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Brief paper

Synchronization of diffusively-coupled limit cycle oscillators☆

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ABSTRACT

We develop analytical and numerical conditions to determine whether limit cycle oscillations synchronize in diffusively coupled systems. We examine two classes of systems: reaction–diffusion PDEs with Neumann boundary conditions, and compartmental ODEs, where compartments are interconnected through diffusion terms with adjacent compartments. In both cases the uncoupled dynamics are governed by a nonlinear system that admits an asymptotically stable limit cycle. We provide two-time scale averaging methods for certifying stability of spatially homogeneous time-periodic trajectories in the presence of sufficiently small or large diffusion and develop methods using the structured singular value for the case of intermediate diffusion. We highlight cases where diffusion stabilizes or destabilizes such trajectories.

1. Introduction

Diffusively coupled models are crucial for understanding the dynamical behavior of a range of engineering and biological systems. In particular, synchronization of diffusively coupled models is an active and rich research area (Hale, 1997). Conversely, developing conditions that rule out synchrony is also important, as they can facilitate study of spatial pattern formation. One of the major ideas behind pattern formation in cells and organisms is based on diffusion-driven instability (Segel & Jackson, 1972; Turing, 1952), which occurs when higher-order spatial modes in a reaction–diffusion partial differential equation (PDE) are destabilized by diffusion (Cross & Hohenberg, 1993; Hsia, Holtz, Huang, Arcak, & Maharbiz, 2012; Jovanović, Arcak, & Sontag, 2008; Murray, 2002; Othmer, Painter, Umulis, & Xue, 2009).

The majority of synchronization studies address phase coupled oscillators (Chopra & Spong, 2009; Dörfler & Bullo, 2012; Kuramoto, 1975; Strogatz, 2000), which rely on the assumption of weak coupling to be able to represent the subsystems with a single phase variable. Full state models have been studied in Arcak (2011), Pogromsky and Nijmeijer (2001), Russo and Di Bernardo (2009), Scardovi, Arcak, and Sontag (2010), Stan and Sepulchre (2007) and Wang and Slotine (2005); however, these references derive global results that may be conservative when synchronization of trajectories close to a specific attractor, such as a limit cycle, is of interest. The Ref. Pecora and Carroll (1998) gives a method to determine synchronization applicable to a wide class of coupled oscillators; however, it does not allow direct determination of synchronization for intervals of diffusion coefficients and may encounter loss of accuracy due to difficulties in numerically computing the state transition matrix.

In this paper, we study diffusively coupled nonlinear systems that exhibit limit cycles in the absence of diffusion. We develop analytical and numerical tools to determine whether diffusion stabilizes the spatially homogeneous limit cycle trajectories, thereby synchronizing the oscillations across the spatial domain. Our methods apply to reaction–diffusion PDEs with Neumann boundary conditions as well as compartmental ODEs. In the latter case, each compartment has identical dynamics and represents a well-mixed spatial domain wherein like components in adjacent compartments are coupled by diffusion.

We first linearize the system about an asymptotically stable limit cycle trajectory and then study the resulting periodic linear time varying system. In both the PDE and ODE cases, synchrony amounts to stability of an auxiliary system of the form

$$\dot{x} = (A(t) - \lambda_k D)x, \tag{1}$$

where $A(t)$ is periodic, D is a matrix of diffusion coefficients, and λ_k is the kth eigenvalue of the Laplacian operator (for PDEs) or

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matrix (for ODEs). In the case of sufficiently small or large diffusion, we use Floquet theory to decompose the linearized system into fast and slow time scales, and present results using *two-time scale averaging theory* (Sastry & Bodson, 1989; Teel, Moreau, & Nešić, 2003) that guarantee synchrony. In the case of diffusion coefficients of intermediate strength, we turn to a numerical approach, in which we use *harmonic balance* (Wereley & Hall, 1990; Zhou & Hagiwara, 2002) to represent the linearized system as an infinite-dimensional linear time invariant system.

We make use of concepts from robust control, in particular the structured singular value (SSV) (Packard & Doyle, 1993), to determine stability of the linearized system in the presence of diffusion coefficients spanning a specified finite interval. In particular, our method extends the notion of structured singular value to the infinite-dimensional harmonic transfer operators that may be used to describe the frequency domain behavior of periodic linear time-varying systems. We apply our tests to a relaxation oscillator system and find that large enough diffusion can indeed lead to loss of synchrony. Unlike standard examples of diffusion-driven instability of a homogeneous steady-state (Murray, 2002; Segel & Jackson, 1972; Turing, 1952), this example demonstrates destabilization of a spatially homogeneous periodic orbit by diffusion.

The stability of (1) in which the matrix $A(t)$ is constant has been studied in the literature. In Casten and Holland (1977), the authors showed that the stability of a homogeneous steady-state in a reaction–diffusion PDE with Neumann boundary conditions is equivalent to the simultaneous stability of a family of matrices of the form (1). In this case, the matrix $A(t) = A$ is constant because it represents the Jacobian linearization of the reaction terms at the steady-state. In the typical case where the matrix D of diffusion coefficients is diagonal, a sufficient condition for the desired simultaneous stability property is that A be an additively D-stable matrix (Kaszkurewicz & Bhaya, 2000), which means that $A - D$ is Hurwitz for all diagonal $D \geq 0$. While recent work has sought to characterize additive D-stability for constant matrices (Ge & Arcak, 2009; Kim & Braatz, 2012; Wang & Li, 2001), the periodic time-varying case addressed here has not been studied.

The paper is organized as follows. In Section 2, we formulate the problem and present an example of a system with an asymptotically stable limit cycle that loses spatial synchrony in the presence of diffusion. In Section 3, we outline tests for synchrony in the case of sufficiently small or large perturbations. In Section 4, we develop a method to verify synchrony for an interval of diffusion coefficients. We present relaxation and ring oscillator examples in Section 5 and give the conclusions in Section 6. Derivations of our application of structured singular value to periodic linear time-varying systems and harmonic transfer operators are given in the Appendix.

2. Problem formulation

In this section, we formulate the problem of synchronization of limit cycle oscillations in diffusively-coupled systems. For both reaction–diffusion systems of PDEs with Neumann boundary conditions and compartmental systems of ODEs, we show that determining synchrony, in the sense of a system exhibiting spatially homogeneous oscillations, amounts to examining stability of a linear system with time-periodic coefficients. To motivate our developments, we also provide an example of a system with an asymptotically stable limit cycle that loses spatial synchrony in the presence of large enough diffusion.

We first discuss systems governed by reaction–diffusion PDEs and define the spatial domain $\Omega \in \mathbb{R}^r$ with smooth boundary $\partial\Omega$, spatial variable $\xi \in \Omega$, and outward normal vector $n(\xi)$ for $\xi \in \partial\Omega$. The PDE model is

$$\frac{\partial x}{\partial t} = f(x) + D\nabla^2 x, \tag{2}$$

subject to Neumann boundary conditions $\nabla x_i(t, \xi) \cdot n(\xi) = 0$ for all $\xi \in \partial\Omega$, where $x(t, \xi) \in \mathbb{R}^n$, $D \in \mathbb{R}^{n\times n}$, and

$$\nabla^2 x = [\nabla^2 x_1 \cdots \nabla^2 x_n]^T \tag{3}$$

is a vector of Laplacian operators with respect to the spatial variable ξ applied to each entry of x. In a reaction–diffusion system, $x(t, \xi)$ represents a vector of concentrations for the reactants and D is a diagonal matrix of diffusion coefficients. However, for generality of our derivations, we will not assume D to be diagonal unless we state otherwise.

We say that a solution $x(t, \xi)$ of (1) synchronizes if $x(t, \xi_j) - x(t, \xi_k) \to 0$ for any two points ξ_j and ξ_k in Ω. We assume that the lumped system $\dot{x} = f(x)$ has an asymptotically stable limit cycle and that $\bar{x}(t)$ is a solution of $\dot{x} = f(x)$ along the limit cycle. Then $x(t, \xi) = \bar{x}(t)$ for all $\xi \in \Omega$ is a solution of (2). In the absence of diffusion ($D = 0$), the system (2) admits out-of-phase oscillations, that is, solutions of the form $x(t, \xi) = \bar{x}(t + \varphi(\xi))$, where $\varphi(\xi)$ is a phase that depends on the location ξ. To determine whether diffusion eliminates such spatial phase differences, we examine the Jacobian linearization about the limit cycle trajectory $\bar{x}(t, \xi)$:

$$\frac{\partial \tilde{x}}{\partial t} = (A(t) + D\nabla^2)\tilde{x} \tag{4}$$

where $\tilde{x}(t, \xi) = x(t, \xi) - \bar{x}(t)$ and

$$A(t) = J(\bar{x}(t)) = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}(t)}. \tag{5}$$

Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots$ denote the eigenvalues and $\phi_1(\xi),\ \phi_2(\xi),\ \ldots$ denote the corresponding orthogonal eigenfunctions of the operator $L = -\nabla^2$ on Ω with Neumann boundary conditions:

$$L\phi_i(\xi) = \lambda_i\phi_i(\xi), \qquad \nabla\phi_i(\xi)\cdot n(\xi) = 0 \quad \text{for all } \xi \in \partial\Omega. \tag{6}$$

The solution to (4) can be expressed as

$$\tilde{x}(t, \xi) = \sum_{i=1}^{\infty} \sigma_i(t)\phi_i(\xi), \tag{7}$$

where $\sigma_i(t) \in \mathbb{R}^n$ satisfy the decoupled system of ODEs:

$$\dot{\sigma}_i = (A(t) - \lambda_i D)\sigma_i, \quad i = 1, 2, \ldots. \tag{8}$$

Since the eigenfunction $\phi_1(\xi)$ for $\lambda_1 = 0$ is constant, the term corresponding to $i = 1$ represents a spatially homogeneous mode σ_1 governed by $\dot{\sigma}_1 = A(t)\sigma_1$. When the subsystems (8) are asymptotically stable for $i = 2, 3, \ldots$, the contributions of the inhomogeneous modes $\phi_2(\xi),\ \phi_3(\xi), \ldots$ to the solution $\tilde{x}(t, \xi)$ decay to zero in time, which implies that $x(t, \xi)$ synchronizes.

We also study a compartmental ODE model, where each compartment represents a well-mixed spatial domain interconnected with the other compartments over an undirected graph:

$$\dot{x}_i = f(x_i) + D\sum_{j\in N_i}(x_j - x_i), \quad i = 1, \ldots, N. \tag{9}$$

The vector $x_i \in \mathbb{R}^n$ represents each compartment's state, $\mathcal{N}_i$ denotes the neighbors of compartment i, and $D \in \mathbb{R}^{n\times n}$. We say that a solution $(x_1(t), \ldots, x_N(t))$ synchronizes if $x_j(t) - x_k(t) \to 0$ for any pair (j, k). We take the Jacobian linearization about a limit cycle trajectory $\bar{x}(t)$ and aggregate the dynamics of the subsystems using the state variable $\tilde{x} = [\tilde{x}_1^T \cdots \tilde{x}_N^T]^T$, $\tilde{x}_i(t) = x_i(t) - \bar{x}(t)$. We represent the interaction between state variables by a graph Laplacian matrix $L = L^T \in \mathbb{R}^{N\times N}$, defined as

$$L = EE^T, \tag{10}$$

where E is an incidence matrix whose rows represent vertices (compartments) and columns represent edges (couplings between the compartments). The dynamics of the aggregated system may be written as

$$\dot{\tilde{x}} = (I \otimes A(t) - L \otimes D)\tilde{x}, \tag{11}$$

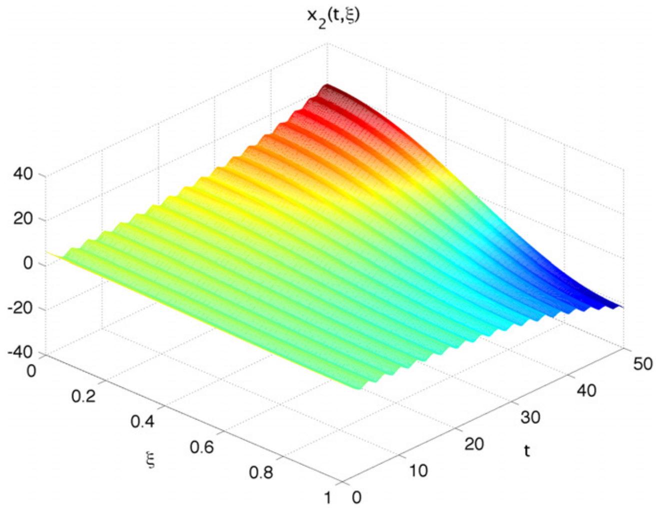

Fig. 1. Spatio-temporal evolution of x_2 for system (2)–(14) with $d_1 = 100$, $d_2 = 0$, and $\mu = 0.1$ on the one-dimensional spatial domain $\Omega = [0, 1]$ with initial condition $x_2(0, \xi) = 5 + \cos(\pi\xi)$ and Neumann boundary conditions. The oscillations do not synchronize, and in fact growth of the spatial mode $\phi_2(\xi)$ is observed.

where $A(t)$ is as in (5) and $\otimes$ denotes the Kronecker product. Let $U \in \mathbb{R}^{N\times N}$ be a unitary similarity transformation that brings L into the diagonal matrix of its eigenvalues $\Sigma \in \mathbb{R}^{N\times N}$: $L = U\Sigma U^T$. Choosing $\tilde{y} = (U^{-1} \otimes I)\tilde{x}$, we rewrite (11) as a block diagonal system, making use of the Kronecker product identity $(M \otimes S)(T \otimes W) = MT \otimes SW$ for matrices M, T and S, W of conformable dimensions, respectively. We then have

$$\dot{\tilde{y}} = (I \otimes A(t) - \Sigma \otimes D)\tilde{y}, \tag{12}$$

which is decoupled into the subsystems

$$\dot{\tilde{y}}_l = (A(t) - \lambda_l D)\tilde{y}_l, \quad l = 1, \ldots, N, \tag{13}$$

where $\tilde{y}_l \in \mathbb{R}^n$ and λ_l is the lth eigenvalue of the Laplacian matrix, respectively. In particular, $\lambda_1 = 0$ and $\lambda_l > 0$, $l = 2, 3, \ldots, N$ when the graph is connected. Note that (13) is analogous to (8) except that it consists of finitely many modes $l = 1, \ldots, N$. If the subsystems (13), $l = 2, \ldots, N$, are asymptotically stable, then for any pair $(j, k) \in \{1, \ldots, N\}\times\{1, \ldots, N\}$, we have $x_j(t)-x_k(t) \to 0$ exponentially as $t \to \infty$, which implies that $(x_1(t), \ldots, x_N(t))$ synchronizes.

Motivating example

To see that a diagonal $D \succeq 0$ does not necessarily guarantee synchronization, consider the system (2) with the dynamics

$$f(x) = \begin{bmatrix} \dfrac{1}{\mu}\left(x_1 - \dfrac{1}{3}x_1^3 - x_2\right) \\ x_1 + \mu x_2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} d_1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{14}$$

with $d_1 > 0$. When $\mu > 0$ is sufficiently large, the vector field $f(x)$ has the behavior of a *relaxation oscillator* (Khalil, 2002) and admits a stable limit cycle. The Jacobian linearization about the limit cycle trajectory $\bar{x}(t)$ is given by

$$A(t) = \begin{bmatrix} \dfrac{1}{\mu}(1 - \bar{x}_1^2(t)) & \dfrac{1}{\mu} \\ 1 & \mu \end{bmatrix}. \tag{15}$$

When $\lambda_i d_1 \gg 1/\mu$, system (8) exhibits two-time scale behavior, with the slow dynamics unstable:

$$\dot{\sigma}_{i2} = \mu\sigma_{i2}. \tag{16}$$

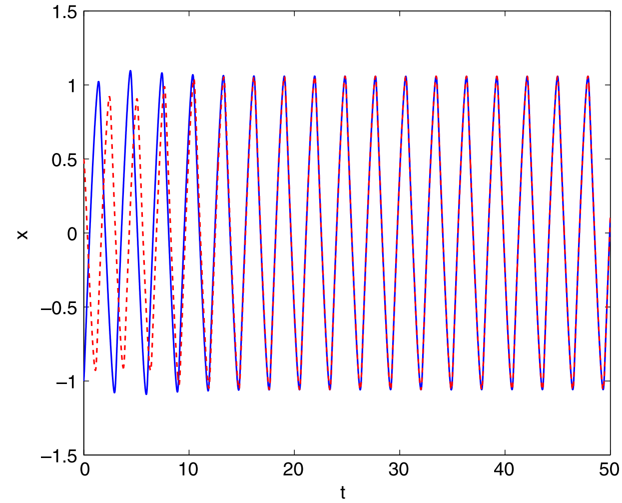

Fig. 2. Trajectories of x_{12} (blue, solid) and x_{22} (red, dashed) of (9) and (14) for two compartments synchronize under small diffusion coefficient $d_1 = .5$ and initial conditions $(x_{12}(0), x_{22}(0)) = (-1, .5)$.

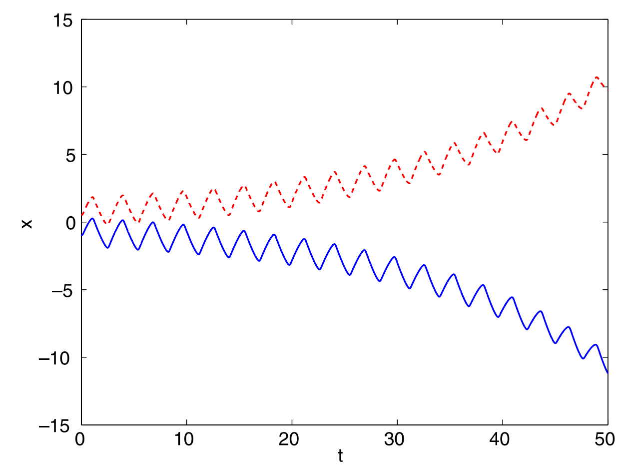

Fig. 3. Trajectories of x_{12} (blue, solid) and x_{22} (red, dashed) of (9) and (14) for two compartments do not synchronize under larger diffusion coefficient $d_1 = 100$ and initial conditions $(x_{12}(0), x_{22}(0)) = (-1, .5)$.

Thus, we expect the system (2), with $f(x)$ and D as in (14), to be unstable when $\lambda_i d_1$ is sufficiently large. Indeed, for $d_1 = 100$ and $\mu = 0.1$, the simulations over the spatial domain $[0, 1]$ demonstrate the growth of the spatial mode $\phi_2(\xi) = \cos(\pi\xi)$; see Fig. 1. Unlike standard examples of diffusion-driven instability of a homogeneous steady-state (Murray, 2002; Segel & Jackson, 1972; Turing, 1952), this example demonstrates destabilization of a spatially homogeneous periodic orbit by diffusion.

Similar behavior can be observed for the compartmental model (9) with two compartments, and $f(x)$ and D given by (14). The two-node graph representing the interconnection of the two compartments has Laplacian eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$. When d_1 is small, we find that oscillations synchronize spatially, as shown in Fig. 2. When d_1 is large, the trajectories corresponding to compartments one and two diverge from each other, as shown in Fig. 3.

3. Synchronization under weak or strong coupling

As shown in the previous section, for both the PDE (2) and the compartmental ODE (9), synchrony is determined by the stability of the time-varying system (1). For simplicity of notation we drop λ_k from (1) and analyze

$$\dot{x} = (A(t) - D)x, \tag{17}$$

since D can be appropriately scaled to account for λ_k.

When D is sufficiently small or large, we use Floquet theory to decompose (17) into fast and slow time scales and develop stability conditions using two-time scale averaging theory.

Recall that $A(t) = \frac{\partial f}{\partial x}\big|_{\bar{x}(t)}$ is the linearization of $f(x)$ about a limit cycle trajectory $\bar{x}(t)$, and let T denote the period of oscillations: $A(t+T) = A(t)$ for all t. We first consider the case with $D = 0$, that is

$$\dot{x} = A(t)x, \tag{18}$$

and note that it admits the periodic solution $x(t) = \dot{\bar{x}}(t)$. To see this, observe the following:

$$\dot{\bar{x}}(t) = f(\bar{x}(t)) \Longrightarrow \ddot{\bar{x}}(t) = \frac{\partial f}{\partial x}\bigg|_{\bar{x}(t)} \dot{\bar{x}}(t) = A(t)\dot{\bar{x}}(t). \tag{19}$$

Floquet's Theorem (Farkas, 1994, Thm. 2.2.5) implies that the state transition matrix $\Phi(t, t_0)$ of (18) is periodic and can be written as

$$\Phi(t, t_0) = U(t)\exp(F(t - t_0))V(t_0), \tag{20}$$

where $F \in \mathbb{R}^{n\times n}$ is a constant matrix, $U(t+T) = U(t) \in \mathbb{R}^{n\times n}$ and $U(t) = V^{-1}(t) \in \mathbb{R}^{n\times n}$, with the columns of $U(t)$ given by $u_i(t)$ and the rows of $V(t)$ given by $v_j^T(t)$. Since (18) results from linearization about a stable limit cycle, F can be written as

$$F = \begin{bmatrix} 0 & 0 \\ 0 & F_2 \end{bmatrix}, \tag{21}$$

where F_2 is an $(n-1)\times(n-1)$ Hurwitz matrix and $u_1(t) = \dot{\bar{x}}(t)$. The eigenvalues of F are called *Floquet exponents*, and the evaluation of the state transition matrix over one period with initial condition t_0, $\Phi(t_0 + T, t_0) = \exp(FT)$, is called the *monodromy matrix*.

In what follows, we derive a condition that relates the stability of (17) with sufficiently small D to $u_1(t)$ and $v_1(t)$. First, we review properties of $u_1(t)$ and $v_1(t)$ that follow from Floquet theory. The definition of $U(t)$ and $V(t)$ implies that $v_j^T(t)u_i(t) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. In particular, $v_1(t)$ is a periodic solution of the adjoint system:

$$\dot{\rho} = -A^T(t)\rho. \tag{22}$$

To compute $u_1(t)$ and $v_1(t)$, we follow (Demir, Mehrotra, & Roychowdhury, 2000) and numerically integrate

$$\frac{\partial}{\partial t}\Phi(t, t_0) = A(t)\Phi(t, t_0) \tag{23}$$

over one period with the initial condition $\Phi(t_0, t_0) = I$. We then compute the eigenvector of the monodromy matrix corresponding to its eigenvalue at one:

$$u_1(t_0) = \Phi(t_0 + T, t_0)u_1(t_0). \tag{24}$$

Using the numerically-computed state transition matrix $\Phi(t, t_0)$, we then calculate the trajectory $u_1(t) = \Phi(t, t_0)u_1(t_0)$. To obtain $v_1(t)$, we begin by computing the left eigenvector of the monodromy matrix corresponding to its eigenvalue at one:

$$v_1^T(t_0)\Phi(t_0 + T, t_0) = v_1^T(t_0). \tag{25}$$

We scale $v_1(t_0)$ such that $v_1^T(t_0)u_1(t_0) = 1$. Finally, to obtain $v_1(t)$, we numerically integrate the adjoint system (22) backwards in time with the terminal condition $\rho(t_0 + T) = v_1(t_0)$.

Having reviewed the case $D = 0$, we now prove a result about the stability of (18) with sufficiently small D.

Proposition 3.1. *Let $v_1^T(t)$ be the first row of $V(t)$ and $u_1(t)$ be the first column of $U(t)$, where $\Phi(t, t_0) = U(t)\exp(F(t - t_0))V(t_0)$ as described above. Given a matrix $D_0 \in \mathbb{R}^{n\times n}$, if the inequality*

$$\int_{t_0}^{t_0+T} v_1^T(t)D_0u_1(t)\,dt > 0 \tag{26}$$

holds, then the origin of the system

$$\dot{x} = (A(t) - \epsilon D_0)x \tag{27}$$

is exponentially stable for sufficiently small $\epsilon > 0$.

Proof. Floquet theory implies that the time-varying change of coordinates $y = V(t)x$ transforms (18) into a linear time invariant system:

$$\dot{y} = Fy, \tag{28}$$

where F is as in (21). Introducing the decomposition $y = [w^T \;\; z^T]^T$, we rewrite (28) as

$$\begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}. \tag{29}$$

When applied to system (27), the preceding change of coordinates yields

$$\begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix} = \left(\begin{bmatrix} 0 & 0 \\ 0 & F_2 \end{bmatrix} - \epsilon V(t)D_0U(t) \right) \begin{bmatrix} w \\ z \end{bmatrix}. \tag{30}$$

For small ϵ, this time-varying periodic system exhibits two-time scale behavior, which allows us to exploit the theory of two-time scale averaging (Sastry & Bodson, 1989; Teel et al., 2003). The *averaged slow system* corresponding to (30) is given by

$$\dot{w} = -\epsilon a w, \qquad a = \frac{1}{T}\int_{t_0}^{t_0+T} v_1^T(t)D_0u_1(t)dt. \tag{31}$$

Since F_2 is Hurwitz, an application of Lemma A.1 in Appendix A shows that if $a > 0$, then the equilibrium $y = 0$ is exponentially stable for sufficiently small ϵ. □

Note that Proposition 3.1 does not require D_0 to be diagonal. When D_0 is diagonal, the test (26) can be simplified as follows:

Corollary 3.2. *Let u_{1i} and v_{1j}^T be the ith and jth components of u_1 and v_1^T, respectively. If the inequalities*

$$\int_{t_0}^{t_0+T} v_{1i}^T(t)u_{1i}(t)\,dt > 0, \quad i = 1, \ldots, n \tag{32}$$

hold, then given any diagonal matrix $D_0 \succeq 0$, $D_0 \neq 0$, the periodic solution of the linearized system (27) is stable for sufficiently small $\epsilon > 0$.

We now turn to the case where D is large. Standard results from perturbation theory (Khalil, 2002) guarantee stability of (17) when D is nonsingular and sufficiently large. When D is singular, we again leverage two-time scale arguments to derive a condition that guarantees stability of (17):

Proposition 3.3. *Given a matrix $D_0 \in \mathbb{R}^{n\times n}$, consider the linear time varying system:*

$$\dot{x} = (A(t) - \epsilon^{-1}D_0)x \tag{33}$$

$$A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix}, \qquad D_0 = \begin{bmatrix} 0 & 0 \\ 0 & D_2 \end{bmatrix}, \tag{34}$$

where $x \in \mathbb{R}^n$, $A(t+T) = A(t)$ for all t, $A_{22}(t)$ and D_2 have the same dimension, $-D_2$ is Hurwitz, and $\epsilon > 0$. If

$$\bar{A}_{11} = \frac{1}{T}\int_{t_0}^{t_0+T} A_{11}(t)\,dt \tag{35}$$

is Hurwitz, then $x = 0$ is an exponentially stable equilibrium of (33) for sufficiently small ϵ.

The proof follows from Lemma A.1. Note that if D_0 is not block diagonal, but is singular with trivial Jordan blocks corresponding to its eigenvalues at zero and all remaining eigenvalues in the closed right half plane, there exists a similarity transformation that will bring (33) to the form required by (34).

The tests that we have derived, while analytic in nature, may be applied to problems of interest by computing a linearization about a periodic solution, and numerically integrating the resulting differential equation (23) in order to obtain u_1 and v_1. We demonstrate the application of these tests in Section 5.

4. Numerical verification of synchronization using SSV

In this section, we develop numerical tools to determine the stability of (17) for a family of matrices D parametrized as

$$D = M + B\Delta C, \tag{36}$$

where $M \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{m\times n}$ are fixed matrices, and $\Delta \in \mathbb{R}^{m\times m}$ is a diagonal matrix whose entries take values in $[-1, 1]$. For example, suppose that the system (17) has one diffusible component, with

$$D = \mathrm{diag}([d_1 0 \dots 0]), \tag{37}$$

where $d_1 \in [r, R]$. Then D can be written as in (36) with $M = \frac{R+r}{2} e_1 e_1^T$ where e_i is a standard basis vector, $B = \left[\frac{R-r}{2}\ \ 0 \dots 0\right]^T$, $C = [1\ \ 0 \dots 0]$, and $\Delta = \delta$ is a scalar. The problem is then to ascertain that the system (17) is stable for all values of δ on the interval $[-1, 1]$.

Structured singular value (SSV) analysis provides a useful test for determining the robustness of a stable linear time invariant system to structured modeling uncertainty. However, since (38) is time-varying, in order to apply SSV analysis directly we must first bring the system to an equivalent time invariant form. For such analysis, it is useful to rewrite the system (17) as

$$\begin{aligned} \dot{x} &= (A(t) - M)x - Bq \\ y &= Cx \\ q &= \Delta y. \end{aligned} \tag{38}$$

Previous efforts to apply SSV analysis to time-varying systems have focused on the time-domain *lifting* idea of Bamieh, Pearson, Francis, and Tannenbaum (1991) and Chen and Francis (1995), outlined in Dullerud and Glover (1996), Kim, Bates, and Postlethwaite (2006) and Ma and Iglesias (2002), where system (38) is discretized and converted to a continuous time invariant system.

Instead, we pursue an SSV analysis that makes use of the *harmonic balance* approach (Wereley & Hall, 1990) and frequency domain lifting as in Fardad, Jovanovic, and Bamieh (2008) and avoids the numerical difficulties and sensitivity of computing the state transition matrix and discretizing with an adequate number of samples in the lifting approach. Our computational experiments show that the harmonic balance approach frequently leads to less conservative results in establishing the values of diffusion coefficients that lead to instabilities. We first give a brief summary of harmonic balance and then outline its application to the problem of determining the stability of (17).

We assume that each entry of the matrix $A(t)$ is a continuous function of t that has an absolutely convergent Fourier series, and so $A(t)$ may be expressed as

$$A(t) = \sum_{m\in\mathbb{Z}} A_m e^{jm\omega_p t}, \tag{39}$$

where ω_p is the fundamental frequency. Define doubly infinite vectors representing the harmonics of the state:

$$X = [\cdots x_{-1}^T\, x_0^T\, x_1^T\ \cdots]^T, \tag{40}$$

and do the same for the input Q and output Y. The doubly infinite block Toeplitz matrix $\mathcal{A}$ is determined by the harmonics of $A(t)$:

$$\mathcal{A} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \dots & A_0 & A_{-1} & A_{-2} & \dots \\ \dots & A_1 & A_0 & A_{-1} & \dots \\ \dots & A_2 & A_1 & A_0 & \dots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{41}$$

We define the doubly infinite matrices $\mathcal{I} = \mathrm{blkdiag}(I)$, $\mathcal{B} = \mathrm{blkdiag}(B)$, and $\mathcal{C} = \mathrm{blkdiag}(C)$, and define the modulation frequency matrix as

$$\mathcal{N} = \mathrm{blkdiag}\{jm\omega_p I\}, \quad \forall m \in \mathbb{Z}. \tag{42}$$

We define the matrix $\tilde{\Delta} = \mathrm{blkdiag}(\Delta)$ to be block diagonal with copies of the diagonal matrix Δ in each block, and the matrix $\mathcal{M} = \mathrm{blkdiag}(M)$ to be a block diagonal scaling matrix with copies of the matrix M in each block. We now introduce the harmonic state space model, where $s = j\omega$:

$$\begin{aligned} sX &= (\mathcal{A} - \mathcal{M} - \mathcal{N})X - \mathcal{B}Q \\ Y &= \mathcal{C}X \\ Q &= \tilde{\Delta}Y. \end{aligned} \tag{43}$$

We perform SSV analysis to determine if there exist matrices D such that (17) is unstable. For the precise definition of the structured singular value in the context of periodic linear-time varying systems represented by a harmonic state space model, we refer the reader to Appendix B. To obtain a computationally tractable test, we truncate the doubly infinite system. As shown in Appendix B, we may approximate (43) arbitrarily well. In the examples we consider there exist fewer than ten significant harmonics, and we represent the doubly infinite system by a finite dimensional system. We then perform SSV analysis on the truncated version of (43) to determine the range of matrices Δ for which (17) remains stable. In particular, we use the MATLAB command *mussv* in the Robust Control Toolbox, which performs SSV analysis to test if there exists Δ such that (43) is unstable. We summarize our procedure in Algorithm 1.

Algorithm 1 Numerical verification of synchrony using harmonic balance

1: Using the parametric decomposition (36) for the given family of matrices D under consideration, determine the matrices B and C in order to express (17) in the form of (38).
2: Determine the Fourier series coefficients of $A(t)$.
3: Define the truncated linear time invariant harmonic state space model as in (43).
4: Compute the structured singular value μ of the harmonic state space model.

Following the completion of Algorithm 1, if $\mu > 1$, we compute the corresponding matrix Δ with smallest norm such that the truncated harmonic state space model is unstable. We then use the computed matrix Δ and (36) to compute a candidate for a matrix D that makes (17) unstable. If $\mu \leq 1$, appealing to the convergence properties in Appendix B provides evidence that system (17) is stable. The choice of the number of terms in the truncation resulting from Step 3 involves a tradeoff between numerical accuracy and computation time.

Since the problem of computing the structured singular value of a system is NP complete (Packard & Doyle, 1993), the Robust Control Toolbox employs linear matrix inequality relaxations as well as a discretization of the continuous frequency domain, which can lead to numerical inaccuracies and conservatism. This conservatism can pose a problem in certifying stability over large intervals. In many cases, it may be necessary to perform SSV analysis on smaller intervals, and to certify the remaining (possibly infinite) interval using the perturbation arguments of Section 3.

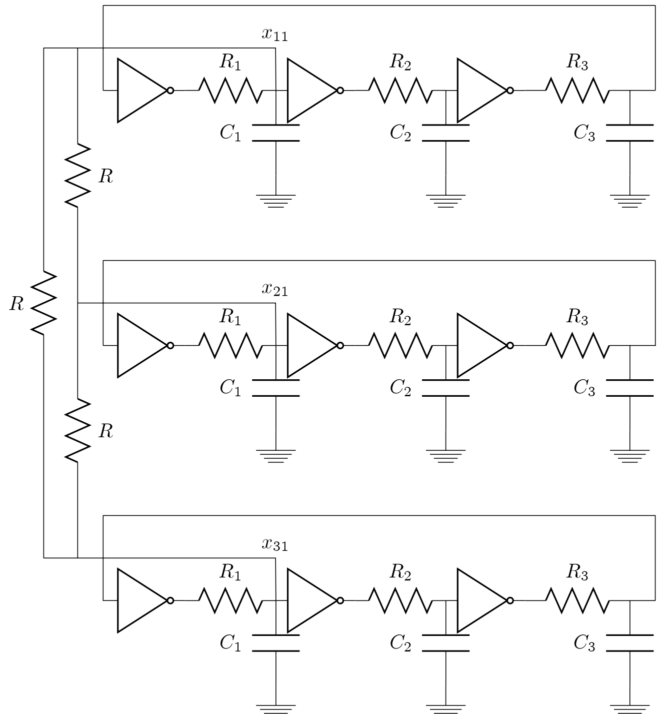

Fig. 4. Three-stage ring oscillators as in (44) coupled through node 1.

5. Examples

Example 1 (*Relaxation Oscillator*)**.** We first discuss numerical results for the *relaxation oscillator* example given by (14) in Section 2. We set the parameter $\mu = 0.1$, and first study the two compartment ODE model (9). When D is small, the techniques of Section 3 apply, and we can easily check that the conditions of Corollary 3.2 are satisfied for nonnegative $\lambda_i d_1 < \epsilon^*$, where ϵ^* is computed from the proof of Lemma A.1. In Fig. 2, we show the oscillations of the solution of x_2 synchronizing spatially under small D, as expected.

We next examine the case of larger D for both (9) and (2). To apply the harmonic balance method, we compute the harmonic components of $x_1(t)$ and find that eight harmonics are sufficient to represent the signal. We then use the harmonic expansion to generate a corresponding finite dimensional approximation of the matrix $\mathcal{A}$. Because D is diagonal and nonnegative, we set $M = \frac{r+\epsilon^*}{2} e_1 e_1^T$, $B = [\frac{r-\epsilon^*}{2} \;\; 0]^T$, $C = [1 \;\; 0]$, and $\Delta = \delta$, and perform SSV analysis to determine values of d_1 that lead to instabilities. We find that at $\lambda_i d_1 \geq 87.6$, stability is lost.

Indeed, when the product $\lambda_i d_1 \geq 87.6$, the two compartment ODE, with $\lambda_2 = 2$, will exhibit trajectories that diverge, and the reaction–diffusion PDE model, with $\lambda_i = (i-1)^2$, $i = 2, 3, \ldots$, will lose spatial uniformity for initial spatial modes with large enough wavenumber i regardless of d_1. In Figs. 1 and 3, we show that the oscillations of the solution of x_2 do not synchronize spatially for large D and observe increasing spatial inhomogeneity over time.

Example 2 (*Ring Oscillator*)**.** We next study a coupled three-stage *ring oscillator* model (Fig. 4), with the dynamics of each circuit given by

$$\begin{aligned} \dot{x}_{i1} &= -\eta_1 x_{i1} - \alpha_1 \tanh(\beta_1 x_{i3}) + w_{i1} \\ \dot{x}_{i2} &= -\eta_2 x_{i2} - \alpha_2 \tanh(\beta_2 x_{i1}) \\ \dot{x}_{i3} &= -\eta_3 x_{i3} - \alpha_3 \tanh(\beta_3 x_{i2}), \end{aligned} \tag{44}$$

with coupling at node 1 of each circuit. The parameters $\eta_i = \frac{1}{R_i C_i}$, α_1, and β_1 correspond to the gain of each inverter. The coupling is defined by

$$w_{i1} = -d_1 \left(\sum_{j \in \mathcal{N}_i} (x_{i1} - x_{j1}) \right), \tag{45}$$

where $d_1 = 1/(RC_1)$ and $\mathcal{N}_i$ denotes the set of circuits to which circuit i is connected. The Laplacian matrix describing the interaction between three coupled circuits as in Fig. 4 is given by

$$L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}. \tag{46}$$

Following (9), the vector field $f(x)$ is given by (44) with $D = \operatorname{diag}([d_1 \; 0 \; 0])$.

In order for (44) with w_{i1} to admit a limit cycle, it must have $\alpha_i \beta_i > 2$ (Ge, Arcak, & Salama, 2010). We set $\eta_i = 1$, $\alpha_i = 2$, and $\beta_i = 1.2$ for all i. When d_1 is small, we use the techniques of Section 3 to study the effects of small D on (9). Upon computing $u_1(t)$ and $v_1(t)$, it is readily seen that the conditions of Corollary 3.2 are satisfied. Thus, the equilibrium at $x = 0$ is exponentially stable for nonnegative $\lambda_i d_1 < \epsilon^*$, where ϵ^* is computed from the proof of Lemma A.1.

We next apply Proposition 3.3 to study the effects of large D on (9). Linearization about a limit cycle trajectory $\bar{x}(t)$ brings (44) to the form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = -\left(\begin{bmatrix} \lambda_i d_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & \gamma_1(\bar{x}_1) \\ \gamma_2(\bar{x}_2) & 1 & 0 \\ 0 & \gamma_3(\bar{x}_3) & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \tag{47}$$

with $\gamma_1(\bar{x}_1) = \alpha_1 \beta_1 \operatorname{sech}(\beta_1 \bar{x}_3)^2$, $\gamma_2(\bar{x}_2) = \alpha_2 \beta_2 \operatorname{sech}(\beta_2 \bar{x}_1)^2$, and $\gamma_3(\bar{x}_3) = \alpha_3 \beta_3 \operatorname{sech}(\beta_3 \bar{x}_2)^2$. When d_1 is large, the system exhibits two-time scale behavior. Since $D \succeq 0$ is diagonal and the averaged slow system corresponding to $[x_2 \;\; x_3]^T$, given by

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = -\left(\int_{t_0}^{t_0+T} \begin{bmatrix} 1 & 0 \\ \gamma_3(\bar{x}_3) & 1 \end{bmatrix} dt \right) \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}, \tag{48}$$

has an exponentially stable equilibrium at zero, we conclude from Proposition 3.3 that the equilibrium at $x = 0$ is exponentially stable for $\lambda_i d_1 > m^*$, where m^* is computed from the proof of Lemma A.1.

We use SSV analysis to certify synchrony for the remaining interval $[\epsilon^*, m^*]$. Following Section 3, we set $M = \frac{m^*+\epsilon^*}{2} e_1 e_1^T$, $B = \left[\frac{m^*-\epsilon^*}{2} \;\; 0 \ldots 0\right]^T$, $C = [1 \;\; 0 \ldots 0]$, and $\Delta = \delta$. A discrete Fourier transform of a periodic trajectory $\bar{x}(t)$ suggests that eight harmonics are sufficient to represent the truncated harmonic state space model for the time-varying system. SSV analysis indicates that coefficients $\lambda_i d_1 \in [\epsilon^*, m^*]$ will result in (9) being stable. In Fig. 5 we show an example of a coefficient $d_1 \in [\epsilon^*, m^*]$ with synchronized oscillations.

6. Conclusion

We have studied diffusively coupled compartmental ODEs as well as reaction–diffusion PDEs that admit stable limit cycles. We have established analytic tests using two-time scale averaging theory to study the case of weak or strong coupling. We then presented a numerical method applying the harmonic balance and structured singular value analysis on intervals of intermediate coupling strength to determine whether limit cycle oscillations synchronize. Finally, we applied our tests to examples, where we identified cases in which diffusion leads to loss of spatial synchrony. The effect of truncation of bi-infinite matrices on the accuracy of the numerical results is currently being studied. Our results

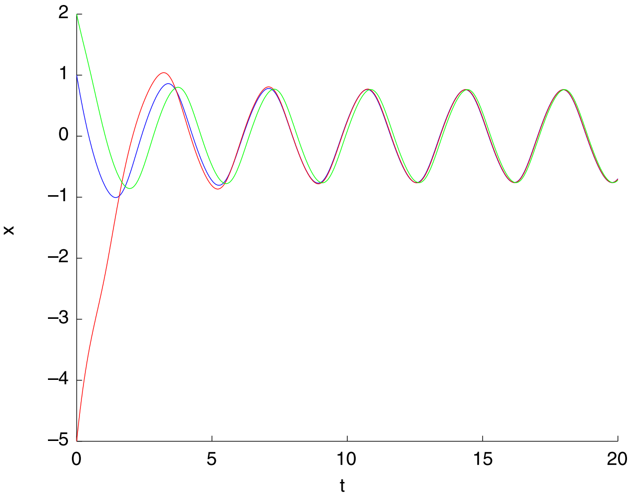

Fig. 5. Coupled identical three-staged ring oscillators as in (44) with $\eta_i = 1, \alpha_i = 2$, and $\beta_i = 1.2$ for all i. Oscillations synchronize with initial conditions $x_{11} = 1$ (blue), $x_{21} = -5$ (red), and $x_{31} = 2$ (green). For brevity we show only the first component.

could also be used to decide on coupling strengths in diffusively coupled systems, such as multiagent systems and voltage controlled oscillators, to guarantee synchrony.

Appendix A. Two-time scale averaging

We state a lemma that follows from standard results in two-time scale averaging (see, e.g., Sastry and Bodson (1989, Thm. 4.4.3)).

Lemma A.1. *Let $w \in \mathbb{R}^p$ and $z \in \mathbb{R}^q$, and consider the linear time varying system:*

$$\begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix} = \left(\begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix} - \epsilon \begin{bmatrix} H_{11}(t) & H_{12}(t) \\ H_{21}(t) & H_{22}(t) \end{bmatrix} \right) \begin{bmatrix} w \\ z \end{bmatrix}, \tag{A.1}$$

where each $H_{ij}(t)$, $i, j \in \{1, 2\}$ is a bounded piecewise continuous matrix-valued function of time such that $H_{ij}(t + T) = H_{ij}(t)$, $G \in \mathbb{R}^{q\times q}$, and $\epsilon > 0$. Define the associated averaged slow system:

$$\dot{w} = -\epsilon \bar{H}_{11} w, \qquad \bar{H}_{11} = \frac{1}{T}\int_{t_0}^{t_0+T} H_{11}(t)\,dt. \tag{A.2}$$

If $-\bar{H}_{11}$ and G are Hurwitz, then there exists ϵ^* such that $[w^T \ \ z^T]^T = 0$ is an exponentially stable equilibrium of (A.1) for $0 < \epsilon < \epsilon^*$.

Proof. We provide a proof for completeness and to exhibit a procedure for obtaining ϵ^*. We begin by introducing a change of coordinates:

$$w = \left[I - \epsilon \int_0^t \left(H_{11}(\tau) - \bar{H}_{11} \right) d\tau \right] \hat{w}. \tag{A.3}$$

Upon substitution, we have

$$\left[I - \epsilon \int_0^t \left(H_{11}(\tau) - \bar{H}_{11} \right) d\tau \right] \dot{\hat{w}} - \epsilon [H_{11}(t) - \bar{H}_{11}] \hat{w}$$
$$= -\epsilon H_{11}(t) \left[I - \epsilon \int_0^t \left(H_{11}(\tau) - \bar{H}_{11} \right) d\tau \right] \hat{w} - \epsilon H_{12}(t) z. \tag{A.4}$$

Since each $H_{ij}(t)$ is bounded, we have $|H_{ij}(t)| \le \hat{h}_{ij}$ for all t. Furthermore, since each $\int_0^t \left(H_{11}(\tau) - \bar{H}_{11} \right) d\tau$ is periodic, we have

$$\left| \int_0^t \left(H_{11}(\tau) - \bar{H}_{11} \right) d\tau \right| \le 2T\hat{h}_{11}. \tag{A.5}$$

Then for $\epsilon < \epsilon_1 \triangleq \frac{1}{2T\hat{h}_{11}}$, the change of coordinates is invertible. Rewriting, we have

$$\begin{aligned} \dot{\hat{w}} = & -\epsilon \left[I - \epsilon \int_0^t \left(H_{11}(\tau) - \bar{H}_{11} \right) d\tau \right]^{-1} \\ & \times H_{11}(t) \left[I - \epsilon \int_0^t \left(H_{11}(\tau) - \bar{H}_{11} \right) d\tau \right] \hat{w} \\ & - \epsilon \left[I - \epsilon \int_0^t \left(H_{11}(\tau) - \bar{H}_{11} \right) d\tau \right]^{-1} H_{12}(t) z \\ & - \epsilon \left[I - \epsilon \int_0^t \left(H_{11}(\tau) - \bar{H}_{11} \right) d\tau \right]^{-1} \\ & \times [\bar{H}_{11} - H_{11}(t)] \hat{w}. \end{aligned} \tag{A.6}$$

With a similar change of coordinates, we have

$$\dot{z} = Gz - \epsilon H_{21}(t) \left[I - \epsilon \int_0^t \left(H_{11}(\tau) - \bar{H}_{11} \right) d\tau \right] \hat{w} - \epsilon H_{22}(t) z. \tag{A.7}$$

Define the positive definite matrices P_w and P_z such that

$$\begin{aligned} & P_w \bar{H}_{11} + \bar{H}_{11}^T P_w = I \\ & P_z G + G^T P_z = -I. \end{aligned} \tag{A.8}$$

We next consider the candidate Lyapunov function

$$V = \hat{w}^T P_w \hat{w} + z^T P_z z. \tag{A.9}$$

Define the scalar $\gamma \triangleq \frac{1}{1 - 2\epsilon_1 T \hat{h}_{11}}$. Differentiating V, we have

$$\begin{aligned} \dot{V} \le & -(\epsilon\gamma - 2\epsilon^2 \gamma T \hat{h}_{11}^2) |\hat{w}|^2 + \epsilon\gamma \hat{h}_{12} |\hat{w}||z| \\ & - (1 - \epsilon \hat{h}_{22}) |z|^2 + \epsilon \hat{h}_{21} |z||\hat{w}| \\ & + \epsilon^2 (2T \hat{h}_{21} \hat{h}_{11}) |z||\hat{w}|. \end{aligned} \tag{A.10}$$

Because $\epsilon < \epsilon_1$, the first term of (A.10) is negative. Similarly, choosing $\epsilon < \epsilon_2 \triangleq \frac{1}{\hat{h}_{22}}$ guarantees that the second term of (A.10) is negative. If the condition

$$\begin{aligned} & \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} \prec 0, \quad \text{with} \\ & M_{11} = -(\epsilon\gamma - 2\epsilon^2 \gamma T \hat{h}_{11}^2) \\ & M_{12} = \frac{1}{2} \left(\epsilon\gamma \hat{h}_{12} + \epsilon \hat{h}_{21} + \epsilon^2 (2T \hat{h}_{21} \hat{h}_{11}) \right) \\ & M_{22} = -(1 - \epsilon \hat{h}_{22}), \end{aligned} \tag{A.11}$$

is satisfied, then (A.10) is negative. Using the Schur complement, we rewrite the condition as

$$-(\epsilon\gamma - 2\epsilon^2 \gamma T \hat{h}_{11}^2) + \frac{1}{4} \left(\epsilon\gamma \hat{h}_{12} + \epsilon \hat{h}_{21} + \epsilon^2 (2T \hat{h}_{21} \hat{h}_{11}) \right)^2 \times (1 - \epsilon \hat{h}_{22})^{-1} < 0. \tag{A.12}$$

Therefore, there exists $\epsilon_3 > 0$ such that for $0 < \epsilon < \epsilon_3$, (A.12) holds. Thus, with $\epsilon^* = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$, $[w^T \ \ z^T]^T = 0$ is an exponentially stable equilibrium of (A.1). □

Appendix B. Structured singular value for periodic systems using harmonic balance

To justify the SSV analysis proposed in Section 4, in this appendix we extend the concept of structured singular value to periodic systems using a generalized Nyquist result and a classic result from robust control. We begin by reviewing the Nyquist criterion for periodic linear time varying systems studied by Hall and Wereley (1990) and Zhou and Hagiwara (2005). We consider the system

$$\begin{aligned} \dot{x} &= A(t)x + B(t)q \\ y &= C(t)x \\ q &= -\Delta y, \end{aligned} \tag{B.1}$$

with $A(t), B(t), C(t) \in \mathbb{R}^{n\times n}$. We assume that $A(t), B(t)$, and $C(t)$ are periodic with period T and continuous with absolutely convergent Fourier series, and $\Delta \in \mathbb{R}^n$ with $|\Delta_{ij}| \leq 1$. The bi-infinite harmonic state space model of (B.1) is given by

$$\begin{aligned} (s\mathcal{I} + \mathcal{N})X &= \mathcal{A}X + \mathcal{B}Q \\ Y &= \mathcal{C}X \\ Q &= -\tilde{\Delta}Y, \end{aligned} \tag{B.2}$$

where the matrix $\tilde{\Delta} = \text{blkdiag}(\Delta)$, and $\mathcal{N}$, $\mathcal{I}$, and $\mathcal{A}$ as in Section 4, and $\mathcal{B}$ and $\mathcal{C}$ defined similarly to $\mathcal{A}$. The (infinite-dimensional) harmonic open loop transfer operator of the system is

$$G(s) = \mathcal{C}[s\mathcal{I} - (\mathcal{A} - \mathcal{N})]^{-1}\mathcal{B}. \tag{B.3}$$

We rewrite (B.2), substituting for Y:

$$\begin{bmatrix} I - (s\mathcal{I} + \mathcal{N})^{-1}\mathcal{A} & (s\mathcal{I} + \mathcal{N})^{-1}\mathcal{B} \\ \tilde{\Delta}\mathcal{C} & \mathcal{I} \end{bmatrix} \begin{bmatrix} X \\ Q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{B.4}$$

Following Zhou and Hagiwara (2005), suppose $\mathcal{M}$ is a linear compact operator. Denote by $\lambda_i(\mathcal{M})$ and $\sigma_i(\mathcal{M})$ the ith eigenvalue and singular value of $\mathcal{M}$, respectively. Denote by $C_p(l_2)$ the set of linear compact operators $\mathcal{M} : l_2 \to l_2$ that satisfy

$$\|\mathcal{M}\|_p = \left(\sum_i \sigma_i(\mathcal{M})^p\right)^{\frac{1}{p}} < \infty. \tag{B.5}$$

Elements in $C_1(l_2)$ are called trace-class operators, and the determinant

$$\det(\mathcal{I} + \mathcal{M}) \triangleq \Pi_{k=-\infty}^{\infty}(1 + \lambda_k) \tag{B.6}$$

of a trace-class operator is well-defined in the sense that it converges absolutely. Elements in $C_2(l_2)$ are called Hilbert–Schmidt operators and may not have absolutely-convergent determinants.

We next apply the Schur determinant lemma to the left hand side of (B.4):

$$\begin{aligned} \psi_{cl}(\Delta) &\triangleq \det\left(\mathcal{I} - (s\mathcal{I} + \mathcal{N})^{-1}\mathcal{A} - (s\mathcal{I} + \mathcal{N})^{-1}\mathcal{B}\tilde{\Delta}\mathcal{C}\right) \\ &= \det\left(\mathcal{I} - (s\mathcal{I} + \mathcal{N})^{-1}(\mathcal{A} - \mathcal{B}\tilde{\Delta}\mathcal{C})\right). \end{aligned} \tag{B.7}$$

The operator $\mathcal{H}(s) = (s\mathcal{I} + \mathcal{N})^{-1}(\mathcal{A} - \mathcal{B}\tilde{\Delta}\mathcal{C})$ is not in $C_1(l_2)$, and in particular, $\psi_{cl}(\Delta)$ does not converge absolutely. Thus, it is not possible to develop a Nyquist criterion making use of the standard infinite determinant $\psi_{cl}(\Delta)$. To deal with this problem, Zhou and Hagiwara (2005) proposed an approach using the 2-*regularized determinant*. Since $\mathcal{H}(s) \in C_2(l_2)$, it holds that $R_2(\mathcal{H}) = (\mathcal{I} + \mathcal{H})\exp(-\mathcal{H}) - \mathcal{I} \in C_1(l_2)$, and so the 2-regularized determinant $\det_2(\mathcal{I} + \mathcal{H}) \triangleq \det(\mathcal{I} + R_2(\mathcal{H}))$ is well-defined. We note that the Sylvester determinant lemma holds for the 2-regularized determinant: $\det_2(\mathcal{I} + \mathcal{K}\mathcal{H}) = \det_2(\mathcal{I} + \mathcal{H}\mathcal{K})$. Making use of the 2-regularized determinant, it can be shown that

$$\begin{aligned} \psi_G(\Delta) &\triangleq \det{}_2(\mathcal{I} + \tilde{\Delta}G(s))\exp(-r(s+\rho)) \\ &= \frac{\det_2(\mathcal{I} - ((s+\rho)\mathcal{I} + \mathcal{N})^{-1}(\mathcal{A} - \mathcal{B}\tilde{\Delta}\mathcal{C} + \rho\mathcal{I}))}{\det_2(\mathcal{I} - ((s+\rho)\mathcal{I} + \mathcal{N})^{-1}(\mathcal{A} + \rho\mathcal{I}))}, \end{aligned} \tag{B.8}$$

with $r(s+\rho) = -\text{tr}[((s+\rho)\mathcal{I} + \mathcal{N})^{-1}(\mathcal{A} + \rho\mathcal{I})(s\mathcal{I} + \mathcal{N} - \mathcal{A})^{-1}\mathcal{B}\tilde{\Delta}\mathcal{C}]$, and $\rho > 0$. The generalized Nyquist stability criterion for periodic linear time varying systems follows (see Zhou and Hagiwara (2005, Theorem 3.1)).

Theorem B.1. *Assume that $A(t)$, $B(t)$, and $C(t)$ are piecewise-continuously differentiable and have absolutely convergent Fourier series expansions. Let $\rho > 0$ be an arbitrary positive number, and let n_p be the number of unstable eigenvalues in the fundamental strip of the open loop operator $\mathcal{A} - \mathcal{N}$. The closed loop system (B.2) is asymptotically stable if and only if the Nyquist locus of $\psi_G(\Delta)$ is not zero for all points along the Nyquist contour about the closed right half plane intersected with the fundamental strip (Hall & Wereley, 1990):*

$$S \triangleq \left\{s \in \mathbb{C} : -\frac{\omega_p}{2} < \text{Im}(s) \leq \frac{\omega_p}{2}\right\}, \tag{B.9}$$

and $\psi_G(\Delta)$ encircles the origin n_p times in the counterclockwise direction.

We now generalize a key result from MIMO robust control theory (Doyle & Stein, 1981; Zames, 1966) to periodic time-varying systems, making use of Theorem B.1:

Theorem B.2. *Given a proper harmonic transfer operator $G(s) \in C_2(l_2)$ with no unstable poles and a bounded uncertainty set $\Phi \in \mathbb{R}^{n\times n}$ with a given sparsity structure, the closed loop system (B.2) is asymptotically stable if and only if $\det_2(\mathcal{I} + \tilde{\Delta}G(j\omega)) \neq 0$ for all $\Delta \in \Phi$ and $\omega \in \left(-\frac{\omega_p}{2}, \frac{\omega_p}{2}\right]$.*

Proof. First, we shall prove necessity by contradiction. Suppose (B.2) is not asymptotically stable. By the Nyquist criterion in Theorem B.2, the Nyquist plot of $\psi_G(\Delta) = \det_2(\mathcal{I} + \tilde{\Delta}G(s))\exp(-r(s+\rho))$ encircles or touches the origin for some $\omega \in \left(-\frac{\omega_p}{2}, \frac{\omega_p}{2}\right]$. Consider the homotopy

$$h(\epsilon) = \det{}_2(\mathcal{I} + \epsilon\tilde{\Delta}G(s))\exp(-r_\epsilon(s+\rho)), \tag{B.10}$$

with $\epsilon \in [0, 1]$, $\text{Im}(s) = \omega$, and $r_\epsilon(s+\rho) = -\text{tr}[((s+\rho)\mathcal{I} + \mathcal{N})^{-1}(\mathcal{A} + \rho\mathcal{I})(s\mathcal{I} + \mathcal{N} - \mathcal{A})^{-1}\mathcal{B}\epsilon\tilde{\Delta}\mathcal{C}]$. Because $\psi_G(\Delta)$ is meromorphic (analytic except at a countable number of points) (Zhou & Hagiwara, 2005), it holds by Lemma A.1.18 in Curtain and Zwart (1995) that if $h(\epsilon)$ vanishes nowhere, then $h(0)$ and $h(1)$ must have the same winding number, or Nyquist index. However, the curve at $\epsilon = 0$ is a point at 1, while the curve at $\epsilon = 1$ encircles or touches the origin. Thus $h(\epsilon)$ must vanish for some $\epsilon_0 \in (0, 1]$. Now since $\Delta \in \Phi$ implies that $\Delta_0 \triangleq \epsilon\Delta \in \Phi$, and because $\exp(-r_\epsilon(s+\rho)) \neq 0$ since $r(s+\rho)$ is bounded (see Zhou and Hagiwara (2005), Lemma 2.3), it must hold that $\det_2(\mathcal{I} + \tilde{\Delta}_0 G(j\omega)) = 0$ for some $\Delta_0 \in \Phi$.

Next, we prove sufficiency, also by contradiction. Suppose that there exist $\omega \in \left(-\frac{\omega_p}{2}, \frac{\omega_p}{2}\right]$ and $\Delta \in \Phi$ such that $\det_2(\mathcal{I} + \tilde{\Delta}G(j\omega)) = 0$. Then (B.2) has a pole on the imaginary axis and is not asymptotically stable. □

We now define the structured singular value for periodic linear time-varying systems in terms of the harmonic transfer operator:

$$\mu_{\mathbf{\Delta}}(G) \triangleq \frac{1}{\min\{\bar{\sigma}(\Delta) : \Delta \in \mathbf{\Delta},\ \det_2(\mathcal{I} + G(s)\tilde{\Delta}) = 0\}}, \tag{B.11}$$

where $\mathbf{\Delta}$ denotes a bounded structured uncertainty set and where we have invoked the Sylvester determinant lemma. Then Theorem B.2 may be recast using the structured singular value.

Theorem B.3. *Let $\Phi = \{\Delta : \Delta \in \mathbf{\Delta},\ \bar{\sigma}(\Delta) \leq \gamma\}$. Then the closed loop system (B.2) is stable if and only if $\mu_{\mathbf{\Delta}}(G(s)) < \frac{1}{\gamma}$ for all $s \in S$ given in (B.9).*

The final step of the analysis is developing a computationally tractable test, which requires a finite dimensional truncation of the infinite-dimensional operator $G(s)$. In Sandberg, Mollerstedt, and Bemhardsson (2005), the authors showed that $G(s)$ could be approximated arbitrarily well by a finite truncated operator consisting of only the first N terms of the Fourier series expansion of $A(t)$, $B(t)$, and $C(t)$. We denote by $G_N(s)$ the truncated operator given by

$$G_N(s) = \mathcal{C}_N[s\mathcal{I} - (\mathcal{A}_N - \mathcal{N})]^{-1}\mathcal{B}_N \tag{B.12}$$

where $\mathcal{A}_N$ is the $(N+1) \times (N+1)$ submatrix of $\mathcal{A}$ centered along the A_0 diagonal, and $\mathcal{B}_N$ and $\mathcal{C}_N$ defined similarly. In particular, it was shown that $\|G - G_N\|_2 \leq K(N) \triangleq O(N^{-1})$. From this fact and following the proof technique of Lemma 4.1 in Zhou and Hagiwara (2005), it then holds that

$$|\det{}_2(\mathcal{I} + G) - \det{}_2(\mathcal{I} + G_N)| \\ \leq \|G - G_N\|_2 \exp\left\{\frac{1}{2}\left[\|G\|_2 + \|G_N\|_2 + 1\right]\right\} \leq K(N) \cdot M, \tag{B.13}$$

where the second inequality follows from the boundedness of G and G_N. With the observation that the matrix Δ may be incorporated into the matrix $B(t)$ or $C(t)$, we see that the term $\det_2(\mathcal{I} + G\tilde{\Delta})$ appearing in Theorem B.2 and (B.11) may be asymptotically approximated by $\det_2(\mathcal{I} + G_N\tilde{\Delta}_N)$, where $\tilde{\Delta}_N$ is the truncated version of $\tilde{\Delta}$.

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