y) \;\ge\; (2/\pi^2)\, y. \tag{21}$$

2) Given d integers $n_1, \ldots, n_d$, we have the bound

$$(n_1 + \cdots + n_d)^2 \;\le\; (2d+1)\,(n_1^2 + \cdots + n_d^2) \tag{22}$$

3) In the limit of large N,

$$\sum_{\substack{n \neq 0 \\ n \in \mathbb{Z}_N^d}} \frac{1}{(n_1^2 + \cdots + n_d^2)} \;\approx\; \begin{cases} \frac{1}{d-2}(N^{d-2} - 1) & d \neq 2 \\ N^{d-2}\log(N) & d = 2 \end{cases}, \tag{23}$$

where $f(N) \;\approx\; g(N)$ is notation for

$$\underline{c}\; g(N) \;\le\; f(N) \;\le\; \bar{c}\; g(N),$$

for some constants $\bar{c}$ and $\underline{c}$ and all $N \ge \bar{N}$ for some $\bar{N}$.